Exercise walk-through

Numerical solution of equations ▪ 3.6

eXercise

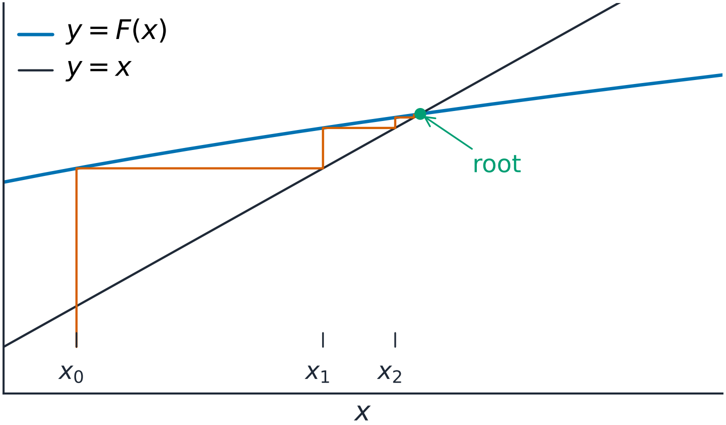
You must be able to

- show a root lies in an interval by a sign change
- show an equation rearranges into a given iterative form
- iterate $x_{n+1} = F(x_n)$ to the required accuracy

Method — Iteration to a root

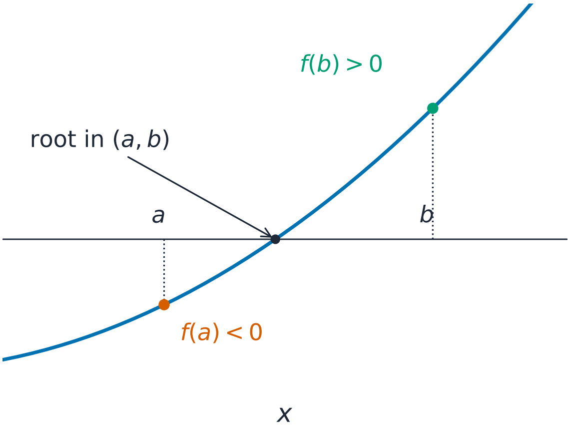
$x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, ... $\rightarrow$ root ≈ -1.26 (3 s.f.).

Method — Iteration to a root



A staircase between y=F(x) and y=x closing in on the root

Method — Iteration to a root



Locate the root first by a sign change

Practice 2.1 ▪ 2 marks

$f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$.

Solution 2.1

Method: Iteration to a root

Worked steps

$f(1) = e - 3 = -0.282$, $f(2) = e^2 - 6 = 1.389$ **M1**; opposite signs $\rightarrow$ a root lies between 1 and 2 **A1**.

Practice 2.2 ▪ 1 mark

Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration.

Solution 2.2

Worked steps

e.g. $x = \sqrt[3]{5 - x}$ **B1**.

Exam-style 3.1 ▪ 1 mark

An iteration $x_{n+1} = F(x_n)$ from a suitable start **converges** to a root if the values:

A grow without limit

B settle towards a fixed number

C alternate between two values forever

D stay equal to x_0

Solution 3.1

Method: Iteration to a root

A grow without limit

B settle towards a fixed number ✓

C alternate between two values forever

D stay equal to x_0

Worked steps

B. Convergence means the iterates settle towards a fixed value.

Exam-style 3.2 ▪ 2 marks

The equation $x^3 = 4x + 1$ has a root α near $x = 2$.
(a) Verify there is a root between $x = 2$ and $x = 2.2$.
(b) Show the equation can be written $x = \sqrt[3]{4x + 1}$ and use $x_{n+1} = \sqrt[3]{4x_n + 1}$, $x_0 = 2.1$, to find α to 2 d.p.

Solution 3.2

Method: Iteration to a root

Worked steps

(a) $f(x) = x^3 - 4x - 1$: $f(2) = -1 < 0$, $f(2.2) = 10.648 - 8.8 - 1 = 0.848 > 0$ **M1**; sign change $\rightarrow$ root between **A1**.
(b) $x^3 = 4x + 1 \Rightarrow x = \sqrt[3]{4x+1}$ **M1**; $x_1 = \sqrt[3]{9.4} = 2.110$, $x_2 = 2.113$, $x_3 = 2.114$ **M1** **M1**; $\alpha = 2.11$ **A1**.

Exam-style 3.3 ▪ 3 marks

By sketching $y = e^{-x}$ and $y = x$ on the same axes, state the number of roots of $e^{-x} = x$, and give an interval of width 0.5 containing the root.

Solution 3.3

Worked steps

the decreasing curve $y = e^{-x}$ meets the line $y = x$ exactly **once**, so there is 1 root **M1** **A1**; $g(x) = e^{-x} - x$ has $g(0.5) = 0.607 - 0.5 > 0$ and $g(1) = 0.368 - 1 < 0$, so the root lies between 0.5 and 1 **A1**.