# Exercise walk-through

## Numerical solution of equations ▪ 2.6

eXercise

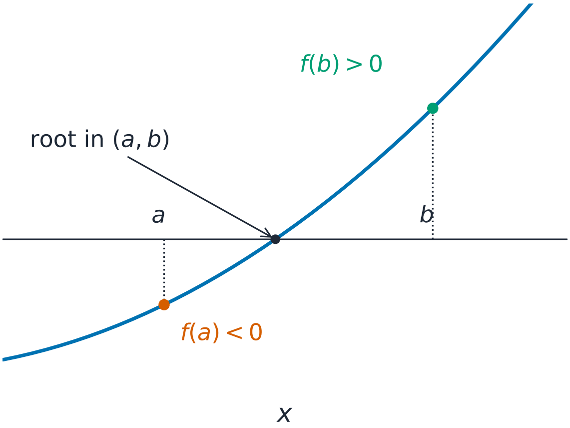
# You must be able to

- use a sign change to show a root lies in an interval
- carry out an iteration $x_{n+1} = F(x_n)$ until it converges
- show a given equation rearranges into a stated iterative form

# Method — Sign change & iteration

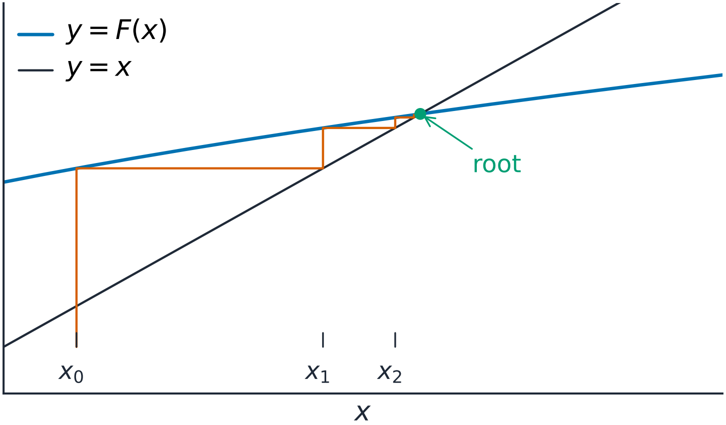
**Iterate** $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, ... $\to$ $\beta = -1.26$ (3 s.f.).

# Method — Sign change & iteration




*A curve crossing the x-axis between a and b*

# Method — Sign change & iteration




*A staircase between y=F(x) and y=x closing in on the root*

# Practice 2.1 ▪ 2 marks

$f(x) = x^3 - 5x + 1$. Evaluate $f(0)$ and $f(1)$, and state what they tell you.

# Solution 2.1

**Worked steps**

$f(0) = 1$, $f(1) = 1 - 5 + 1 = -3$ **M1**; opposite signs, so a root lies between $0$ and $1$ **A1**.

# Practice 2.2 ▪ 1 mark

State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$.

# Solution 2.2

## Worked steps

$x_{n+1} = \cos x_n$ $\boxed{\textbf{B1}}$.

# Exam-style 3.1 ▪ 1 mark

A root of $f(x) = 0$ lies between $x = 2$ and $x = 3$ if:

**A** $f(2)$ and $f(3)$ are both positive

**B** $f(2)$ and $f(3)$ have opposite signs

**C** $f(2) = f(3)$

**D** $f'(2) = 0$

# Solution 3.1

Method: Sign change & iteration

**A** $f(2)$ and $f(3)$ are both positive

**B** $f(2)$ and $f(3)$ have opposite signs ✓

**C** $f(2) = f(3)$

**D** $f'(2) = 0$

## Worked steps

**B**. Opposite signs (with a continuous graph) trap a root between them.

# Exam-style 3.2 ▪ 2 marks

The equation $x^3 - 6x + 2 = 0$ has a root $\alpha$ between $0$ and $1$.

(a) Verify this by a sign change.

(b) Show that the equation can be written as $x = \dfrac{x^3 + 2}{6}$, and use the iteration $x_{n+1} = \dfrac{x_n^3 + 2}{6}$ with $x_0 = 0.3$ to find $\alpha$ to 2 d.p.

# Solution 3.2

Method: Sign change & iteration

## Worked steps

(a) $f(0) = 2 > 0,\ f(1) = 1 - 6 + 2 = -3 < 0$ **M1**; opposite signs $\rightarrow$ root between $0$ and $1$ **A1**.

(b) $x^3 - 6x + 2 = 0 \Rightarrow 6x = x^3 + 2 \Rightarrow x = \dfrac{x^3 + 2}{6}$ **M1**; $x_1 = \dfrac{0.027 + 2}{6} = 0.3378,$ $x_2 = 0.3397,\ x_3 = 0.3398$ **M1** **M1**; $\alpha = 0.34$ **A1**.

# Exam-style 3.3 ▪ 2 marks

Explain why a sign change between $a$ and $b$ does **not** guarantee exactly one root in that interval.

# Solution 3.3

Method: Sign change & iteration

**Worked steps**

there could be several roots between $a$ and $b$ — the graph may cross the axis more than once (an odd number of times) **M1** **A1**.