Past-paper walk-through

Integration ▪ 2.5

eXercise

Questions in this set

- 9709/21 N25 Q1 ▪ 3 marks
- 9709/21 N25 Q5 ▪ 7 marks
- 9709/22 N25 Q5 ▪ 11 marks
- 9709/22 N25 Q6 ▪ 9 marks
- 9709/21 J25 Q4 ▪ 7 marks
- 9709/21 J25 Q7 ▪ 9 marks
- 9709/22 J25 Q1 ▪ 3 marks
- 9709/22 J25 Q4 ▪ 8 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 1

9709/21 N25 ▪ Q1 ▪ 3 marks

Find $\int 6\sin^2 x\,dx$. **[3]**

Solution 1

Mark scheme

- $6\sin^2 x = 3 - 3\cos 2x$, so $\int 6\sin^2 x\,dx = 3x - \frac{3}{2}\sin 2x + C$.

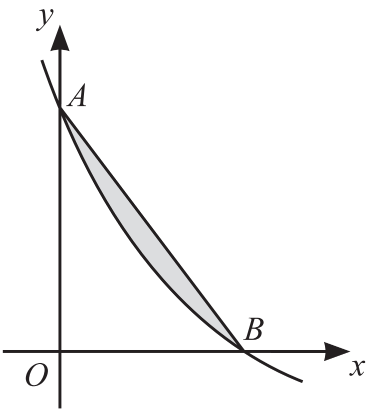
Question 5

9709/21 N25 ▪ Q5 ▪ 7 marks

The diagram shows the curve with equation $y = 8e^{-\frac{1}{2}x} - 1$. The curve meets the axes at the points A and B. The shaded region is bounded by the curve and the line segment AB.

(a) Show that the x-coordinate of B is $6\ln 2$. **[2]**

(b) Find the area of the shaded region. Give your answer in the form $p\ln 2 - q$, where p and q are positive integers. **[5]**



Solution 5

(a)

Mark scheme

- $y = 0 \Rightarrow 8e^{-x/2} = 1 \Rightarrow e^{-x/2} = \frac{1}{8} \Rightarrow -\frac{x}{2} = -3\ln 2 \Rightarrow x = 6\ln 2.$

Solution 5

(b)

Mark scheme

- $A = (0,7)$, $B = (6\ln 2, 0)$. Triangle area $= \frac{1}{2}(6\ln 2)(7) = 21\ln 2$; $\int_0^{6\ln 2}(8e^{-x/2} - 1)\,dx = 14 - 6\ln 2$. Shaded area $= 21\ln 2 - (14 - 6\ln 2) = 27\ln 2 - 14$.

Question 5

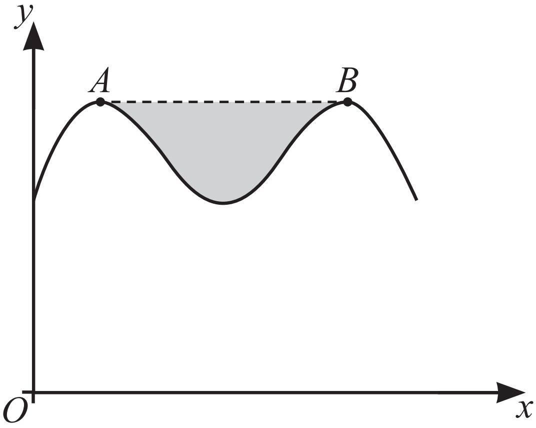
9709/22 N25 ▪ Q5 ▪ 11 marks

The diagram shows the curve with equation $y = 4\cos 2x + 8\sin x$ for $0 \leqslant x \leqslant \pi$. The maximum points on the curve are denoted by A and B, and the shaded region is bounded by the line segment AB and the curve.

(a) Find the coordinates of A and B. **[5]**

(b) Find the exact area of the shaded region. **[6]**

Question 5



Solution 5

(a) Worked steps

Differentiate, factorise, and use the **double-angle derivative** carefully.

- $y = 4\cos 2x + 8\sin x$
- $\dfrac{dy}{dx} = -8\sin 2x + 8\cos x$ — the $\cos 2x$ gives $-2\sin 2x$, times 4.
- Use $\sin 2x = 2\sin x\cos x$: $-16\sin x\cos x + 8\cos x = 8\cos x(1 - 2\sin x)$

Setting that to zero gives $\cos x = 0$ or $\sin x = \frac{1}{2}$:

Solution 5

- $\cos x = 0$ at $x = \frac{\pi}{2}$ — that is the **minimum** between the two peaks.
- $\sin x = \frac{1}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ — the two **maxima**.

At both, $y = 4\cos\frac{\pi}{3} + 8\left(\frac{1}{2}\right) = 2 + 4 = 6$:

Solution 5

$$A = \left(\frac{\pi}{6}, 6\right), \qquad B = \left(\frac{5\pi}{6}, 6\right)$$

Solution 5

Factorise rather than solving numerically — the product form hands you all three stationary points at once. And **use the whole range**: $\sin x = \frac{1}{2}$ has two solutions in $0 \le x \le \pi$, and reporting only $\frac{\pi}{6}$ loses half the answer.

Mark scheme

- $\dfrac{dy}{dx} = -8\sin 2x + 8\cos x = 8\cos x(1 - 2\sin x) = 0 \Rightarrow \sin x = \frac{1}{2}$ at the maxima. $A = \left(\frac{\pi}{6}, 6\right)$, $B = \left(\frac{5\pi}{6}, 6\right)$.

Solution 5

(b) Worked steps

A and B have the **same y-value**, so the chord AB is the horizontal line $y = 6$, and the area is the integral of **(line − curve)** between them.

$$\text{Area} = \int_{\pi/6}^{5\pi/6} (6 - 4\cos 2x - 8\sin x)\, dx$$

- Antiderivative: $6x - 2\sin 2x + 8\cos x$
- At $\frac{5\pi}{6}$: $5\pi - 2\left(-\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{\sqrt{3}}{2}\right) = 5\pi + \sqrt{3} - 4\sqrt{3} = 5\pi - 3\sqrt{3}$
- At $\frac{\pi}{6}$: $\pi - 2\left(\frac{\sqrt{3}}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right) = \pi - \sqrt{3} + 4\sqrt{3} = \pi + 3\sqrt{3}$
- Difference: $(5\pi - 3\sqrt{3}) - (\pi + 3\sqrt{3}) = \mathbf{4\pi - 6\sqrt{3}}$

Solution 5

Three things. **Line minus curve**, because the chord lies above the curve between the maxima — getting it the wrong way round gives the negative of the answer. $\int \cos 2x \, dx = \frac{1}{2} \sin 2x$, so the 4 becomes 2: dividing by the inner coefficient is the mark most often missed. And the answer is **exact**, so keep the π and the surd. Sense-check: $4\pi - 6\sqrt{3} \approx 12.57 - 10.39 = 2.18$, a small positive area, which matches a shallow region between a chord and a curve close beneath it.

Mark scheme

- Area
$$= \int_{\pi/6}^{5\pi/6} \left(6 - 4\cos 2x - 8\sin x\right) dx = \left[6x - 2\sin 2x + 8\cos x\right]_{\pi/6}^{5\pi/6} = 4\pi - 6\sqrt{3}.$$

Question 6

9709/22 N25 ▪ Q6 ▪ 9 marks

It is given that $\int_{-2a}^{a} \left(\frac{1}{2}e^{2x} + \frac{1}{4}e^{-x}\right) dx = 5$, where a is a positive constant.

(a) Show that $a = \frac{1}{2}\ln\left(10 + \frac{1}{2}e^{-a} + \frac{1}{2}e^{-4a}\right)$. **[4]**

(b) Hence show by calculation that the value of a lies between 1.0 and 1.2. **[2]**

(c) Use the iterative formula $a_{n+1} = \frac{1}{2}\ln\left(10 + \frac{1}{2}e^{-a_n} + \frac{1}{2}e^{-4a_n}\right)$ to find the value of a correct to 4 significant figures. Give the result of each iteration to 6 significant figures. **[3]**

Solution 6

(a)

Mark scheme

- $\int \left(\frac{1}{2}e^{2x} + \frac{1}{4}e^{-x}\right) dx = \frac{1}{4}e^{2x} - \frac{1}{4}e^{-x}$. Evaluating $-2a$ to a gives $\frac{1}{2}e^{2a} - \frac{1}{4}e^{-a} - \frac{1}{4}e^{-4a} = 5 \Rightarrow e^{2a} = 10 + \frac{1}{2}e^{-a} + \frac{1}{2}e^{-4a} \Rightarrow a = \frac{1}{2}\ln(\cdots)$. AG.

Solution 6

(b)

Mark scheme

- $F(a) = \frac{1}{2}\ln(10 + \frac{1}{2}e^{-a} + \frac{1}{2}e^{-4a}) - a$: $F(1.0) = +0.161 > 0$ and $F(1.2) = -0.041 < 0$; sign change $\Rightarrow$ root between 1.0 and 1.2.

Solution 6

(c)

Mark scheme

- Iterating from $a_0 = 1.1$: $a_1 = 1.15986$, $a_2 = 1.15930$, $a_3 = 1.15936, \ldots \Rightarrow a = 1.159$ (4 s.f.).

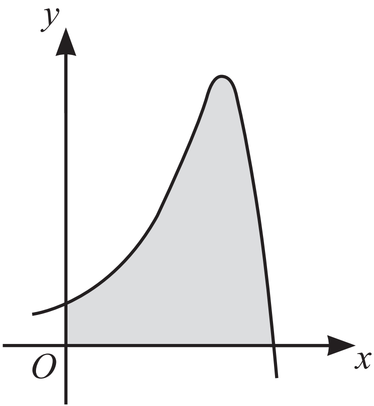
Question 4

9709/21 J25 ▪ Q4 ▪ 7 marks

The diagram shows the curve with equation $y = 6e^{2x} - e^{3x}$. The shaded region is bounded by the axes and the curve.

(a) Find the exact x-coordinate of the maximum point. **[3]**

(b) Find the area of the shaded region. Give your answer in the form $\frac{p}{q}$, where p and q are integers. **[4]**



Solution 4

(a)

Mark scheme

- $\dfrac{dy}{dx} = 12e^{2x} - 3e^{3x} = 0 \Rightarrow e^{x} = 4 \Rightarrow x = \ln 4$ (i.e. $2\ln 2$).

Solution 4

(b)

Mark scheme

- Curve meets x-axis at $x = \ln 6$. Area $= \int_0^{\ln 6} \left(6e^{2x} - e^{3x}\right)\, dx = \left[3e^{2x} - \frac{1}{3}e^{3x}\right]_0^{\ln 6} = \dfrac{100}{3}$.

Question 7

9709/21 J25 ▪ Q7 ▪ 9 marks

(a) Prove that $\sin^2 2x + 4\cos^2 x \cos 2x \equiv 4\cos^4 x$. **[3]**

(b) Find the set of possible values of the constant k for which the equation $\sin^2 2x + 4\cos^2 x \cos 2x + 5 = k$ has no real solutions. **[2]**

(c) Find the exact value of $\displaystyle\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi} \sqrt{\sin^2 t + 4\cos^2\left(\tfrac{1}{2}t\right)\cos t}\, dt$. **[4]**

Solution 7

(a)

Mark scheme

- $\sin^2 2x + 4\cos^2 x \cos 2x = 4\sin^2 x \cos^2 x + 4\cos^2 x(\cos^2 x - \sin^2 x) = 4\cos^4 x$ (AG).

Solution 7

(b)

Mark scheme

- $4\cos^4 x \in [0,4]$, so LHS $+5 \in [5,9]$; no real solutions when $k < 5$ or $k > 9$.

Solution 7

(c)

Mark scheme

- By part (a) (with $x = \frac{t}{2}$) the integrand $= 2\cos^2\frac{t}{2} = 1 + \cos t$; $\int_{-\pi/3}^{\pi/3}(1 + \cos t)\,dt = \left[t + \sin t\right]_{-\pi/3}^{\pi/3} = \frac{2}{3}\pi + \sqrt{3}$.

Question 1

9709/22 J25 ▪ Q1 ▪ 3 marks

Show that $\displaystyle\int_2^{11} \frac{8}{4x+1}\,dx = \ln a$, where a is an integer to be found. **[3]**

Solution 1

Mark scheme

- $\int_2^{11} \frac{8}{4x+1}\,dx = \left[2\ln(4x+1)\right]_2^{11} = 2\ln 45 - 2\ln 9 = 2\ln 5 = \ln 25$, so $a = 25$.

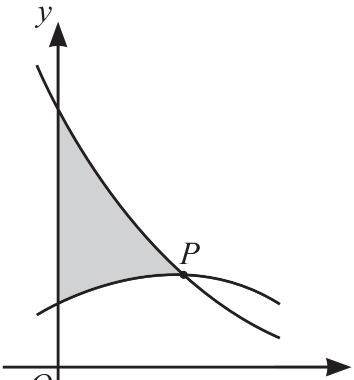
Question 4

9709/22 J25 ▪ Q4 ▪ 8 marks

The diagram shows parts of the curves with equations $y = 4e^{-2x}$ and $y = 1 + 0.5\sin 3x$. Point P is a point of intersection of the curves, and the shaded region is bounded by the two curves and the y-axis.

(a) Show that the x-coordinate of P satisfies the equation $x = -0.5\ln(0.25 + 0.125\sin 3x)$. **[1]**

(b) Use an iterative formula, based on the equation in part (a), to find the x-coordinate of P correct to 4 significant figures. Use an initial value of 0.5 and



Solution 4

(a)

Mark scheme

- At P: $4e^{-2x} = 1 + 0.5\sin 3x \Rightarrow e^{-2x} = 0.25 + 0.125\sin 3x \Rightarrow x = -0.5\ln(0.25 + 0.125\sin 3x)$ (AG).

Solution 4

(b)

Mark scheme

- Iterating $x_{n+1} = -0.5\ln(0.25 + 0.125\sin 3x_n)$ from $x_0 = 0.5$ converges to $x = 0.4912$ (4 s.f.).

Solution 4

(c)

Mark scheme

- Area $= \int_0^{0.4912} \left[4e^{-2x} - (1 + 0.5\sin 3x)\right] dx = \left[-2e^{-2x} - x + \frac{1}{6}\cos 3x\right]_0^{0.4912} = 0.61$ (2 s.f.).