Past-paper walk-through

Differentiation ▪ 2.4

eXercise

Questions in this set

- 9709/21 N25 Q6 ▪ 9 marks
- 9709/21 N25 Q8 ▪ 6 marks
- 9709/22 N25 Q5 ▪ 11 marks
- 9709/22 N25 Q7 ▪ 8 marks
- 9709/21 J25 Q1 ▪ 2 marks
- 9709/21 J25 Q4 ▪ 7 marks
- 9709/21 J25 Q6 ▪ 10 marks
- 9709/22 J25 Q3 ▪ 5 marks
- 9709/22 J25 Q6 ▪ 9 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

9709/21 N25 ▪ Q6 ▪ 9 marks

A curve has parametric equations $x = \tan\theta$, $y = \sin\theta - 2\sin^3\theta$, for $0 < \theta < \frac{1}{2}\pi$.

(a) Show that $\dfrac{dy}{dx} = 6\cos^5\theta - 5\cos^3\theta$. **[4]**

(b) Find the equation of the normal to the curve at the point where it crosses the x-axis. Give your answer in the form $y = mx + c$, where m and c are exact constants. **[5]**

Solution 6

(a) Worked steps

Parametric differentiation is one rule: **differentiate each equation with respect to the parameter, then divide.**

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

- $x = \tan\theta$, so $\dfrac{dx}{d\theta} = \sec^2\theta$.
- $y = \sin\theta - 2\sin^3\theta$, so $\dfrac{dy}{d\theta} = \cos\theta - 6\sin^2\theta\cos\theta = \cos\theta\left(1 - 6\sin^2\theta\right)$.
- Dividing, and using $\dfrac{1}{\sec^2\theta} = \cos^2\theta$:

Solution 6

$$\frac{dy}{dx} = \cos^3\theta\left(1 - 6\sin^2\theta\right)$$

Now convert to cosines with $\sin^2\theta = 1 - \cos^2\theta$:

- $1 - 6\left(1 - \cos^2\theta\right) = 6\cos^2\theta - 5$
- $\dfrac{dy}{dx} = \cos^3\theta\left(6\cos^2\theta - 5\right) = \boldsymbol{6\cos^5\theta - 5\cos^3\theta}$ **(AG)**

Solution 6

Two things carry the marks. **Dividing by** $\sec^2\theta$ **multiplies by** $\cos^2\theta$ — that is where the $\cos^3$ comes from, and writing $\dfrac{1}{\sec^2\theta}$ explicitly keeps it visible. And the **target form is all cosines**, which tells you which identity to use: the answer's shape is the instruction.

Solution 6

A *show that* needs the final line written out even though it is printed in the question.

Mark scheme

- $\dfrac{dx}{d\theta} = \sec^2\theta$, $\dfrac{dy}{d\theta} = \cos\theta(1 - 6\sin^2\theta)$, so $\dfrac{dy}{dx} = \cos^3\theta(1 - 6\sin^2\theta) = \cos^3\theta(6\cos^2\theta - 5) = 6\cos^5\theta - 5\cos^3\theta$.

Solution 6

(b)

Mark scheme

- $y = 0 \Rightarrow \sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}$, point $(1, 0)$. There $\dfrac{dy}{dx} = -\frac{1}{\sqrt{2}}$, so the normal gradient is $\sqrt{2}$: $y = \sqrt{2}\,x - \sqrt{2}$.

Question 8

9709/21 N25 ▪ Q8 ▪ 6 marks

The equation of a curve is $y = 4e^{1-2x}\sqrt{3x-1}$. Find the exact coordinates of the stationary point of the curve. **[6]**

Solution 8

Worked steps

A **product** of $4e^{1-2x}$ and $(3x-1)^{1/2}$, each needing the **chain rule**.

- $u = 4e^{1-2x}$, so $u' = -8e^{1-2x}$ (the inner function differentiates to -2).
- $v = (3x-1)^{1/2}$, so $v' = \dfrac{3}{2\sqrt{3x-1}}$.

Product rule, then factorise:

$$\frac{dy}{dx} = 4e^{1-2x}\left[-2\sqrt{3x-1} + \frac{3}{2\sqrt{3x-1}}\right]$$

Set it to zero. **An exponential is never zero**, so only the bracket can vanish:

- $2\sqrt{3x-1} = \dfrac{3}{2\sqrt{3x-1}}$

Solution 8

Worked steps

- $4(3x-1)=3$, so $3x-1=\frac{3}{4}$ and $x=\mathbf{\frac{7}{12}}$

Then the y-coordinate:

- $\sqrt{3x-1}=\dfrac{\sqrt{3}}{2}$, and $1-2x=1-\frac{7}{6}=-\frac{1}{6}$
- $y=4e^{-1/6}\cdot\dfrac{\sqrt{3}}{2}=\mathbf{2\sqrt{3}\,e^{-1/6}}$

Three points:

- **Factor the exponential out** before solving. It is common to both terms, never zero, and taking it outside turns a messy equation into one line.
- **Both chain-rule factors** — the -2 from e^{1-2x} and the 3 from $(3x-1)$ — are marks.

Solution 8

Worked steps

- **Exact coordinates** are asked for, so leave the surd and the exponential.

Check the domain: $\sqrt{3x-1}$ needs $x > \frac{1}{3}$, and $\frac{7}{12}$ satisfies it. **Mark scheme**

- $\dfrac{dy}{dx} = 4e^{1-2x}\left[-2\sqrt{3x-1} + \dfrac{3}{2\sqrt{3x-1}}\right] = 0 \Rightarrow 3x-1 = \frac{3}{4} \Rightarrow x = \frac{7}{12}$,
  $y = 2\sqrt{3}\,e^{-1/6}$. Stationary point $\left(\frac{7}{12},\ 2\sqrt{3}\,e^{-\frac{1}{6}}\right)$.

Question 5

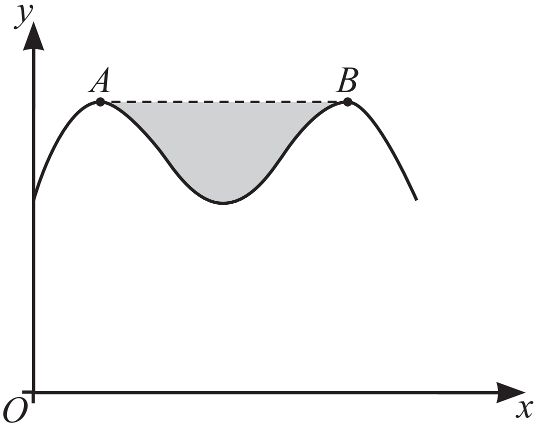
9709/22 N25 ▪ Q5 ▪ 11 marks

The diagram shows the curve with equation $y = 4\cos 2x + 8\sin x$ for $0 \leqslant x \leqslant \pi$. The maximum points on the curve are denoted by A and B, and the shaded region is bounded by the line segment AB and the curve.

(a) Find the coordinates of A and B. **[5]**

(b) Find the exact area of the shaded region. **[6]**

Question 5



Solution 5

(a) Worked steps

Differentiate, factorise, and use the **double-angle derivative** carefully.

- $y = 4\cos 2x + 8\sin x$
- $\dfrac{dy}{dx} = -8\sin 2x + 8\cos x$ — the $\cos 2x$ gives $-2\sin 2x$, times 4.
- Use $\sin 2x = 2\sin x\cos x$: $-16\sin x\cos x + 8\cos x = 8\cos x(1 - 2\sin x)$

Setting that to zero gives $\cos x = 0$ or $\sin x = \frac{1}{2}$:

Solution 5

- $\cos x = 0$ at $x = \frac{\pi}{2}$ — that is the **minimum** between the two peaks.
- $\sin x = \frac{1}{2}$ at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ — the two **maxima**.

At both, $y = 4\cos\frac{\pi}{3} + 8\left(\frac{1}{2}\right) = 2 + 4 = 6$:

Solution 5

$$A = \left(\frac{\pi}{6}, 6\right), \qquad B = \left(\frac{5\pi}{6}, 6\right)$$

Solution 5

Factorise rather than solving numerically — the product form hands you all three stationary points at once. And **use the whole range**: $\sin x = \frac{1}{2}$ has two solutions in $0 \leq x \leq \pi$, and reporting only $\frac{\pi}{6}$ loses half the answer.

Mark scheme

- $\dfrac{dy}{dx} = -8\sin 2x + 8\cos x = 8\cos x(1 - 2\sin x) = 0 \Rightarrow \sin x = \frac{1}{2}$ at the maxima. $A = \left(\frac{\pi}{6}, 6\right)$, $B = \left(\frac{5\pi}{6}, 6\right)$.

Solution 5

(b) Worked steps

A and B have the **same y-value**, so the chord AB is the horizontal line $y = 6$, and the area is the integral of **(line − curve)** between them.

$$\text{Area} = \int_{\pi/6}^{5\pi/6} (6 - 4\cos 2x - 8\sin x)\,dx$$

- Antiderivative: $6x - 2\sin 2x + 8\cos x$
- At $\frac{5\pi}{6}$: $5\pi - 2\left(-\frac{\sqrt{3}}{2}\right) + 8\left(-\frac{\sqrt{3}}{2}\right) = 5\pi + \sqrt{3} - 4\sqrt{3} = 5\pi - 3\sqrt{3}$
- At $\frac{\pi}{6}$: $\pi - 2\left(\frac{\sqrt{3}}{2}\right) + 8\left(\frac{\sqrt{3}}{2}\right) = \pi - \sqrt{3} + 4\sqrt{3} = \pi + 3\sqrt{3}$
- Difference: $(5\pi - 3\sqrt{3}) - (\pi + 3\sqrt{3}) = \mathbf{4\pi - 6\sqrt{3}}$

Solution 5

Three things. **Line minus curve**, because the chord lies above the curve between the maxima — getting it the wrong way round gives the negative of the answer. $\int \cos 2x \, dx = \frac{1}{2} \sin 2x$, so the 4 becomes 2: dividing by the inner coefficient is the mark most often missed. And the answer is **exact**, so keep the π and the surd. Sense-check: $4\pi - 6\sqrt{3} \approx 12.57 - 10.39 = 2.18$, a small positive area, which matches a shallow region between a chord and a curve close beneath it.

Mark scheme

- Area

$$= \int_{\pi/6}^{5\pi/6} (6 - 4\cos 2x - 8\sin x) \, dx = \left[6x - 2\sin 2x + 8\cos x\right]_{\pi/6}^{5\pi/6} = 4\pi - 6\sqrt{3}.$$

Question 7

9709/22 N25 ▪ Q7 ▪ 8 marks

A curve has equation $5x^2y + 4e^{2y} - 7x + 10 = 0$.

(a) Find an expression in terms of x and y for $\dfrac{dy}{dx}$ and hence find the gradient of the curve at the point for which $y = 0$. **[6]**

(b) Show that there is no point on the curve at which the tangent is parallel to the y-axis. **[2]**

Solution 7

(a) Worked steps

Differentiate **every term with respect to** x, remembering that a term containing y carries a factor of $\frac{dy}{dx}$ by the chain rule, and that $5x^2y$ needs the **product rule**.

- $5x^2y \to 10xy + 5x^2\frac{dy}{dx}$
- $4e^{2y} \to 8e^{2y}\frac{dy}{dx}$
- $-7x \to -7$; the constant $10 \to 0$

Solution 7

So $10xy + 5x^2\dfrac{dy}{dx} + 8e^{2y}\dfrac{dy}{dx} - 7 = 0$, and collecting:

$$\frac{dy}{dx} = \frac{7 - 10xy}{5x^2 + 8e^{2y}}$$

The gradient at $y = 0$ needs the x-value first — put $y = 0$ into the **original** equation: $0 + 4 - 7x + 10 = 0$, so $x = 2$.

- $\dfrac{dy}{dx} = \dfrac{7 - 0}{20 + 8} = \dfrac{7}{28} = \frac{1}{4}$

Two habits. **Every** y **term gets** $\dfrac{dy}{dx}$ — the term most often missed is the one inside a product. And **find the missing coordinate from the original equation**, not from the derivative. Mark scheme

Solution 7

- $10xy + 5x^2\dfrac{dy}{dx} + 8e^{2y}\dfrac{dy}{dx} - 7 = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{7 - 10xy}{5x^2 + 8e^{2y}}$. At $y = 0$ the curve gives $x = 2$, so gradient $= \dfrac{7}{20 + 8} = \frac{1}{4}$.

Solution 7

(b) Worked steps

A tangent parallel to the y-axis is **vertical**, which means $\dfrac{dy}{dx}$ is undefined — so the question is whether the **denominator** can be zero.

- Denominator: $5x^2 + 8e^{2y}$
- $5x^2 \geqslant 0$ for every real x.
- $8e^{2y} > 0$ for every real y — an exponential is **never** zero or negative.
- So the sum is **strictly positive** and can never be zero: there is no such point. **Shown.**

Solution 7

Both inequalities are needed, and note they are different: the square is $\geqslant 0$ (it can be zero, at $x = 0$), while the exponential is **strictly** > 0. It is the strict one that makes the total positive.

The general technique: a vertical tangent is a **zero denominator**, a horizontal one is a **zero numerator**. Naming which you are testing is the first line of any question of this shape.

Mark scheme

- A vertical tangent needs $5x^2 + 8e^{2y} = 0$, but $5x^2 \geqslant 0$ and $8e^{2y} > 0$ for all real x, y, so the denominator is never zero — no such point exists.

Question 1

9709/21 J25 ▪ Q1 ▪ 2 marks

Given that $y = 6x\cos(x^2 + 1)$, find an expression for $\dfrac{dy}{dx}$. **[2]**

Solution 1

Worked steps

Two rules at once. $y = 6x\cos(x^2+1)$ is a **product** of $6x$ and $\cos(x^2+1)$, and the second factor needs the **chain rule**.

- $u = 6x$, so $u' = 6$.
- $v = \cos(x^2+1)$. The chain rule: differentiate the cosine, then multiply by the derivative of the inside — $v' = -\sin(x^2+1) \times 2x = -2x\sin(x^2+1)$.
- Product rule, $u'v + uv'$:

Solution 1

$$\frac{dy}{dx} = 6\cos(x^2+1) + 6x\left(-2x\sin(x^2+1)\right) = \mathbf{6}\cos(\mathbf{x^2}+\mathbf{1}) - \mathbf{12x^2}\sin(\mathbf{x^2}+\mathbf{1})$$

Two marks, and the two things they check are the two rules. The **chain-rule factor** $2x$ is what most often goes missing — without it the second term is $-6x\sin(x^2+1)$. The sign is negative because $\dfrac{d}{dx}\cos = -\sin$. Writing $+12x^2\sin$ is the other half of the same slip.

Solution 1

Identify the structure before differentiating anything: *product, with a chain inside the second factor*. Naming it takes five seconds and settles which rule wraps which.

Mark scheme

- $\dfrac{dy}{dx} = 6\cos(x^2+1) - 12x^2\sin(x^2+1)$ (product rule with chain rule).

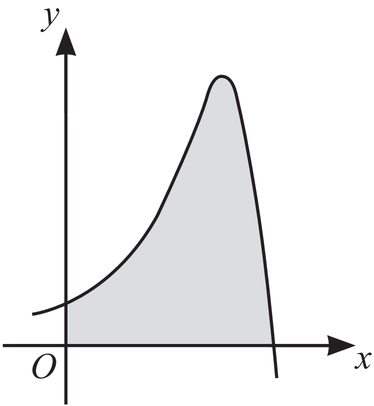
Question 4

9709/21 J25 ▪ Q4 ▪ 7 marks

The diagram shows the curve with equation $y = 6e^{2x} - e^{3x}$. The shaded region is bounded by the axes and the curve.

(a) Find the exact x-coordinate of the maximum point. **[3]**

(b) Find the area of the shaded region. Give your answer in the form $\frac{p}{q}$, where p and q are integers. **[4]**



Solution 4

(a)

Mark scheme

- $\frac{dy}{dx} = 12e^{2x} - 3e^{3x} = 0 \Rightarrow e^{x} = 4 \Rightarrow x = \ln 4$ (i.e. $2\ln 2$).

Solution 4

(b)

Mark scheme

- Curve meets x-axis at $x = \ln 6$. Area $= \int_0^{\ln 6} \left(6e^{2x} - e^{3x}\right)\,dx = \left[3e^{2x} - \frac{1}{3}e^{3x}\right]_0^{\ln 6} = \dfrac{100}{3}$.

Question 6

9709/21 J25 ▪ Q6 ▪ 10 marks

The parametric equations of a curve are $x = \dfrac{2t+1}{3t+4}$, $y = 2\ln(3t+4)$, where $t > -\frac{4}{3}$.

(a) Show that $\dfrac{dy}{dx}$ can be expressed in the form $c(3t+4)$ and state the value of the constant c. **[5]**

(b) It is given that the gradient of the curve at the point $(a, \ln 100)$ is m. Find the values of a and m. **[4]**

(c) State whether the curve represents a decreasing function or an increasing function or neither. Give a reason for your answer. **[1]**

Solution 6

(a) Worked steps

To show **no point exists**, substitute the value and show the resulting equation has **no real solution** — that is what the discriminant is for.

- Put $y = e^{-1}$, so $\ln y = -1$.
- The relation becomes $-(x^2 - 3) + 6x = 14$, that is $x^2 - 6x + 11 = 0$.
- Discriminant: $(-6)^2 - 4(1)(11) = 36 - 44 = -8 < 0$.
- No real x, so there is no such point. **Shown.**

Solution 6

The discriminant answers "are there solutions?" without finding them, which is exactly what is being asked. Attempting to solve and reporting an error on the calculator is not a proof.
Simplify $\ln\left(e^{-1}\right) = -1$ first. Carrying the exponential through the algebra makes a straightforward quadratic look intractable. **Mark scheme**

- $\dfrac{dx}{dt} = \dfrac{5}{(3t+4)^2}$, $\dfrac{dy}{dt} = \dfrac{6}{3t+4}$, so $\dfrac{dy}{dx} = \dfrac{6}{3t+4} \cdot \dfrac{(3t+4)^2}{5} = \frac{6}{5}(3t+4)$; $c = \frac{6}{5}$.

Solution 6

(b) Worked steps

Differentiate implicitly, then use the point.

- $x^2 \ln y$ is a **product**, so it gives $2x \ln y + x^2 \cdot \frac{1}{y}\frac{dy}{dx}$ — the second factor by the chain rule, since $\frac{d}{dx}\ln y = \frac{1}{y}\frac{dy}{dx}$.
- The full derivative: $2x \ln y + \frac{x^2 - 3}{y}\frac{dy}{dx} + 6 = 0$
- At $(2, e^2)$, where $\ln y = 2$: $8 + \frac{1}{e^2}\frac{dy}{dx} + 6 = 0$
- $\frac{dy}{dx} = -14e^2$

Solution 6

Then the tangent through $(2, e^2)$ with that gradient:

- $y - e^2 = -14e^2(x - 2)$
- $y = -14e^2x + 28e^2 + e^2 = \mathbf{-14e^2x + 29e^2}$

Solution 6

Three points. $\ln y$ **differentiates to** $\dfrac{1}{y}\dfrac{dy}{dx}$, not to $\dfrac{1}{y}$ — the chain-rule factor is the mark.

Substitute the point immediately after differentiating rather than rearranging for $\dfrac{dy}{dx}$ in general; the numbers are far simpler. And the question wants **exact** constants, so leave the e^2 terms alone.

Mark scheme

- $2\ln(3t+4) = \ln 100 \Rightarrow 3t+4 = 10 \Rightarrow t = 2$; then $a = \dfrac{2(2)+1}{3(2)+4} = \frac{1}{2}$ and $m = \frac{6}{5}(10) = 12$.

Solution 6

(c)

Mark scheme

- Increasing function: the gradient $\frac{6}{5}(3t+4) > 0$ for all $t > -\frac{4}{3}$.

Question 3

9709/22 J25 ▪ Q3 ▪ 5 marks

Find the coordinates of the stationary points of the curve with equation

$y = \dfrac{8x}{2x+3} - 6x + 5.$ **[5]**

Solution 3

Mark scheme

- $\dfrac{dy}{dx} = \dfrac{24}{(2x+3)^2} - 6 = 0 \Rightarrow (2x+3)^2 = 4 \Rightarrow x = -\frac{1}{2}$ or $-\frac{5}{2}$. Stationary points $\left(-\frac{1}{2}, 6\right)$ and $\left(-\frac{5}{2}, 30\right)$.

Question 6

9709/22 J25 ▪ Q6 ▪ 9 marks

A curve has equation $(x^2 - 3)\ln y + 6x = 14$.

(a) Show that there is no point on the curve at which the y-coordinate is e^{-1}. **[3]**

(b) Find the equation of the tangent to the curve at the point $(2, e^2)$. Give your answer in the form $y = mx + c$, where m and c are exact constants. **[6]**

Solution 6

(a)

Mark scheme

- Put $y = e^{-1}$ ($\ln y = -1$): $-(x^2 - 3) + 6x = 14 \Rightarrow x^2 - 6x + 11 = 0$; discriminant $36 - 44 = -8 < 0$, so there is no real x — no such point (AG).

Solution 6

(b) Mark scheme

- Implicit differentiation: $2x\ln y + \dfrac{x^2-3}{y}\dfrac{dy}{dx} + 6 = 0$. At $(2, e^2)$: $8 + \dfrac{1}{e^2}\dfrac{dy}{dx} + 6 = 0 \Rightarrow \dfrac{dy}{dx} = -14e^2$. Tangent: $y = -14e^2x + 29e^2$.