Past-paper walk-through

Integration ▪ 1.8

eXercise

Questions in this set

- 9709/11 N25 Q8 ▪ 7 marks
- 9709/11 N25 Q11 ▪ 11 marks
- 9709/12 N25 Q4 ▪ 6 marks
- 9709/12 N25 Q5 ▪ 8 marks
- 9709/11 J25 Q2 ▪ 6 marks
- 9709/11 J25 Q4 ▪ 5 marks
- 9709/12 J25 Q6 ▪ 6 marks
- 9709/12 J25 Q9 ▪ 12 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 8

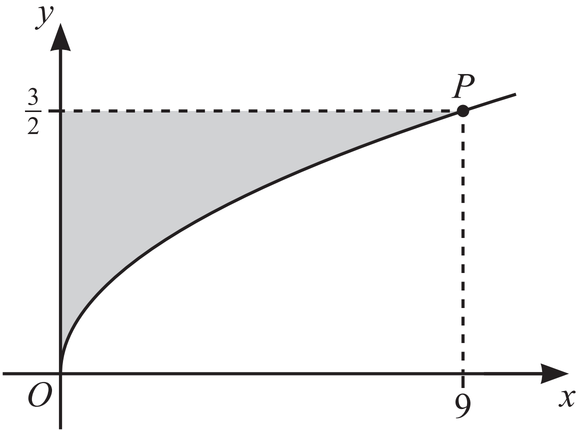
9709/11 N25 ▪ Q8 ▪ 7 marks

The diagram shows the curve with equation $y = \frac{1}{2}\sqrt{x}$ and the point P with coordinates $\left(9, \frac{3}{2}\right)$. The shaded region is bounded by the curve and the lines $x = 0$ and $y = \frac{3}{2}$.

(a) Find the area of the shaded region. **[3]**

(b) The shaded region is rotated through 360 about the y-axis. Find the exact volume of the solid produced. **[4]**

Question 8



Solution 8

(a) Worked steps

The region is bounded **above** by $y = \frac{3}{2}$ and **below** by the curve, so it is the rectangle minus the area under the curve — not an integral of the curve alone.

- Area under the curve from 0 to 9: $\int_0^9 \frac{1}{2}x^{1/2}\,dx = \left[\frac{1}{3}x^{3/2}\right]_0^9 = \frac{1}{3}(27) = 9$
- Rectangle: $9 \times \frac{3}{2} = \frac{27}{2}$
- Shaded area: $\frac{27}{2} - 9 = \mathbf{\frac{9}{2}}$

Solution 8

Sketch the region and mark which boundary is on top before integrating anything. That one habit prevents the standard error here, which is to report 9 — the area under the curve, a different region entirely.

The alternative is to integrate with respect to y — $x = 4y^2$, from $y = 0$ to $\frac{3}{2}$ — which gives $\frac{9}{2}$ directly. Either route scores; the first is safer if the sketch is in front of you.

Mark scheme

- $\int_0^9 \frac{1}{2}\sqrt{x}\,dx = \left[\frac{1}{3}x^{3/2}\right]_0^9 = 9$; shaded area $= \left(9 \times \frac{3}{2}\right) - 9 = \frac{9}{2}$.

Solution 8

(b) Worked steps

Rotation about the y-**axis** means integrating πx^2 with respect to y, so the equation has to be rearranged first.

- From $y = \frac{1}{2}\sqrt{x}$: $\sqrt{x} = 2y$, so $x = 4y^2$ and $x^2 = 16y^4$.
- $V = \pi \int_0^{3/2} 16y^4 \, dy = \pi \left[\frac{16}{5}y^5\right]_0^{3/2}$
- $= \pi \times \frac{16}{5} \times \frac{243}{32} = \mathbf{\dfrac{243}{10}}\pi$

Solution 8

Three things decide it:

- **Rearrange for x in terms of y.** Rotating about the y-axis and integrating $\pi y^2\,dx$ is the single commonest error in this topic.
- **The limits are y-values**, 0 to $\frac{3}{2}$ — not the x-values 0 to 9.
- **Exact**, so leave the π and the fraction. A decimal loses the mark.

Solution 8

$\left(\frac{3}{2}\right)^5 = \frac{243}{32}$, and $\frac{16}{32} = \frac{1}{2}$, which is what makes the arithmetic collapse to $\frac{243}{10}$.

Mark scheme

- $x^2 = 16y^4$; volume $= \pi \int_0^{3/2} 16y^4\, dy = \pi \left[\frac{16}{5} y^5\right]_0^{3/2} = \dfrac{243}{10}\pi$.

Question 11

9709/11 N25 ▪ Q11 ▪ 11 marks

A curve passes through the point $P(4, 3)$ and is such that $\dfrac{dy}{dx} = \dfrac{8}{x^2} - \dfrac{10}{(2x-3)^2}$.

(a) Find the equation of the normal to the curve at P. Give your answer in the form $y = mx + c$. **[3]**

(b) Find the rate of change of the gradient of the curve when $x = 4$. **[3]**

(c) Given that the curve also passes through the point $(-1, q)$, find the value of q. **[5]**

Solution 11

(a)

Mark scheme

- At $x = 4$, gradient $= \dfrac{8}{16} - \dfrac{10}{25} = \frac{1}{10}$; normal gradient -10, so $y = -10x + 43$.

Solution 11

(b) Worked steps

Differentiate again, treating $(2x-3)^{-2}$ with the **chain rule** — the inner function differentiates to 2.

- $\dfrac{d^2y}{dx^2} = -16x^{-3} + 40(2x-3)^{-3}$
- At $x = 4$: $-16(4)^{-3} + 40(5)^{-3} = -\frac{16}{64} + \frac{40}{125} = -0.25 + 0.32 = \mathbf{\frac{7}{100}}$

Solution 11

The chain-rule factor is where these go wrong. Differentiating $(2x-3)^{-2}$ gives $-2(2x-3)^{-3}\times 2=-4(2x-3)^{-3}$, and the extra 2 is the mark.
At $x=4$ the second derivative is **positive**, so the stationary point there is a **minimum** — worth stating if the question asks for the nature of the point.

Mark scheme

- $\dfrac{d^2y}{dx^2}=-16x^{-3}+40(2x-3)^{-3}$; at $x=4$ this is $\dfrac{7}{100}$.

Solution 11

(c) Worked steps

Going from $\dfrac{dy}{dx}$ back to y is integration, and the **constant** is the whole question.

- $y = \displaystyle\int \left(8x^{-2} - 10(2x-3)^{-2}\right) dx = -8x^{-1} + 5(2x-3)^{-1} + c$
- Use the known point $(4,3)$: $3 = -\frac{8}{4} + \frac{5}{5} + c = -2 + 1 + c$, so $c = 4$.
- At $x = -1$: $q = -\dfrac{8}{-1} + \dfrac{5}{-5} + 4 = 8 - 1 + 4 = \mathbf{11}$

Solution 11

Two chain-rule divisions to watch. Integrating $(2x-3)^{-2}$ gives $\dfrac{(2x-3)^{-1}}{-1\times 2} = -\frac{1}{2}(2x-3)^{-1}$, so the -10 becomes $+5$. Dividing by the inner coefficient is the step most often skipped.

And **find c before substituting the second point**. Without it there is one equation and two unknowns, and the answer cannot be pinned down.

Mark scheme

- $y = -8x^{-1} + 5(2x-3)^{-1} + c$; through $(4,3)$: $3 = -2 + 1 + c \Rightarrow c = 4$. At $x = -1$: $q = 8 - 1 + 4 = 11$.

Question 4

9709/12 N25 ▪ Q4 ▪ 6 marks

The equation of a curve is such that $\dfrac{dy}{dx} = kx^3 + \dfrac{2}{x^2}$, where k is a constant. The curve passes through the point $S(2, 20)$ and the gradient of the curve at S is $\dfrac{65}{2}$.

(a) Find the value of k. **[1]**

(b) The coordinates of a point T on the curve are $(1, t)$. Find the value of t. **[5]**

Solution 4

(a)

Mark scheme

- At S $(x = 2)$: $8k + \frac{1}{2} = \frac{65}{2} \Rightarrow 8k = 32 \Rightarrow k = 4$.

Solution 4

(b) Worked steps

The gradient function is known, so integrate to get the curve — and the arbitrary constant is what the given point is for.

- $y = \int (4x^3 + 2x^{-2})\,dx = x^4 - \dfrac{2}{x} + c$
- Through $S(2, 20)$: $16 - 1 + c = 20$, so $c = 5$.
- At $T(1, t)$: $t = 1 - 2 + 5 = \mathbf{4}$

Solution 4

Watch the sign on the second term: $\int 2x^{-2}\,dx = \dfrac{2x^{-1}}{-1} = -\dfrac{2}{x}$. Integrating a negative power raises it by one, which here takes -2 to -1, and the division by the new power flips the sign.

Find c first. Substituting T into an expression that still contains c leaves one equation with two unknowns.

Solution 4

The check that costs one line: differentiate your y back and confirm it returns the gradient function you started with.

Mark scheme

- $y = x^4 - \dfrac{2}{x} + c$; through $S(2, 20)$: $16 - 1 + c = 20 \Rightarrow c = 5$. At $T(1, t)$: $t = 1 - 2 + 5 = 4$.

Question 5

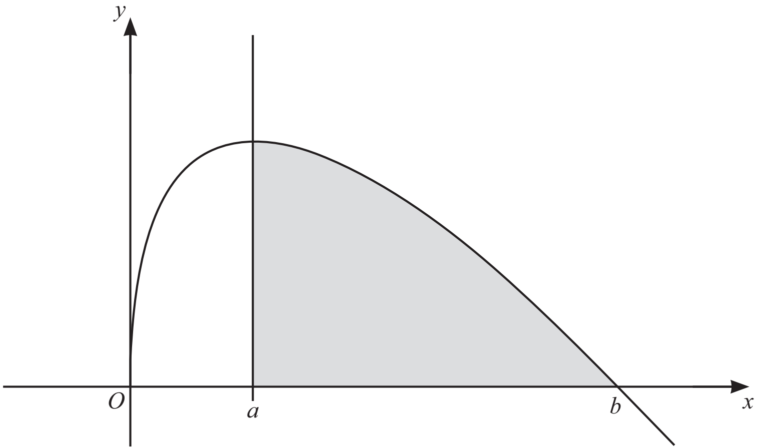
9709/12 N25 ▪ Q5 ▪ 8 marks

The equation of a curve is $y = 4x^{\frac{1}{2}} - x$. The curve has a maximum point when $x = a$ and crosses the x-axis at the point with coordinates $(b, 0)$, where $b > 0$. The shaded region is bounded by the curve, the line $x = a$ and the x-axis (see diagram).

(a) Find the value of a. **[3]**

(b) Find the exact area of the shaded region. **[5]**

Question 5



Solution 5

(a) Worked steps

Two separate values, found by two different methods — do not mix them up.
The maximum, $x = a$: set the derivative to zero.

- $y = 4x^{1/2} - x$, so $\dfrac{dy}{dx} = 2x^{-1/2} - 1$.
- $2x^{-1/2} = 1$ gives $x^{1/2} = 2$, so $a = 4$.

The crossing, $x = b$: set y itself to zero.

- $4x^{1/2} = x$, so $x^{1/2}\left(4 - x^{1/2}\right) = 0$, giving $x = 0$ or $x^{1/2} = 4$.
- $b > 0$, so $b = 16$.

Solution 5

The distinction is the whole of part (a): a **maximum** is where the *gradient* is zero, a **root** is where the *function* is zero. Solving the wrong one is the standard error, and both equations look similar enough to confuse under time.

Confirm the stationary point is a maximum — $\dfrac{d^2y}{dx^2} = -x^{-3/2}$, negative for $x > 0$ — if the question asks for its nature.

Mark scheme

- $\dfrac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow x^{1/2} = 2 \Rightarrow a = 4.$

Solution 5

(b) Worked steps

The region runs from $x = a$ to $x = b$, so integrate between those limits.

- $\int_4^{16} \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_4^{16}$
- At 16: $\frac{8}{3}(64) - \frac{1}{2}(256) = \frac{512}{3} - 128$
- At 4: $\frac{8}{3}(8) - \frac{1}{2}(16) = \frac{64}{3} - 8$
- Difference: $\left(\frac{512}{3} - 128\right) - \left(\frac{64}{3} - 8\right) = \frac{448}{3} - 120 = \dfrac{88}{3}$

Solution 5

Take the limits from part (a) rather than from the diagram: the shaded region is bounded by the line $x = a$ on one side and the curve's root on the other, and those are the numbers just computed.

Solution 5

$16^{3/2} = \left(\sqrt{16}\right)^3 = 64$ — take the root first, then cube. Cubing 16 and then rooting is the same answer with far more arithmetic.

Mark scheme

- Curve meets x-axis at $b = 16$. Area $= \int_4^{16} \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_4^{16} = \frac{128}{3} - \frac{40}{3} = \frac{88}{3}$.

Question 2

9709/11 J25 ▪ Q2 ▪ 6 marks

The equation of a curve is such that $\dfrac{dy}{dx} = 4(2x-5)^3 - 9x^{\frac{1}{2}}$. The curve passes through the point $A\left(4, -\frac{11}{2}\right)$.

(a) Find the gradient of the normal to the curve at the point A. **[2]**

(b) Find the equation of the curve. **[4]**

Solution 2

(a)

Mark scheme

- At A $(x = 4)$: $\dfrac{dy}{dx} = 4(3)^3 - 9(2) = 90$, so the gradient of the normal is $-\frac{1}{90}$.

Solution 2

(b)

Mark scheme

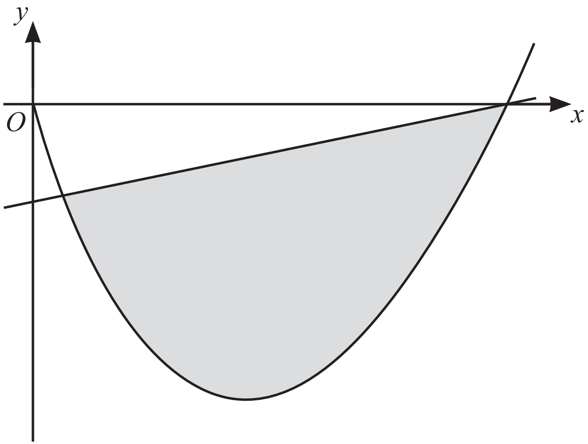
- $y = \frac{1}{2}(2x-5)^4 - 6x^{\frac{3}{2}} + C$; through $A(4, -\frac{11}{2})$ gives $C = 2$, so $y = \frac{1}{2}(2x-5)^4 - 6x^{\frac{3}{2}} + 2$.

Question 4

9709/11 J25 ▪ Q4 ▪ 5 marks

The diagram shows the curve with equation $y = 5x^{\frac{3}{2}} - 20x$ and the line with equation $y = x - 16$. The x-coordinates of the points of intersection of the curve and line are 1 and 16. Find the area of the shaded region between the curve and the line. **[5]**

Question 4



Solution 4

Mark scheme

- Area $= \int_1^{16} \left[(x-16) - (5x^{\frac{3}{2}} - 20x)\right] dx = \int_1^{16} (21x - 16 - 5x^{\frac{3}{2}})\, dx = \left[\frac{21}{2}x^2 - 16x - 2x^{\frac{5}{2}}\right]_1^{16} = 384 - (-7.5) = 391.5.$

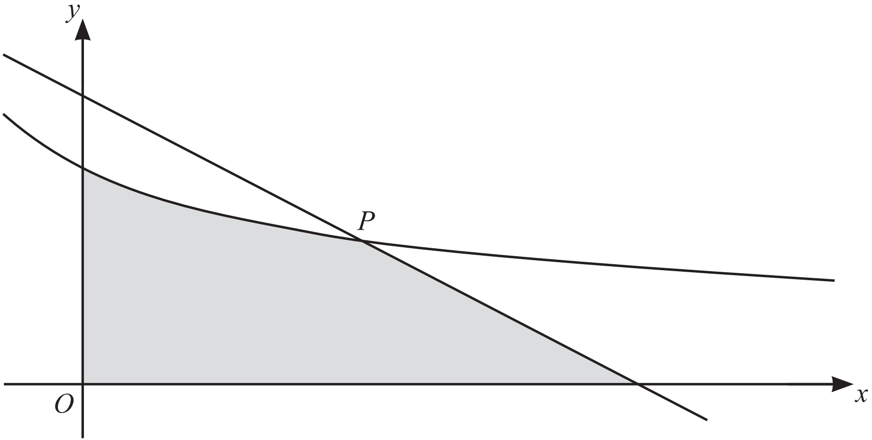
Question 6

9709/12 J25 ▪ Q6 ▪ 6 marks

The diagram shows the curve with equation $y = \dfrac{9}{(5x+4)^{\frac{1}{2}}}$ and the line $y = 6 - 3x$.

The line and the curve intersect at the point P which has y-coordinate 3. Find the area of the shaded region. **[6]**

Question 6



Solution 6

Mark scheme

- $6 - 3x = 3 \Rightarrow P$ has x-coordinate 1. Area under curve $= \int_0^1 9(5x+4)^{-1/2}\,dx = \left[\frac{18}{5}(5x+4)^{1/2}\right]_0^1 = \frac{18}{5}$. The line meets the x-axis at $x = 2$, giving a triangle of area $\frac{1}{2} \times 1 \times 3 = \frac{3}{2}$. Total shaded area $= \frac{18}{5} + \frac{3}{2} = \frac{51}{10}$.

Question 9

9709/12 J25 ▪ Q9 ▪ 12 marks

The equation of a curve is such that $\dfrac{d^2y}{dx^2} = -\dfrac{24}{x^3}$. It is given that the curve has a stationary point at $(-2, 19)$.

(a) Find an expression for $\dfrac{dy}{dx}$. **[3]**

(b) Find the x-coordinate of the other stationary point of the curve, and determine the nature of this stationary point. **[2]**

(c) Find the equation of the curve. **[3]**

(d) Find the equation of the normal to the curve at the point where $\dfrac{dy}{dx} = -\dfrac{9}{4}$ and x is positive. Express your answer in the form $px + qy + r = 0$, where p, q and r are integers. **[4]**

Solution 9

(a)

Mark scheme

- $\dfrac{dy}{dx} = \int -24x^{-3}\,dx = 12x^{-2} + c$. Stationary at $x = -2$: $\frac{12}{4} + c = 0 \Rightarrow c = -3$. So $\dfrac{dy}{dx} = \dfrac{12}{x^2} - 3$.

Solution 9

(b)

Mark scheme

- $\dfrac{12}{x^2} - 3 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$. Since $\dfrac{d^2y}{dx^2} = -\dfrac{24}{2^3} = -3 < 0$, it is a maximum.

Solution 9

(c)

Mark scheme

- $y = \int \left(12x^{-2} - 3\right)\, dx = -12x^{-1} - 3x + d$. Through $(-2, 19)$: $19 = 6 + 6 + d \Rightarrow d = 7$. So $y = -\dfrac{12}{x} - 3x + 7$.

Solution 9

(d)

Mark scheme

- $\dfrac{12}{x^2} - 3 = -\dfrac{9}{4} \Rightarrow x^2 = 16 \Rightarrow x = 4$ (positive), $y = -8$. Gradient of normal $= \frac{4}{9}$. $\dfrac{y+8}{x-4} = \frac{4}{9} \Rightarrow 4x - 9y - 88 = 0$.