Past-paper walk-through

Differentiation ▪ 1.7

eXercise

Questions in this set

- 9709/11 N25 Q11 ▪ 11 marks
- 9709/12 N25 Q4 ▪ 6 marks
- 9709/12 N25 Q5 ▪ 8 marks
- 9709/12 N25 Q9 ▪ 8 marks
- 9709/11 J25 Q2 ▪ 6 marks
- 9709/11 J25 Q7 ▪ 9 marks
- 9709/12 J25 Q4 ▪ 5 marks
- 9709/12 J25 Q9 ▪ 12 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 11

9709/11 N25 ▪ Q11 ▪ 11 marks

A curve passes through the point $P(4, 3)$ and is such that $\dfrac{dy}{dx} = \dfrac{8}{x^2} - \dfrac{10}{(2x-3)^2}$.

(a) Find the equation of the normal to the curve at P. Give your answer in the form $y = mx + c$. **[3]**

(b) Find the rate of change of the gradient of the curve when $x = 4$. **[3]**

(c) Given that the curve also passes through the point $(-1, q)$, find the value of q. **[5]**

Solution 11

(a)

Mark scheme

- At $x = 4$, gradient $= \dfrac{8}{16} - \dfrac{10}{25} = \frac{1}{10}$; normal gradient -10, so $y = -10x + 43$.

Solution 11

(b) Worked steps

Differentiate again, treating $(2x-3)^{-2}$ with the **chain rule —** the inner function differentiates to 2.

- $\dfrac{d^2y}{dx^2} = -16x^{-3} + 40(2x-3)^{-3}$
- At $x = 4$: $-16(4)^{-3} + 40(5)^{-3} = -\frac{16}{64} + \frac{40}{125} = -0.25 + 0.32 = \mathbf{\frac{7}{100}}$

Solution 11

The chain-rule factor is where these go wrong. Differentiating $(2x-3)^{-2}$ gives $-2(2x-3)^{-3}\times 2=-4(2x-3)^{-3}$, and the extra 2 is the mark.
At $x=4$ the second derivative is **positive**, so the stationary point there is a **minimum** — worth stating if the question asks for the nature of the point.

Mark scheme

- $\dfrac{d^2y}{dx^2}=-16x^{-3}+40(2x-3)^{-3}$; at $x=4$ this is $\dfrac{7}{100}$.

Solution 11

(c) Worked steps

Going from $\dfrac{dy}{dx}$ back to y is integration, and the **constant** is the whole question.

- $y = \displaystyle\int \left(8x^{-2} - 10(2x-3)^{-2}\right) dx = -8x^{-1} + 5(2x-3)^{-1} + c$
- Use the known point $(4,3)$: $3 = -\frac{8}{4} + \frac{5}{5} + c = -2 + 1 + c$, so $c = 4$.
- At $x = -1$: $q = -\dfrac{8}{-1} + \dfrac{5}{-5} + 4 = 8 - 1 + 4 = \mathbf{11}$

Solution 11

Two chain-rule divisions to watch. Integrating $(2x-3)^{-2}$ gives $\dfrac{(2x-3)^{-1}}{-1\times 2} = -\frac{1}{2}(2x-3)^{-1}$, so the -10 becomes $+5$. Dividing by the inner coefficient is the step most often skipped.

And **find c before substituting the second point**. Without it there is one equation and two unknowns, and the answer cannot be pinned down.

Mark scheme

- $y = -8x^{-1} + 5(2x-3)^{-1} + c$; through $(4, 3)$: $3 = -2 + 1 + c \Rightarrow c = 4$. At $x = -1$: $q = 8 - 1 + 4 = 11$.

Question 4

9709/12 N25 ▪ Q4 ▪ 6 marks

The equation of a curve is such that $\frac{dy}{dx} = kx^3 + \frac{2}{x^2}$, where k is a constant. The curve passes through the point $S(2, 20)$ and the gradient of the curve at S is $\frac{65}{2}$.

(a) Find the value of k. **[1]**

(b) The coordinates of a point T on the curve are $(1, t)$. Find the value of t. **[5]**

Solution 4

(a)

Mark scheme

- At S $(x = 2)$: $8k + \frac{1}{2} = \frac{65}{2} \Rightarrow 8k = 32 \Rightarrow k = 4$.

Solution 4

(b) Worked steps

The gradient function is known, so integrate to get the curve — and the arbitrary constant is what the given point is for.

- $y = \int (4x^3 + 2x^{-2})\, dx = x^4 - \dfrac{2}{x} + c$
- Through $S(2, 20)$: $16 - 1 + c = 20$, so $c = 5$.
- At $T(1, t)$: $t = 1 - 2 + 5 = \mathbf{4}$

Solution 4

Watch the sign on the second term: $\int 2x^{-2}\,dx = \frac{2x^{-1}}{-1} = -\frac{2}{x}$. Integrating a negative power raises it by one, which here takes -2 to -1, and the division by the new power flips the sign.

Find c first. Substituting T into an expression that still contains c leaves one equation with two unknowns.

Solution 4

The check that costs one line: differentiate your y back and confirm it returns the gradient function you started with.

Mark scheme

- $y = x^4 - \dfrac{2}{x} + c$; through $S(2, 20)$: $16 - 1 + c = 20 \Rightarrow c = 5$. At $T(1, t)$: $t = 1 - 2 + 5 = 4$.

Question 5

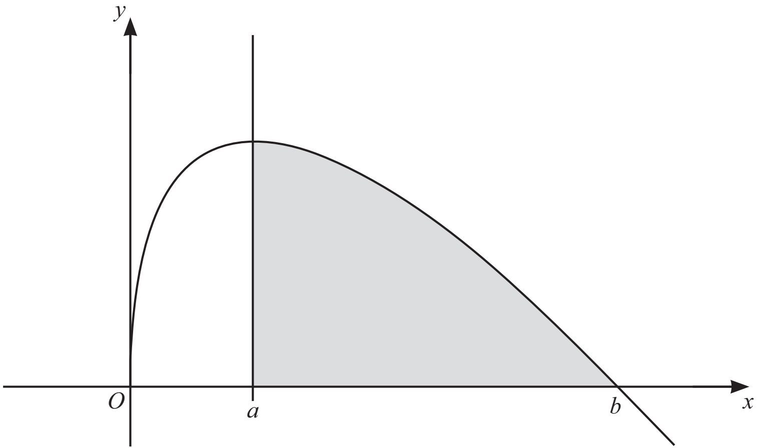
9709/12 N25 ▪ Q5 ▪ 8 marks

The equation of a curve is $y = 4x^{\frac{1}{2}} - x$. The curve has a maximum point when $x = a$ and crosses the x-axis at the point with coordinates $(b, 0)$, where $b > 0$. The shaded region is bounded by the curve, the line $x = a$ and the x-axis (see diagram).

(a) Find the value of a. **[3]**

(b) Find the exact area of the shaded region. **[5]**

Question 5



Solution 5

(a) Worked steps

Two separate values, found by two different methods — do not mix them up.
The maximum, $x = a$: set the derivative to zero.

- $y = 4x^{1/2} - x$, so $\dfrac{dy}{dx} = 2x^{-1/2} - 1$.
- $2x^{-1/2} = 1$ gives $x^{1/2} = 2$, so $a = 4$.

The crossing, $x = b$: set y itself to zero.

- $4x^{1/2} = x$, so $x^{1/2}\left(4 - x^{1/2}\right) = 0$, giving $x = 0$ or $x^{1/2} = 4$.
- $b > 0$, so $b = 16$.

Solution 5

The distinction is the whole of part (a): a **maximum** is where the *gradient* is zero, a **root** is where the *function* is zero. Solving the wrong one is the standard error, and both equations look similar enough to confuse under time.

Confirm the stationary point is a maximum — $\dfrac{d^2y}{dx^2} = -x^{-3/2}$, negative for $x > 0$ — if the question asks for its nature.

Mark scheme

- $\dfrac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow x^{1/2} = 2 \Rightarrow a = 4.$

Solution 5

(b) Worked steps

The region runs from $x = a$ to $x = b$, so integrate between those limits.

- $\int_4^{16} \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_4^{16}$
- At 16: $\frac{8}{3}(64) - \frac{1}{2}(256) = \frac{512}{3} - 128$
- At 4: $\frac{8}{3}(8) - \frac{1}{2}(16) = \frac{64}{3} - 8$
- Difference: $\left(\frac{512}{3} - 128\right) - \left(\frac{64}{3} - 8\right) = \frac{448}{3} - 120 = \frac{88}{3}$

Solution 5

Take the limits from part (a) rather than from the diagram: the shaded region is bounded by the line $x = a$ on one side and the curve's root on the other, and those are the numbers just computed.

Solution 5

$16^{3/2} = \left(\sqrt{16}\right)^3 = 64$ — take the root first, then cube. Cubing 16 and then rooting is the same answer with far more arithmetic.

Mark scheme

- Curve meets x-axis at $b = 16$. Area $= \int_4^{16} \left(4x^{1/2} - x\right)\,dx = \left[\frac{8}{3}x^{3/2} - \frac{1}{2}x^2\right]_4^{16} = \dfrac{128}{3} - \dfrac{40}{3} = \dfrac{88}{3}$.

Question 9

9709/12 N25 ▪ Q9 ▪ 8 marks

The function f is defined by $f(x) = \dfrac{4}{(3x-6)^2} + \dfrac{1}{(3x-6)^3}$ for $x > 2$.

(a) Find an expression for $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither. **[4]**

(b) State whether f^{-1} exists. Give a reason for your answer. **[1]**

(c) The function g is defined by $g(x) = 4x - 3$ for $x > a$. Find the range of g in terms of the constant a. **[1]**

(d) Find the set of values of a for which the composite function fg exists. **[2]**

Solution 9

(a)

Mark scheme

- $f'(x) = -24(3x-6)^{-3} - 9(3x-6)^{-4}$; for $x > 2$, $(3x-6) > 0$ so both terms are negative — f is a decreasing function.

Solution 9

(b)

Mark scheme

- f^{-1} exists because f is a one-to-one (decreasing) function.

Solution 9

(c)

Mark scheme

- $g(x) > 4a - 3$.

Solution 9

(d)

Mark scheme

- fg requires range$(g) \subseteq$ domain$(f) = (2, \infty)$, so $4a - 3 \geqslant 2 \Rightarrow a \geqslant \frac{5}{4}$.

Question 2

9709/11 J25 ▪ Q2 ▪ 6 marks

The equation of a curve is such that $\frac{dy}{dx} = 4(2x - 5)^3 - 9x^{\frac{1}{2}}$. The curve passes through the point $A\left(4, -\frac{11}{2}\right)$.

(a) Find the gradient of the normal to the curve at the point A. **[2]**

(b) Find the equation of the curve. **[4]**

Solution 2

(a)

Mark scheme

- At A $(x = 4)$: $\dfrac{dy}{dx} = 4(3)^3 - 9(2) = 90$, so the gradient of the normal is $-\frac{1}{90}$.

Solution 2

(b)

Mark scheme

- $y = \frac{1}{2}(2x-5)^4 - 6x^{\frac{3}{2}} + C$; through $A(4, -\frac{11}{2})$ gives $C = 2$, so $y = \frac{1}{2}(2x-5)^4 - 6x^{\frac{3}{2}} + 2$.

Question 7

9709/11 J25 ▪ Q7 ▪ 9 marks

The equation of a curve is $y = 4x^2 + \dfrac{9}{x^2} - 8$.

(a) A point P is moving along the curve in such a way that its y-coordinate is decreasing at 5 units per second. Find the rate at which the x-coordinate of point P is changing when $x = 2$. **[4]**

(b) Find the coordinates of the stationary points of the curve and determine their nature. **[5]**

Solution 7

(a) Mark scheme

- $\dfrac{dy}{dx} = 8x - \dfrac{18}{x^3}$; at $x = 2$ this is $\frac{55}{4}$. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$: $-5 = \frac{55}{4} \cdot \dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = -\frac{4}{11}$ units/s.

Solution 7

(b) Mark scheme

- $8x - \dfrac{18}{x^3} = 0 \Rightarrow x^4 = \frac{9}{4} \Rightarrow x = \pm\frac{\sqrt{6}}{2}$, $y = 4$. Since $\dfrac{d^2y}{dx^2} = 8 + \dfrac{54}{x^4} = 32 > 0$, both $\left(\pm\frac{\sqrt{6}}{2}, 4\right)$ are minima.

Question 4

9709/12 J25 ▪ Q4 ▪ 5 marks

A point P is moving along the curve with equation $y = ax^{\frac{3}{2}} - 12x$ in such a way that the x-coordinate of P is increasing at a constant rate of 5 units per second.

(a) Find the rate at which the y-coordinate of P is changing when $x = 9$. Give your answer in terms of the constant a. **[3]**

(b) Given that the curve has a minimum point when $x = \frac{1}{4}$, find the value of a. **[2]**

Solution 4

(a) Mark scheme

- $\dfrac{dy}{dx} = \frac{3}{2}ax^{1/2} - 12$. Then
$\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt} = \left(\frac{3}{2}a \cdot 3 - 12\right) \times 5 = \dfrac{45a}{2} - 60 = \frac{15}{2}(3a - 8)$.

Solution 4

(b)

Mark scheme

- At the minimum $\dfrac{dy}{dx} = 0$ at $x = \frac{1}{4}$: $\frac{3}{2}a\left(\frac{1}{4}\right)^{1/2} - 12 = 0 \Rightarrow a = 16.$

Question 9

9709/12 J25 ▪ Q9 ▪ 12 marks

The equation of a curve is such that $\frac{d^2y}{dx^2} = -\frac{24}{x^3}$. It is given that the curve has a stationary point at $(-2, 19)$.

(a) Find an expression for $\frac{dy}{dx}$. **[3]**

(b) Find the x-coordinate of the other stationary point of the curve, and determine the nature of this stationary point. **[2]**

(c) Find the equation of the curve. **[3]**

(d) Find the equation of the normal to the curve at the point where $\frac{dy}{dx} = -\frac{9}{4}$ and x is positive. Express your answer in the form $px + qy + r = 0$, where p, q and r are integers. **[4]**

Solution 9

(a)

Mark scheme

- $\dfrac{dy}{dx} = \int -24x^{-3}\,dx = 12x^{-2} + c$. Stationary at $x = -2$: $\frac{12}{4} + c = 0 \Rightarrow c = -3$. So $\dfrac{dy}{dx} = \dfrac{12}{x^2} - 3$.

Solution 9

(b)

Mark scheme

- $\frac{12}{x^2} - 3 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$. Since $\frac{d^2y}{dx^2} = -\frac{24}{2^3} = -3 < 0$, it is a maximum.

Solution 9

(c)

Mark scheme

- $y = \int \left(12x^{-2} - 3\right)\, dx = -12x^{-1} - 3x + d$. Through $(-2, 19)$: $19 = 6 + 6 + d \Rightarrow d = 7$. So $y = -\dfrac{12}{x} - 3x + 7$.

Solution 9

(d)

Mark scheme

- $\dfrac{12}{x^2} - 3 = -\dfrac{9}{4} \Rightarrow x^2 = 16 \Rightarrow x = 4$ (positive), $y = -8$. Gradient of normal $= \frac{4}{9}$. $\dfrac{y+8}{x-4} = \frac{4}{9} \Rightarrow 4x - 9y - 88 = 0.$