# Past-paper walk-through

Series ▪ 1.6

eXercise

# Questions in this set

- 9709/11 N25 Q2 ▪ 5 marks
- 9709/11 N25 Q3 ▪ 5 marks
- 9709/11 N25 Q9 ▪ 8 marks
- 9709/12 N25 Q2 ▪ 3 marks
- 9709/12 N25 Q8 ▪ 11 marks
- 9709/11 J25 Q3 ▪ 7 marks
- 9709/11 J25 Q5 ▪ 7 marks
- 9709/12 J25 Q3 ▪ 4 marks
- 9709/12 J25 Q10 ▪ 10 marks

# Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

# Question 2

9709/11 N25 ▪ Q2 ▪ 5 marks

A geometric progression has first term *a* and common ratio $\cos\theta$, where $0 < \theta < \frac{1}{2}\pi$. It is given that the second term is 8 and the fifth term is $\frac{1}{8}$.

**(a)** Find the value of $\theta$. Give your answer correct to 3 significant figures. **[3]**

**(b)** Find the exact value of the sum to infinity. **[2]**

# Solution 2

## (a) Worked steps

Two terms of a GP are given, so **divide them —** that eliminates the first term and leaves only the ratio.

- $\dfrac{u_5}{u_2} = \dfrac{ar^4}{ar} = r^3$, so $r^3 = \dfrac{1/8}{8} = \dfrac{1}{64}$.
- $r = \dfrac{1}{4}$, and here $r = \cos\theta$, so $\cos\theta = 0.25$.
- $\theta = \cos^{-1} 0.25 = 1.32$ rad (3 s.f.).

# Solution 2

Two habits protect the marks. **Divide rather than solve simultaneously** — dividing kills *a* in one line, where substitution takes three. And **count the gap**: from the 2nd term to the 5th is **three** steps, so the ratio is cubed, not raised to the fourth. The answer is in **radians**, because the domain $0 < \theta < \frac{1}{2}\pi$ is written in radians. A calculator left in degrees returns 75.5 and scores nothing.

**Mark scheme**

- $r^3 = \dfrac{1/8}{8} = \dfrac{1}{64}$, so $\cos\theta = 0.25$, giving $\theta = 1.32$ rad (3 s.f.).

# Solution 2

## (b) Worked steps

Work back to $a$ first, then apply the sum to infinity.

- $u_2 = ar$, so $a = \dfrac{8}{0.25} = 32$.
- $S_\infty = \dfrac{a}{1-r} = \dfrac{32}{1-0.25} = \dfrac{32}{0.75} = \dfrac{128}{3}$.

# Solution 2

The question says **exact**, so leave it as $\frac{128}{3}$ — 42.7 is a rounded decimal and loses the mark.

# Solution 2

Check the convergence condition before using the formula: $S_\infty$ exists only when $|r| < 1$, and $r = 0.25$ satisfies it. That check is worth writing down, because a question of this shape sometimes sets a ratio that fails it.

**Mark scheme**

- $a = \dfrac{8}{0.25} = 32$; sum to infinity $= \dfrac{a}{1-r} = \dfrac{32}{1-0.25} = \dfrac{128}{3}$.

# Question 3

9709/11 N25 ▪ Q3 ▪ 5 marks

In the expansion of $(px + 3)^5 - \left(x^3 + \dfrac{p}{x}\right)^4$, the coefficient of $x^4$ is 216. Find the value of the positive constant $p$. **[5]**

# Solution 3

## Worked steps

Neither bracket needs expanding in full — take **one term from each**, the one that produces $x^4$.

**From $(px+3)^5$.** The general term is $\binom{5}{k}(px)^{5-k}3^k$, and $x^4$ needs $5-k=4$, so $k=1$:

- $\binom{5}{1}(px)^4 \cdot 3 = 5p^4 \cdot 3 \cdot x^4 = 15p^4x^4$

# Solution 3

**From** $\left(x^3 + \dfrac{p}{x}\right)^4$**.** The general term is $\binom{4}{k}(x^3)^{4-k}\left(\dfrac{p}{x}\right)^k$, whose power of $x$ is $3(4-k) - k = 12 - 4k$. Setting that to 4 gives $k = 2$:

- $\binom{4}{2}(x^3)^2\left(\dfrac{p}{x}\right)^2 = 6p^2x^4$

**The expression subtracts the second from the first**, so:

- $15p^4 - 6p^2 = 216$, that is $15p^4 - 6p^2 - 216 = 0$.
- Let $u = p^2$: $15u^2 - 6u - 216 = 0$, so $5u^2 - 2u - 72 = 0$, giving $u = 4$ (the other root is negative).
- $p^2 = 4$, and $p$ is **positive**, so $p = 2$.

# Solution 3

Three places this goes wrong, all avoidable:

- **Find $k$ from the power of $x$, algebraically.** Guessing which term gives $x^4$ works for the first bracket and fails for the second, where the negative power of $x$ makes it $k = 2$ rather than $k = 4$.
- **Keep the minus sign** between the two expansions.
- **Take the positive root** — the question says $p$ is positive, and $p^2 = 4$ has two solutions.

# Solution 3

## Mark scheme

- From $(px+3)^5$ the $x^4$ term is $5p^4 \times 3 = 15p^4$; from $\left(x^3 + \frac{p}{x}\right)^4$ the $x^4$ term is $6p^2$. So $15p^4 - 6p^2 - 216 = 0 \Rightarrow p = 2$.

# Question 9

9709/11 N25 ▪ Q9 ▪ 8 marks

An arithmetic progression has first term 2 and common difference $d$. The sum of the first $n$ terms is denoted by $S_n$.

**(a)** It is given that $(S_2 - 1)$, $S_4$, $S_9$ are the first three terms of a second arithmetic progression. Find the value of $d$. **[4]**

**(b)** Hence find the difference between the values of the 15th terms of the two arithmetic progressions. **[4]**

# Solution 9

## (a) Worked steps

Write each of the three quantities in terms of $d$, then use the **defining property of an AP**: consecutive terms differ by a constant.
With first term 2, $S_n = \frac{n}{2}(4 + (n-1)d)$:

- $S_2 = 4 + d$, so $S_2 - 1 = 3 + d$
- $S_4 = 8 + 6d$
- $S_9 = 18 + 36d$

# Solution 9

For three terms of an AP, the **middle is the mean of the outer two** — equivalently, the two gaps are equal:

- $(3 + d) + (18 + 36d) = 2(8 + 6d)$
- $21 + 37d = 16 + 12d$, so $25d = -5$ and $d = -\frac{1}{5}$.

Using $t_2 - t_1 = t_3 - t_2$ rearranges to exactly the same equation, so use whichever you write more reliably. Substituting each $S_n$ separately and solving simultaneously also works but is three times the algebra.

**Mark scheme**

- $S_2 - 1 = 3 + d$, $S_4 = 8 + 6d$, $S_9 = 18 + 36d$. APs: $(3 + d) + (18 + 36d) = 2(8 + 6d) \Rightarrow d = -\frac{1}{5}$.

# Solution 9

## (b) Worked steps

Two different progressions, so find each 15th term separately and subtract.

**The first AP** — first term 2, common difference $-\frac{1}{5}$:

- $u_{15} = 2 + 14\left(-\frac{1}{5}\right) = 2 - 2.8 = -0.8$

**The second AP** — its first term is $S_2 - 1 = 3 + d = \frac{14}{5}$, and its common difference is $S_4 - (S_2 - 1) = (8 + 6d) - (3 + d) = 5 + 5d = 4$:

# Solution 9

- $u_{15} = \frac{14}{5} + 14(4) = 2.8 + 56 = 58.8$

**Difference** $= 58.8 - (-0.8) = 59.6.$

# Solution 9

Two things to be deliberate about. The second AP's first term and common difference come from **part (a)'s three terms**, not from the original progression — they are a different sequence that happens to be built from the first one's sums. And the 15th term uses $u_n = a + (n-1)d$, so the multiplier is **14**, not 15.

## Mark scheme

- First AP 15th term $= 2 + 14(-\frac{1}{5}) = -0.8$. Second AP: first term $\frac{14}{5}$, common difference 4, 15th term $= 58.8$. Difference $= 59.6$.

# Question 2

9709/12 N25 ▪ Q2 ▪ 3 marks

Find the term independent of $x$ in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^6$. **[3]**

# Solution 2

## Worked steps

"Independent of $x$" means the term whose power of $x$ is **zero** — so find which term that is **algebraically** rather than by expanding and looking.

Write the general term of $\left(2x^2 - \frac{3}{x}\right)^6$:

$$\binom{6}{k}\left(2x^2\right)^{6-k}\left(-\frac{3}{x}\right)^k$$

Collect the powers of $x$: $2(6-k) - k = 12 - 3k$. Set that to zero:

- $12 - 3k = 0$, so $k = 4$.

# Solution 2

Now evaluate that one term:

- $\binom{6}{4}\left(2x^2\right)^2\left(-\dfrac{3}{x}\right)^4 = 15 \times 4 \times 81 = \mathbf{4860}$

Three details decide the marks:

- **Find $k$ from the exponent equation.** Expanding all seven terms works and takes ten times as long, with seven chances to slip.
- **Both brackets are raised to powers**, so $\left(2x^2\right)^2 = 4x^4$ — the 2 is squared as well as the $x^2$. Forgetting that gives 1215.
- **The sign is positive**: $(-3)^4$ is even-powered. A negative answer means the power of the minus sign was mishandled.

# Solution 2

$\binom{6}{4} = \binom{6}{2} = 15$ — the symmetry is a useful check on the coefficient.

**Mark scheme**

- General term gives power $x^{12-3k}$; independent when $k = 4$: $\binom{6}{2}(2x^2)^2\left(-\frac{3}{x}\right)^4 = 15 \times 4 \times 81 = 4860.$

# Question 8

9709/12 N25 ▪ Q8 ▪ 11 marks

The first three terms of a geometric progression are $a$, $b$ and $c$ respectively, where $a$, $b$ and $c$ are positive constants. The first three terms of an arithmetic progression are $a$, $b$ and $-3c$ respectively.

**(a)** Show that $a^2 - 10ac + 9c^2 = 0$. **[3]**

**(b)(i)** Find the sum to infinity of the geometric progression. **[5]**

**(b)(ii)** Find the sum of the first 20 terms of the arithmetic progression. **[3]**

# Solution 8

## (a) Worked steps

Two progressions, so write **the defining property of each** and eliminate $b$.

- **GP** $a, b, c$: the ratio between consecutive terms is constant, so $\dfrac{b}{a} = \dfrac{c}{b}$, that is $b^2 = ac$.
- **AP** $a, b, -3c$: the difference between consecutive terms is constant, so $2b = a + (-3c) = a - 3c$.

Square the second and substitute the first:

- $4b^2 = (a - 3c)^2$
- $4ac = a^2 - 6ac + 9c^2$
- $a^2 - 10ac + 9c^2 = 0$ **(AG)**

# Solution 8

The two properties worth having in this exact form: for a GP the **middle term squared is the product of its neighbours**; for an AP the **middle term doubled is the sum of its neighbours**. Almost every "three terms are both a GP and an AP" question is those two lines plus algebra.

This is a *show that*, so every line has to appear — including the substitution of $b^2 = ac$, which is the step being marked.

## Mark scheme

- GP: $b^2 = ac$. AP: $2b = a + (-3c) = a - 3c$, so $4ac = (a - 3c)^2 \Rightarrow a^2 - 10ac + 9c^2 = 0$ (AG).

# Solution 8

## (b)(i) Worked steps

Factorise the given result, then follow the chain.

- With $a = 9$: $81 - 90c + 9c^2 = 0$, so $c^2 - 10c + 9 = 0$ and $(c - 9)(c - 1) = 0$.
- $c = 1$ (the smaller root; $c = 9$ would make all three terms equal and the ratio 1, so the sum to infinity would not exist).
- $b^2 = ac = 9$, so $b = 3$ (positive, as the question states).
- $r = \dfrac{b}{a} = \dfrac{3}{9} = \dfrac{1}{3}$.
- $S_\infty = \dfrac{a}{1 - r} = \dfrac{9}{1 - \frac{1}{3}} = \dfrac{27}{2}$.

# Solution 8

**Check** $|r| < 1$ **before using the formula** — and note that this is what rules the other root out, so saying so is part of the answer rather than an aside.

**Mark scheme**

- $a = 9$: $81 - 90c + 9c^2 = 0 \Rightarrow (c-9)(c-1) = 0$, smaller $c = 1$. Then $b = 3$, $r = \frac{1}{3}$; $S_\infty = \dfrac{9}{1-\frac{1}{3}} = \dfrac{27}{2}$.

# Solution 8

## (b)(ii) Worked steps

A different progression, so extract its own first term and common difference.

- The AP is $a, b, -3c$, that is $9, 3, -3$.
- $d = b - a = 3 - 9 = -6$.
- $S_{20} = \dfrac{20}{2}\,(2(9) + 19(-6)) = 10\,(18 - 114) = 10(-96) = \mathbf{-960}$.

# Solution 8

Two things to watch: the multiplier in $S_n = \frac{n}{2}(2a + (n-1)d)$ is $n - 1 = 19$, not 20; and $d$ is **negative**, so the sum is negative. A positive answer means a sign was dropped.

Check the third term against $-3c = -3$: $9 + 2(-6) = -3$. ✓

**Mark scheme**

- $d = b - a = 3 - 9 = -6$; $S_{20} = \frac{20}{2}\big(2(9) + 19(-6)\big) = 10(-96) = -960.$

# Question 3

9709/11 J25 ▪ Q3 ▪ 7 marks

The third term of a geometric progression is 18 and the sum of the first three terms is 26. It is given that the common ratio is negative.

**(a)** Find the tenth term of the progression. Give your answer correct to 3 significant figures. **[5]**

**(b)** Find the exact value of the sum to infinity of the progression. **[2]**

# Solution 3

## (a) Worked steps

Two facts, two unknowns. Write both in terms of $a$ and $r$, then **eliminate $a$ by division**.

- Third term: $ar^2 = 18$
- Sum of the first three: $a + ar + ar^2 = a\left(1 + r + r^2\right) = 26$

Dividing the second by the first:

# Solution 3

$$\frac{1+r+r^2}{r^2} = \frac{26}{18} = \frac{13}{9}$$

- $9\left(1+r+r^2\right) = 13r^2$
- $4r^2 - 9r - 9 = 0$, so $(4r+3)(r-3) = 0$ and $r = -\frac{3}{4}$ or $r = 3$.
- The question says the ratio is **negative**, so $r = -\frac{3}{4}$.
- Then $a = \dfrac{18}{r^2} = \dfrac{18}{9/16} = 32$.
- Tenth term: $ar^9 = 32\left(-\frac{3}{4}\right)^9 = \mathbf{-2.40}$ (3 s.f.).

# Solution 3

Three things carry the marks. **Divide to eliminate** $a$ rather than substituting — it is one line instead of four. **Use the sign condition** to choose between the roots; both satisfy the quadratic and only one satisfies the question. And the tenth term uses $r^9$, not $r^{10}$ — $u_n = ar^{n-1}$.

The answer is **negative** because an odd power of a negative ratio is negative. A positive answer means the exponent was even.

## Mark scheme

- $ar^2 = 18$, $a(1 + r + r^2) = 26 \Rightarrow 4r^2 - 9r - 9 = 0 \Rightarrow r = -\frac{3}{4}$ (negative), $a = 32$. Tenth term $= 32\left(-\frac{3}{4}\right)^9 = -2.40$.

# Solution 3

## (b) Worked steps

- $S_\infty = \dfrac{a}{1-r} = \dfrac{32}{1-\left(-\frac{3}{4}\right)} = \dfrac{32}{\frac{7}{4}} = \dfrac{128}{7}$

**Exact**, so leave it as a fraction; 18.3 is a rounded decimal and loses the mark.

# Solution 3

Watch the double negative in the denominator: $1-(-\frac{3}{4})$ is $\frac{7}{4}$, not $\frac{1}{4}$. That single sign is the difference between $\frac{128}{7}$ and 128.
And check convergence: $|r| = \frac{3}{4} < 1$, so the sum to infinity exists.

**Mark scheme**

- $S_\infty = \dfrac{a}{1-r} = \dfrac{32}{1+\frac{3}{4}} = \dfrac{128}{7}$.

# Question 5

9709/11 J25 ▪ Q5 ▪ 7 marks

**(a)(i)** $(2 - px)^5$ **[2]**

**(a)(ii)** $\left(1 - \frac{1}{2}x\right)^4$ **[2]**

**(b)** Given that the coefficient of $x^2$ in the expansion of $(2 - px)^5 \left(1 - \frac{1}{2}x\right)^4$ is 93, find the possible values of the constant $p$. **[3]**

# Solution 5

## (a)(i) Worked steps

Expand as far as the $x^2$ term — no further, because that is all part (b) needs.

$$(2 - px)^5 = 2^5 + \binom{5}{1} 2^4 (-px) + \binom{5}{2} 2^3 (-px)^2 + \cdots$$

- $= 32 - 5(16)px + 10(8)p^2x^2 + \cdots$
- $= \mathbf{32 - 80px + 80p^2x^2} + \cdots$

# Solution 5

Two habits: **square the whole of** $-px$, giving $+p^2x^2$ — the sign becomes positive and the $p$ is squared. And keep the powers of 2 straight: $2^4 = 16$ for the $x$ term, $2^3 = 8$ for the $x^2$ term.

**Mark scheme**

- $(2 - px)^5 = 32 - 80px + 80p^2x^2 + \cdots$

# Solution 5

## (a)(ii)

### Worked steps

$$\left(1-\tfrac{1}{2}x\right)^4 = 1 + \binom{4}{1}\left(-\tfrac{1}{2}x\right) + \binom{4}{2}\left(-\tfrac{1}{2}x\right)^2 + \cdots$$

- $= 1 + 4\left(-\tfrac{1}{2}\right)x + 6\left(\tfrac{1}{4}\right)x^2 + \cdots$
- $= \mathbf{1 - 2x + \tfrac{3}{2}x^2} + \cdots$

Again the $x^2$ coefficient is **positive**, because $\left(-\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$.

### Mark scheme

- $\left(1-\tfrac{1}{2}x\right)^4 = 1 - 2x + \tfrac{3}{2}x^2 + \cdots$

# Solution 5

## (b) Worked steps

Multiplying two series, the $x^2$ term collects **three** products — and only three. Pair the powers that add to 2:

$$x^0 \times x^2, \qquad x^1 \times x^1, \qquad x^2 \times x^0$$

- $32 \times \frac{3}{2} = 48$
- $(-80p) \times (-2) = 160p$
- $80p^2 \times 1 = 80p^2$

So the coefficient is $80p^2 + 160p + 48 = 93$:

# Solution 5

- $80p^2 + 160p - 45 = 0$, so $16p^2 + 32p - 9 = 0$
- $(4p - 1)(4p + 9) = 0$, giving $p = \frac{1}{4}$ or $p = -\frac{9}{4}$.

The question asks for the **possible values**, plural, so give **both** — nothing here rules either out. The mark most often lost is a **missing product**: writing only $80p^2 + 48$ and forgetting the $x \times x$ term. Listing the three pairings before multiplying anything is what prevents it.

## Mark scheme

- Coefficient of $x^2$: $32 \cdot \frac{3}{2} + (-80p)(-2) + 80p^2 = 80p^2 + 160p + 48 = 93 \Rightarrow 16p^2 + 32p - 9 = 0 \Rightarrow p = \frac{1}{4}$ or $p = -\frac{9}{4}$.

# Question 3

9709/12 J25 ▪ Q3 ▪ 4 marks

The coefficient of $x^7$ in the expansion of $\left(px^2 + \frac{4}{p}x\right)^5$ is 1280. Find the value of the constant $p$. **[4]**

# Solution 3

## Mark scheme

- The $x^7$ term $(r = 3)$ is $\binom{5}{3}(px^2)^2\left(\frac{4}{p}x\right)^3 = \dfrac{640x^7}{p}$. So $\dfrac{640}{p} = 1280 \Rightarrow p = \frac{1}{2}$. (SC B1: $p = 4$.)

# Question 10

9709/12 J25 ▪ Q10 ▪ 10 marks

**(a)(i)** Find the value of $k$. **[2]**

**(a)(ii)** Find the sum of the first 20 terms of the progression. **[3]**

**(b)** The fourth and sixth terms of a geometric progression are 36 and 6 respectively. The common ratio of the progression is positive. Find the sum to infinity of the progression. Give your answer in the form $\dfrac{a}{\sqrt{b} - c}$, where $a$, $b$ and $c$ are integers. **[5]**

# Solution 10

## (a)(i) Worked steps

Three consecutive terms of an AP have **equal gaps**, so set the two differences equal.

- $k^2 - 4k = 8k - k^2$
- $2k^2 = 12k$, so $2k(k-6) = 0$ and $k = 0$ or $k = 6$.
- $k = \mathbf{6}$, since $k = 0$ would make every term zero and the progression trivial.

The same condition written as "the middle term is the mean of the outer two", $2 \times \text{middle} = \text{first} + \text{third}$, gives the identical equation — use whichever you write more reliably.

# Solution 10

**Reject the degenerate root and say why.** A quadratic here almost always has one root that collapses the progression, and stating the reason is the difference between the mark and a lucky answer.

### Mark scheme

- AP condition $k^2 - 4k = 8k - k^2 \Rightarrow 2k^2 = 12k \Rightarrow k = 6$ (since $k \neq 0$).

# Solution 10

## (a)(ii) Worked steps

Substitute $k = 6$ and read the progression off.

- The terms become $24, 36, 48$, so $a = 24$ and $d = 12$.
- $S_{20} = \frac{20}{2}\,(2(24) + 19(12)) = 10\,(48 + 228) = \mathbf{2760}$

# Solution 10

Check the common difference from **both** gaps — $36-24=12$ and $48-36=12$ — which confirms $k=6$ at the same time. And the multiplier is $n-1=19$, not 20.

**Mark scheme**

- With $k=6$: $a=24$, terms $24, 36, 48$ so $d=12$. $S_{20}=\frac{20}{2}(2\cdot 24+19\cdot 12)=2760$.

# Solution 10

## (b) Worked steps

A different progression. Divide the two given terms to eliminate $a$.

- $ar^3 = 36$ and $ar^5 = 6$, so $r^2 = \dfrac{6}{36} = \dfrac{1}{6}$.
- $r = \dfrac{1}{\sqrt{6}}$, taking the **positive** root as the question requires.
- $a = \dfrac{36}{r^3} = 36\left(\sqrt{6}\right)^3 = 36 \times 6\sqrt{6} = 216\sqrt{6}$.
- $S_\infty = \dfrac{a}{1-r} = \dfrac{216\sqrt{6}}{1 - \dfrac{1}{\sqrt{6}}}$

# Solution 10

The answer is wanted as $\dfrac{a}{\sqrt{b}-c}$, so multiply top and bottom by $\sqrt{6}$:

$$S_\infty = \frac{216\sqrt{6} \times \sqrt{6}}{\sqrt{6}-1} = \frac{1296}{\sqrt{6}-1}$$

# Solution 10

Three points. **Divide the two terms —** the gap is two places, so the ratio is squared, not cubed. **Take the positive root** as instructed. And read the **required form** before simplifying: the instinct is to rationalise the denominator, and here that would take the answer *away* from the form asked for.

## Mark scheme

- $ar^3 = 36,\ ar^5 = 6 \Rightarrow r^2 = \frac{1}{6} \Rightarrow r = \frac{1}{\sqrt{6}}$ (positive). $a = \dfrac{36}{(1/\sqrt{6})^3} = 216\sqrt{6}$.

  $S_\infty = \dfrac{216\sqrt{6}}{1 - 1/\sqrt{6}} = \dfrac{1296}{\sqrt{6} - 1}$.