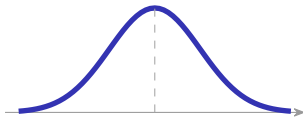


Probability & Statistics 1

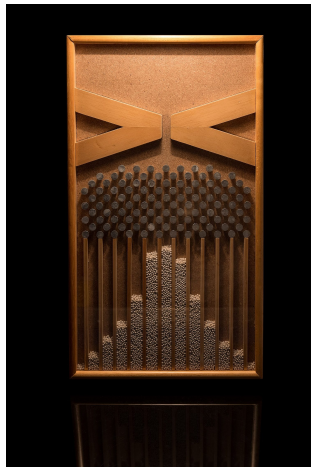
A-Level Mathematics ■ Topic 5

A-Level Mathematics



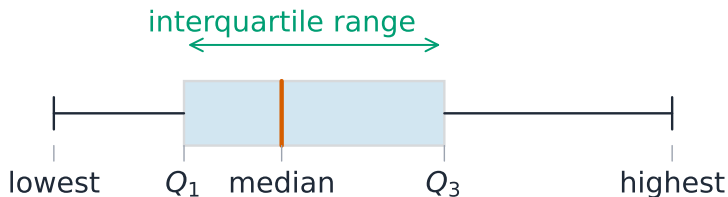
What we will cover

- 1 Representing data
- 2 Permutations & combinations
- 3 Probability
- 4 Discrete random variables
- 5 The normal distribution
- 6 Recap





Box plot: quartiles



box: Q_1 – Q_3 · median line · whiskers: min/max

box plot: quartiles

Choose the diagram that suits the data

Diagram	What it is for
stem-and-leaf	keeps the original values <i>and</i> shows the shape; a back-to-back version compares two sets round a shared stem
histogram	grouped data, where the area of each bar gives the frequency
box-and-whisker	the five-figure summary: min, Q_1 , median, Q_3 , max
cumulative frequency	read the median across from half the total, quartiles from a quarter and three quarters

In a histogram it is **area**, not height, that means frequency – which is why unequal class widths need frequency *density*.

The diagrams, and what each shows

Stem-and-leaf diagram

A **stem-and-leaf diagram** keeps every original value while showing the shape – so the median and quartiles can be read straight off it.

Box-and-whisker plot

A **box-and-whisker plot** shows the five-number summary and makes two distributions easy to compare side by side.

Histogram

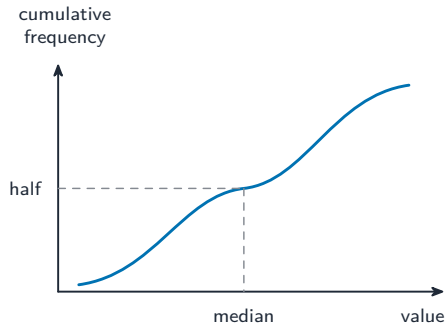
For continuous data in unequal classes: it is the *area* that represents frequency, so the vertical axis is frequency density.

Cumulative frequency

Plot at the upper class boundary and read **percentiles** off the curve.

A **measure of central tendency** says where the data sits; the **range (of data)**, interquartile range and standard deviation say how spread out it is.

Representation of data



A cumulative frequency curve is S-shaped; read the median across from half the total frequency, and the quartiles from one and three quarters

Averages and spread

- **mean** $\bar{x} = \frac{\sum x}{n}$, **median**, **mode**
- **standard deviation** $\sigma = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$
- **interquartile range** $Q_3 - Q_1$

The mean, median and mode are **measures of central tendency** ; the standard deviation and IQR are measures of **variation** . **Coding** makes big numbers easy: put $t = x - a$ for an **assumed mean** a , then $\bar{x} = a + \bar{t}$ and the standard deviation is **unchanged**.

$$n = 10, \sum x = 50, \sum x^2 = 300$$

$$\bar{x} = 5, \sigma = \sqrt{30 - 25} = 2.24$$

The measures of spread

the range

largest minus smallest – and wholly determined by two extreme values

the interquartile range

upper quartile minus lower quartile – the middle 50%, so outliers cannot distort it

the standard deviation

the root-mean-square distance from the mean – uses every value

the variance

its square, which is what the formulas add and subtract

which to quote

the IQR with a median, the standard deviation with a mean

an outlier test

more than $1.5 \times \text{IQR}$ beyond a quartile

Pair the measures correctly. A mean with an IQR, or a median with a standard deviation, is a mismatch the examiner notices.



Permutations and combinations

Permutation (order matters): ${}^nP_r = \frac{n!}{(n-r)!}$.

Combination : ${}^nC_r = \binom{n}{r}$.

NEEDLESS (E×3, S×2)

$$\frac{8!}{3!2!} = 3360$$

permutation (order matters) **combination (order does not)**

AB

BA

$\{A, B\}$

$$NPr = n!/(n-r)!; nCr = nPr/r!$$

Arranging 3 of 4 objects (4P_3)

$$\boxed{4} \times \boxed{3} \times \boxed{2} = 24$$

1st pick 2nd pick 3rd pick

Choosing 3 of 4 (order ignored): ${}^4C_3 = \frac{{}^4P_3}{3!} = 4$

$${}^nP_r = n!/(n-r)! \quad \cdot \quad nC_r = nP_r/r!$$

$$nPr = n!/(n-r)!; nCr = nPr/r!$$

Worked example – arrangements with repeated letters

How many different arrangements are there of the letters of NEEDLESS?

- 8 letters, with E three times and S twice
- divide by the factorial of each repeat count

- $\frac{8!}{3! 2!} = \frac{40320}{12} = 3360$

Repeated letters are indistinguishable, so every arrangement has been counted $3! \times 2!$ times. **Dividing by the repeats** is the whole correction.

Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Independent: $P(A \cap B) = P(A)P(B)$. Conditional: $P(A | B) = P(A \cap B)/P(B)$.

Mutually exclusive $\Rightarrow$ intersection is empty, so you just add. Independent is not the same as exclusive.



Discrete random variables

$E(X) = \sum x P(X=x)$. **Binomial** $B(n, p)$:

$$P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}, \quad E(X) = np.$$

$B(10, 0.3)$

$P(X=2) = 0.233, \quad E(X) = 3$

The named distributions, and the probability rules

binomial

n fixed independent trials, constant p : $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$

geometric

the trial on which the **first success** happens: $P(X = r) = (1 - p)^{r-1} p$

the difference

binomial counts successes in a fixed number of trials; geometric counts trials until the first success

addition

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, and the last term is zero for mutually exclusive events

multiplication

for “and” with **independent** events, $P(A \cap B) = P(A)P(B)$

conditional

$P(A | B) = \frac{P(A \cap B)}{P(B)}$ – and independence means $P(A | B) = P(A)$

Check independence before multiplying. Drawing without replacement makes the second event conditional, and that is the trap these questions set.

The named discrete distributions

Probability distribution table

Any discrete random variable can be written as a **probability distribution table** : every value, with its probability, summing to 1.

Expectation

The **expectation** $E(X) = \sum xP(X = x)$ – the long-run mean, not a value the variable need ever take.

Binomial distribution

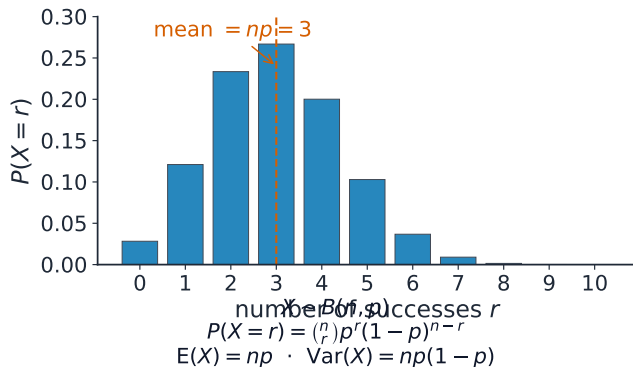
The **binomial distribution** $B(n, p)$: a fixed number of independent trials, each a success or failure with the same p .

Geometric distribution

The **geometric distribution** counts trials *up to and including* the first success, so $P(X = r) = p(1 - p)^{r-1}$.

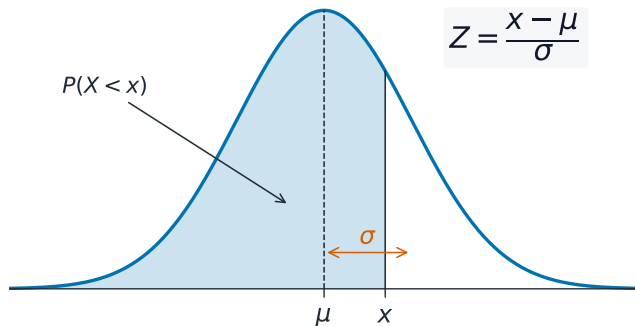
Choose by the question asked: how many successes in n trials, or how many trials until the first success.

Discrete random variables, in detail



The distribution $B(10, 0.3)$: each bar is $P(X = r)$, clustered around the mean $np = 3$.

The normal distribution



Normal $X \sim N(\mu, \sigma^2)$.

Standardise :

$$Z = \frac{X - \mu}{\sigma} \sim N(0, 1).$$

Read $P(Z < z)$ from the Φ table;
a large- n binomial $\approx$ normal
(with a **continuity correction**).

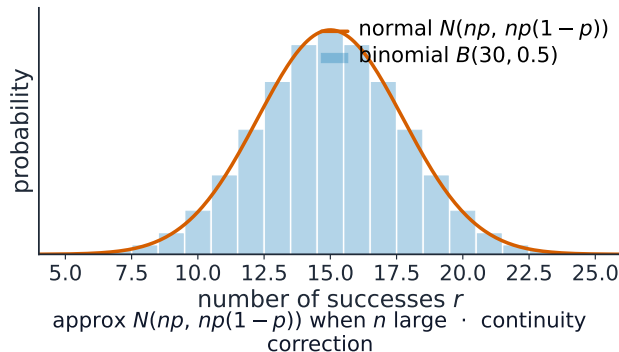
Worked example – finding an unknown mean

Bags of rice have mass $X \sim N(\mu, 0.14^2)$, and $P(X < 1.48) = 0.22$. Find μ .

- standardise: $Z = \frac{X - \mu}{\sigma}$
- from the table, $P(Z < z) = 0.22$ gives $z = -0.772$
- $\frac{1.48 - \mu}{0.14} = -0.772$
- $\mu = 1.48 + 0.772(0.14) = 1.59 \text{ kg}$

The probability is below 0.5, so z must be **negative** – and reading the table's positive value and forgetting the sign is the standard error here.

When n is large the binomial bars



When n is large the binomial bars follow a normal curve of the same mean and variance.

Worked example – mean and standard deviation

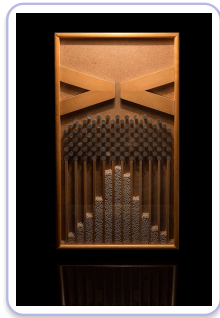
**For 10 values, $\sum x = 50$ and $\sum x^2 = 300$.
Find the mean and standard deviation.**

$$\bar{x} = \frac{50}{10} = 5$$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2} = \sqrt{30 - 25} = 2.24$$

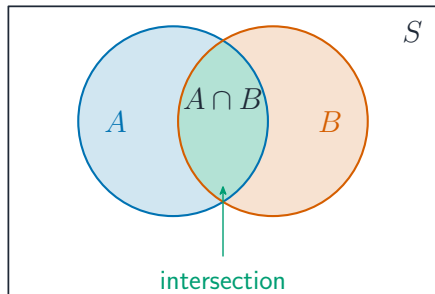
Variance = $\frac{\sum x^2}{n} - \bar{x}^2$ – the
mean of the squares minus
the square of the mean. Sub-
tract in **that** order.

The normal distribution, continued



A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

Probability, continued



The overlap of the two circles is $A \cap B$; the addition rule subtracts it once so it is not counted twice.

Independence, conditioning, and the normal approximation

Independent events

Independent events : $P(A \cap B) = P(A)P(B)$. One happening tells you nothing about the other.

Conditional probability

Conditional probability : $P(A | B) = \frac{P(A \cap B)}{P(B)}$ – restrict to the times B happened, then take A 's share.

Continuous random variable

A **continuous random variable** takes any value in a range, so probability is an area and $P(X = x) = 0$.

Standardisation

Standardisation : $Z = \frac{X - \mu}{\sigma}$ turns any normal variable into the standard one, so a single table serves them all.

A binomial with large n is well approximated by a normal – an **approximation** that needs a continuity correction of ± 0.5 .

Exam tips

- Decide whether order matters: **permutations** (nP_r) **when it does**, **combinations** (nC_r) when it does not.
- For a **discrete random variable**, check the probabilities sum to 1 and use $E(X) = \sum xP(X=x)$.
- For the **normal distribution**, standardise with $z = (x - \mu)/\sigma$, **sketch and shade**, then read the table.
- Apply a **continuity correction** when approximating a discrete variable by the normal.



Topic 5 – key takeaways

1

Data

mean, median, mode; standard deviation

2

Counting

nP_r order matters; nC_r not

3

Probability

add for “or”, multiply for independent “and”

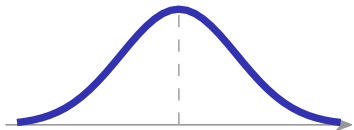
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Distributions

binomial $B(n, p)$; normal $N(\mu, \sigma^2)$

Check yourself

- 1 Does order matter for a combination?
- 2 The test for independence of A and B ?
- 3 $E(X)$ for $X \sim B(20, 0.5)$?
- 4 How do you standardise a normal variable?



Answers 1. no 2. $P(A \cap B) = P(A)P(B)$ 3. $np = 10$ 4. $Z = (X - \mu)/\sigma$