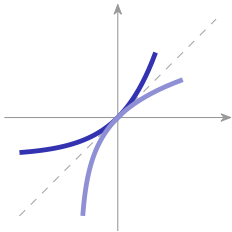


Pure Mathematics 2

A-Level Mathematics ■ Topic 2

A-Level Mathematics



What we will cover

- 1 Modulus & polynomials
- 2 Logarithms & exponentials
- 3 Trigonometry
- 4 Differentiation
- 5 Integration & numerical methods
- 6 Recap



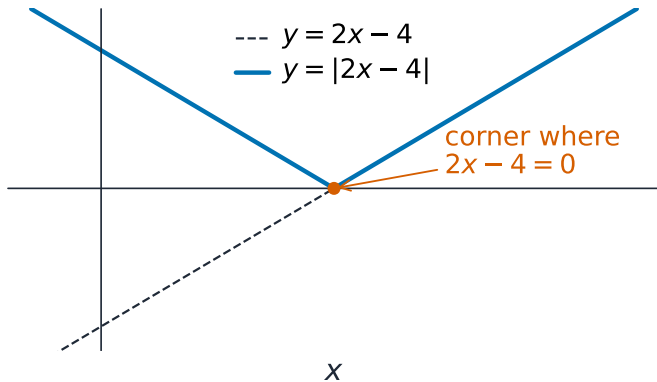


Modulus and polynomials

$|x|$ is x if $x \geq 0$ and $-x$ if $x < 0$. The graph of $y = |f(x)|$ reflects anything below the x -axis above it.

To solve $|f(x)| = k$, do $f(x) = k$ and $f(x) = -k$. Sketch first – a V-shape, vertex on the axis.

Modulus and polynomials, in detail



$y = |f(x)|$ reflects parts of $y = f(x)$ below the x -axis
above it

Taking the modulus folds the part of the line below the axis upward into a V.

Worked example – a modulus inequality

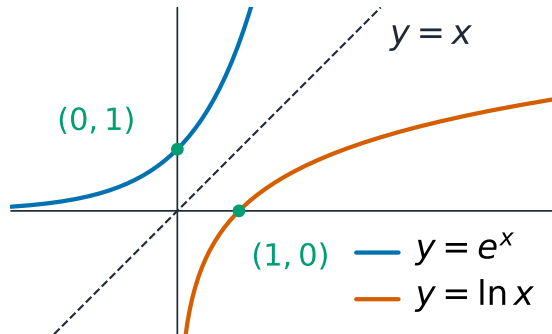
Solve $|3x + 8| < 9$.

- $|A| < b$ means $-b < A < b$
- $-9 < 3x + 8 < 9$
- $-17 < 3x < 1$
- $-\frac{17}{3} < x < \frac{1}{3}$

The double inequality is the safe method: it handles both branches at once, so no case can be forgotten. Squaring both sides also works, but only because both sides are non-negative here.



Logarithms and exponentials



Laws of logs :

$$\log mn = \log m + \log n,$$

$$\log m^k = k \log m.$$

e^x and $\ln x$ are inverses.

Unknown in the power $\Rightarrow$
take logs.

Linear form :

$$y = Ax^n \Rightarrow \ln y = \ln A + n \ln x.$$

ln and exp inverse · reflect in $y = x$



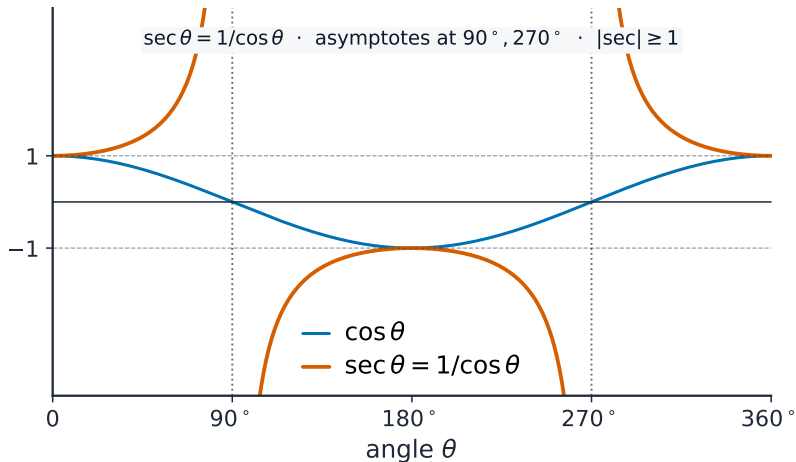
More trigonometry

Reciprocals: $\sec = \frac{1}{\cos}$, $\csc = \frac{1}{\sin}$, $\cot = \frac{1}{\tan}$.

$$\sec^2 \theta \equiv 1 + \tan^2 \theta.$$

Compound & **double angle** ; **R-formula** $a \sin \theta + b \cos \theta = R \sin(\theta + \alpha)$.

More trigonometry, in detail



Reciprocals: $\sec = \frac{1}{\cos}$, $\csc = \frac{1}{\sin}$, $\cot = \frac{1}{\tan}$.

Worked example – reduce to one function first

Solve $2 \tan^2 \theta + 3 \sec \theta = 18$.

- replace $\tan^2 \theta$ with $\sec^2 \theta - 1$
- $2(\sec^2 \theta - 1) + 3 \sec \theta = 18$
- $2 \sec^2 \theta + 3 \sec \theta - 20 = 0$
- a quadratic in $\sec \theta$: factorise, then take $\cos \theta = 1/\sec \theta$

Always reduce to one trig function before solving.

The identity you need is the one that converts the squared term – here $\tan^2 \rightarrow \sec^2$.

The three reciprocal functions, and their identities

secant

$$\sec \theta = \frac{1}{\cos \theta}$$

cosecant

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta}$$

cotangent

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}$$

from $\sin^2 + \cos^2 = 1$ divide by $\cos^2 \theta$: $1 + \tan^2 \theta \equiv \sec^2 \theta$

and by $\sin^2 \theta$

$$1 + \cot^2 \theta \equiv \operatorname{cosec}^2 \theta$$

compound angle

$\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$ – and the double-angle forms that follow

“co” pairs with *sine*, not cosine: **cosecant** is $1/\sin$. Getting that backwards is the standard slip.



Differentiation

Standard: $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$,

$$\frac{d}{dx}\sin x = \cos x.$$

Two rules

product : $(uv)' = u'v + uv'$

quotient : $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Also **parametric** & **implicit** .

Parametric and implicit differentiation

parametric equations

$x = x(t)$, $y = y(t)$ – both coordinates depend on a parameter

the gradient

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ – differentiate each with respect to t , then divide

implicitly defined

an equation in x and y that is not solved for y

the method

differentiate every term with respect to x , using the chain rule on any y term:
 $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$

then

collect the $\frac{dy}{dx}$ terms and make it the subject

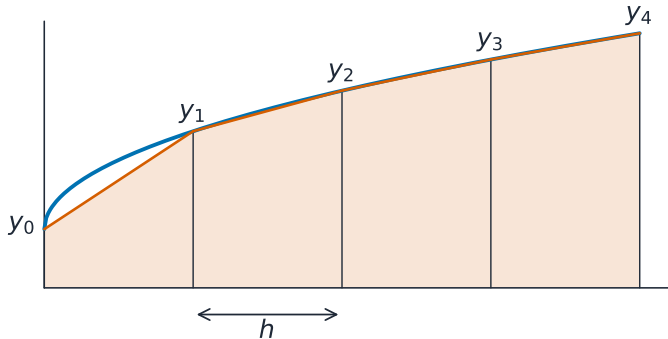
In both cases you never solve for y first. That is the point of the techniques – they work exactly where solving is impossible.

Integration

Reverse each derivative, e.g. $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$.

Trapezium rule estimates an integral: $\frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$.

Integration, in detail



$$\int_a^b y \, dx \approx \frac{h}{2}(y_0 + 2(y_1 + \cdots + y_{n-1}) + y_n)$$

Reverse each derivative, e.g. $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$.

Four techniques worth naming

Product rule

The **product rule** : $(uv)' = u'v + uv'$ – for a product, never for a composition.

Integration by substitution

Integration by substitution reverses the chain rule: change the variable *and* the limits, or convert back at the end.

Natural logarithm

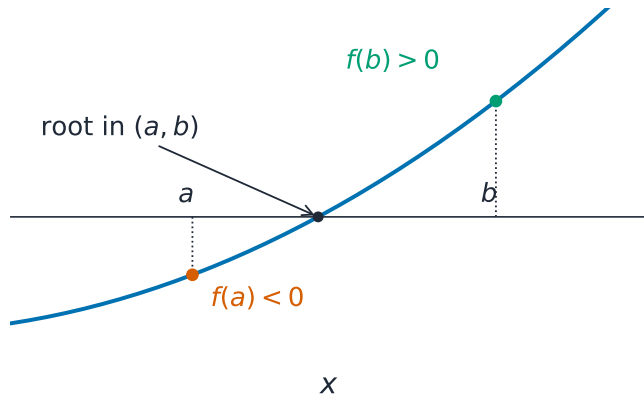
The **natural logarithm** is the inverse of e^x , and $\int \frac{1}{x} dx = \ln |x| + c$.

Remainder theorem

The **remainder theorem** : the remainder when $f(x)$ is divided by $(x - a)$ is $f(a)$ – so a remainder of zero is the factor theorem.

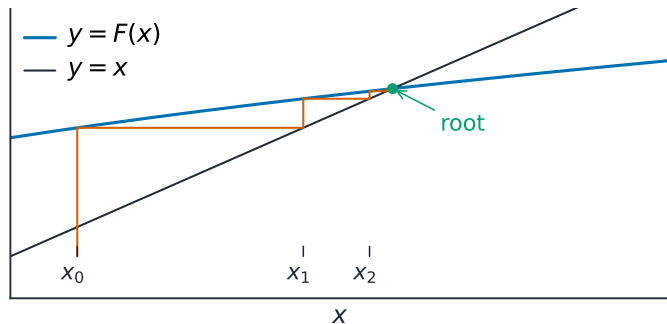
Indices and **trigonometric identities** are what make an integral doable: simplify first, integrate second. A polynomial's **degree** is its highest power.

A sign change traps a root



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

Iteration : $x_{n+1} = F(x_n)$



$x_{n+1} = g(x_n)$ · converges near root if $|g'(x)| < 1$

iteration $x_{n+1} = F(x_n)$

Finding a root when the equation cannot be solved

a root

a solution of $f(x) = 0$ – and many equations have no exact form for one

the sign change rule

if $f(a)$ and $f(b)$ have opposite signs and f is continuous, a root lies between them

iteration

rearrange to $x = F(x)$, then use the **iterative formula** $x_{n+1} = F(x_n)$

starting

begin from a value near the root, found by the sign change

converge

the sequence may converge to the root – or diverge, depending on the rearrangement chosen

the check

a sign change on a small interval either side confirms the root to the required accuracy

Different rearrangements of the *same* equation converge at different speeds, or not at all. The exam gives you the rearrangement to use.

Worked example – the factor theorem

$p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18$ has factor $(x + 2)$. Find k .

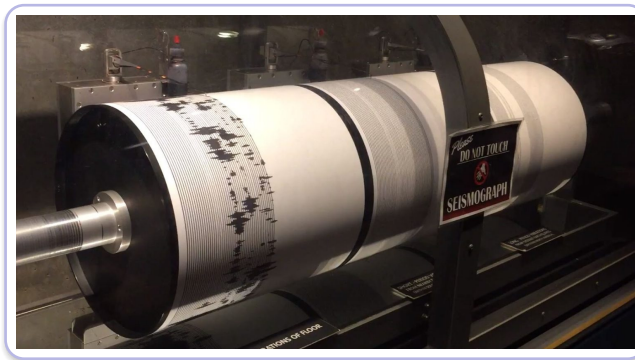
Factor theorem: $p(-2) = 0$.

$$2(16) - 8k + 4k - 34 + 18 = 16 - 4k = 0$$

$$\Rightarrow k = 4$$

$(x + 2)$ a factor $\Leftrightarrow p(-2) = 0$
– substitute the value that makes the bracket zero, not +2.

Logarithmic and exponential functions



Earthquake strength is measured on the logarithmic Richter scale.

Turning a power law into a straight line

the problem

$y = Ax^n$ is a curve, and its constants cannot be read off a plot

take logs

$$\ln y = \ln A + n \ln x$$

the linear form

plotting $\ln y$ against $\ln x$ gives a **straight line**

reading it

the **gradient** is n and the **intercept** is $\ln A$

the other case

$y = Ab^x$ gives $\ln y = \ln A + x \ln b$: plot $\ln y$ against x , not against $\ln x$

Which variable to take logs of is the whole decision. **Power law** $\rightarrow$ log both; **exponential** $\rightarrow$ log y only.

Exam tips

- Use the **laws of logarithms** to solve equations; remember $\ln$ and e^x are inverses.
- For **numerical methods**, show a **sign change** to locate a root, set out the iteration clearly, and give the answer to the stated accuracy.
- Learn the **chain, product and quotient rules** and identify which the function needs.
- Solve modulus equations $|f(x)| = g(x)$ by considering both the positive and negative cases, and sketch to check.



Topic 2 – key takeaways

1

Algebra

factor theorem; modulus inequalities

2

Logs

e^x & $\ln x$ inverse; take logs for powers

3

Trig

$\sec^2 = 1 + \tan^2$; R-formula

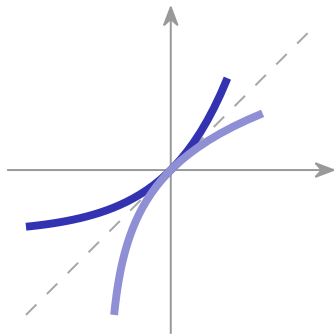
4

Calculus

product/quotient rules; trapezium rule

Check yourself

- 1 What is the remainder when $p(x) \div (x - a)$?
- 2 The derivative of $\ln x$?
- 3 Rewrite $\sec^2 \theta$ using $\tan$.
- 4 What does a sign change between a and b show?



Answers 1. $p(a)$ 2. $1/x$ 3. $1 + \tan^2 \theta$ 4. a root lies between them