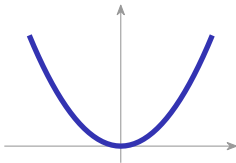


Pure Mathematics 1

A-Level Mathematics ■ Topic 1

A-Level Mathematics



What we will cover

- 1 Quadratics & functions
- 2 Coordinate geometry
- 3 Circular measure & trigonometry
- 4 Series
- 5 Differentiation & integration
- 6 Recap





Quadratics

$ax^2 + bx + c = 0$ has discriminant $\Delta = b^2 - 4ac$.

$$\Delta > 0$$

two distinct real roots

$$\Delta = 0$$

one repeated real root

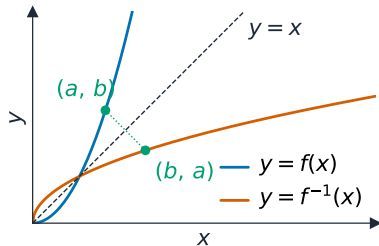
$$\Delta < 0$$

no real roots

Functions

- **domain** (inputs), **range** (outputs)
- **one-one** $\Rightarrow$ has an **inverse**
- **composition** $fg(x) = f(g(x))$
- **transformations** : $f(x) + a$, $f(x + a)$, $af(x)$, $f(ax)$

Functions, in detail



inverse = reflection in $y = x$ · domain of f^{-1} = range of f

f^{-1} reflects f in $y = x$

Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a) .

The vocabulary of a function question

Composition of functions

$fg(x)$ means do g first: the **composition of functions**, and the order matters.

Increasing and decreasing

An **increasing function** has $f'(x) > 0$ throughout an interval; a **decreasing function** has $f'(x) < 0$.

Identity

An **identity** is true for *every* value of x – unlike an equation, which is true for some.

Coefficients

The **coefficients** are the numbers multiplying each power; a **quadratic inequality** is solved from the sketch, not by dividing through.

An inverse exists only for a one-one function – which is why a domain is so often restricted before an inverse is asked for.

Finding an inverse

To find f^{-1} : write $y = f(x)$, make x the subject, then swap the letters.

The domain of f decides which square root you keep.

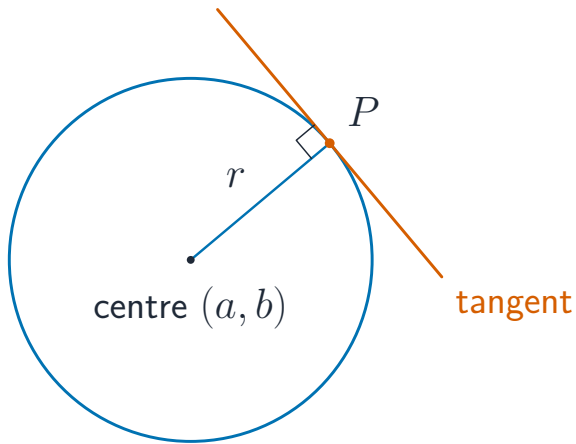
$f(x) = (x + 3)^2 - 12$ for $x \geq 0$. Find f^{-1} .

$(x + 3)^2 = y + 12 \Rightarrow x + 3 = \sqrt{y + 12}$,
taking the **positive** root because $x \geq 0$ means
 $x + 3 \geq 3 > 0$.

$$f^{-1}(x) = \sqrt{x + 12} - 3.$$



Coordinate geometry



Line gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$;
perpendicular $\Rightarrow m_1 m_2 = -1$.

$$(x - a)^2 + (y - b)^2 = r^2.$$

A **tangent** meets the **radius**
at a right angle.

Straight lines

The **gradient** through (x_1, y_1) and (x_2, y_2) :

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Three forms of the same line:

$$y = mx + c, \quad y - y_1 = m(x - x_1), \quad ax + by + c = 0.$$

Parallel : equal gradients. **Perpendicular** : the gradients multiply to -1 .

Circles

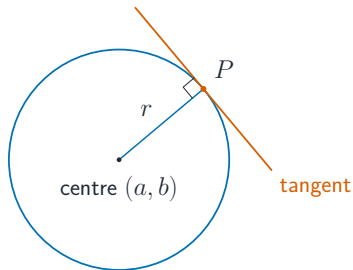
Centre (a, b) , radius r :

$$(x - a)^2 + (y - b)^2 = r^2.$$

An expanded form such as

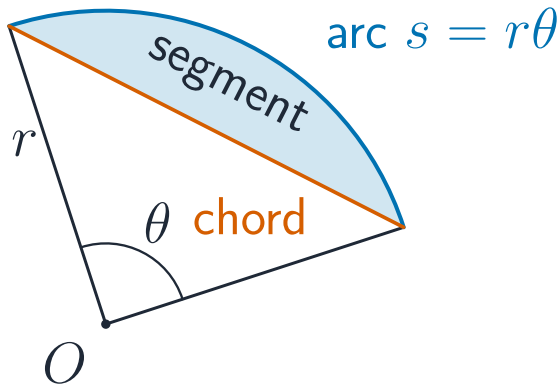
$x^2 + y^2 - 6x + 10y - 27 = 0$ is the **same circle** – complete the square in x and in y to read off the centre and radius.

A **tangent** meets the radius at a **right angle**. That one fact solves most circle problems.





Circular measure



Radians : $\pi = 180^\circ$. For a **sector** (angle in radians):

$$s = r\theta, \quad A = \frac{1}{2}r^2\theta.$$

Segment $= \frac{1}{2}r^2(\theta - \sin \theta).$

Radians, circles, and the trigonometric words

Degrees and radians

$180^\circ = \pi$ radians. Every calculus formula for a trigonometric function assumes radians, never **degrees**.

Circle parts

The **diameter** is twice the radius; a **chord** joins two points on the circumference; arc length is $r\theta$ and sector area $\frac{1}{2}r^2\theta$, with θ in radians.

The functions

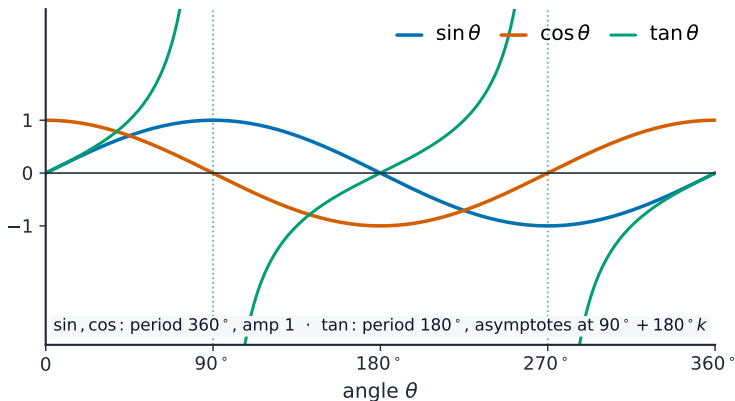
Cosine and the **tangent function** join sine as the three ratios – and the tangent is undefined where the cosine is zero.

Principal value

An inverse trigonometric function returns only the **principal value**; the other solutions come from the graph or the CAST diagram.

The **equation of a straight line** $y - y_1 = m(x - x_1)$ and $\tan \theta = m$ connect this topic to coordinate geometry.

Trigonometry



$\sin, \cos$ repeat every 360° ; learn exact values.

Identities :

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta},$$

$$\sin^2 \theta + \cos^2 \theta \equiv 1.$$

Reduce to one function,
then solve in the
interval.

Solving trigonometric equations

Solve $6 \sin \theta = 1 + \frac{2}{\sin \theta}$ **for** $-180^\circ < \theta < 180^\circ$.

Multiply by $\sin \theta$ – it is a **quadratic in** $\sin \theta$:

$$6 \sin^2 \theta - \sin \theta - 2 = 0 \Rightarrow (3 \sin \theta - 2)(2 \sin \theta + 1) = 0.$$

$$\sin \theta = \frac{2}{3} \Rightarrow \theta = 41.8^\circ, 138.2^\circ; \quad \sin \theta = -\frac{1}{2} \Rightarrow \theta = -30^\circ, -150^\circ.$$

Reduce to **one** function first, then find **every** solution in the interval – the marks are lost by stopping at the first one.



Series

Binomial :

$$(a + b)^n = \sum \binom{n}{r} a^{n-r} b^r.$$

AP : $u_n = a + (n - 1)d.$

GP : $u_n = ar^{n-1}.$

Sum to infinity ($|r| < 1$)

$$S_{\infty} = \frac{a}{1 - r}.$$

e.g. $a = 32$, $r = -\frac{3}{4}$: $S_{\infty} = \frac{128}{7}.$

The binomial expansion

For a positive integer n ,

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n,$$

$$\text{with } \binom{n}{r} = \frac{n!}{r! (n-r)!}.$$

First three terms of $(2 - px)^5$

$$32 - 80px + 80p^2x^2 + \cdots$$

Keep the sign with the term: $(-px)^2$ is **positive**.

Arithmetic and geometric progressions

AP – adds a fixed **common difference** d :

$$u_n = a + (n - 1)d, \quad S_n = \frac{n}{2}(2a + (n - 1)d).$$

GP – multiplies by a fixed **common ratio** r :

$$u_n = ar^{n-1}, \quad S_n = \frac{a(1 - r^n)}{1 - r}.$$

The two progressions, and the binomial

Arithmetic progression

An **arithmetic progression** adds a common difference: $u_n = a + (n - 1)d$, and $S_n = \frac{n}{2}(2a + (n - 1)d)$.

Geometric progression

A geometric progression multiplies by a common ratio: $u_n = ar^{n-1}$, and the sum to infinity exists only when $|r| < 1$.

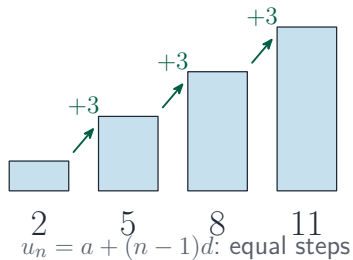
Binomial coefficient

The **binomial coefficient** $\binom{n}{r}$ counts the ways of choosing r from n – and it is the coefficient in the expansion of $(a + b)^n$.

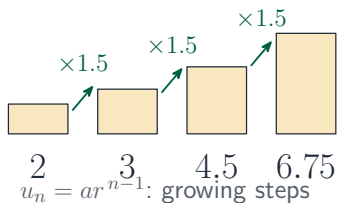
Check which progression before writing anything: a common *difference* or a common *ratio*, tested on the first three terms.

Arithmetic and geometric progressions, in detail

arithmetic: add d each step



geometric: multiply by r each step



An AP climbs in equal steps; a GP's steps grow by the same ratio

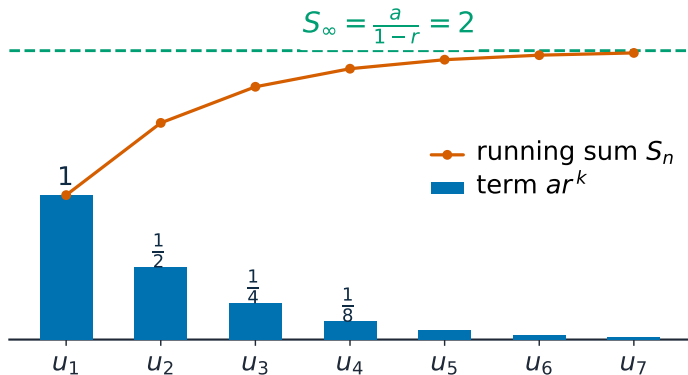
The sum to infinity

A GP **converges** when $|r| < 1$, and then

$$S_{\infty} = \frac{a}{1-r}.$$

$|r| < 1$ is a **condition**, not a formality – state it before you use the formula.

The sum to infinity, in detail



$$S_n = \frac{a(1-r^n)}{1-r} \cdot S_{\infty} = \frac{a}{1-r} \text{ only if } |r| < 1$$

The bars shrink to zero while the running sum approaches $a/(1-r)$

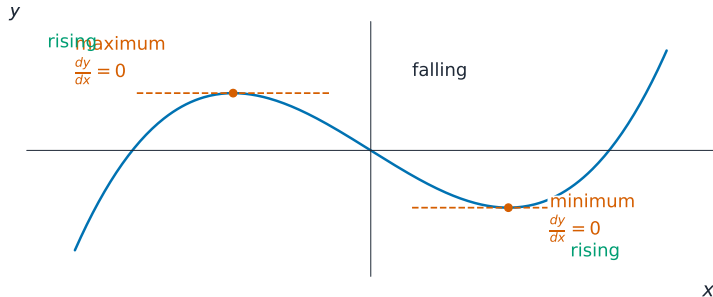


Differentiation

$$\frac{d}{dx}x^n = nx^{n-1}; \quad \text{chain: } f'(g)g'.$$

Stationary point : $\frac{dy}{dx} = 0$. Then $f'' > 0$ min, $f'' < 0$ max.

Differentiation, in detail



stationary points: $\frac{dy}{dx} = 0$ · use $\frac{d^2y}{dx^2}$ or first-derivative sign to classify

At a **stationary point** the tangent is horizontal. The second derivative then tells you which: $f'' > 0$ is a minimum, $f'' < 0$ a maximum.

The rules of differentiation

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Chain rule. $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ **Product rule.**

$$(uv)' = u'v + uv'$$

Quotient rule.

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Read the function's **shape** first:
a power, a function of a function,
a product, or a quotient.
The shape names the rule.

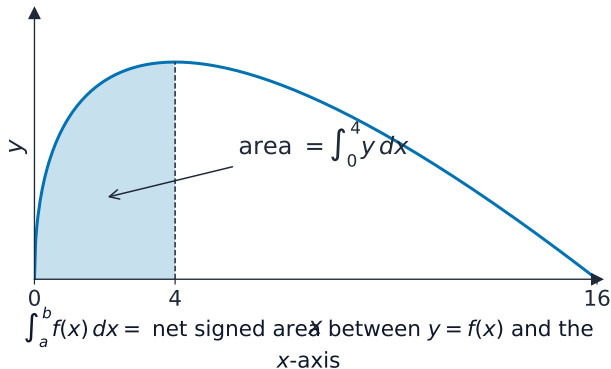
Integration

Reverse of **differentiation** (+C):

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + C.$$

$$\text{Area} = \int_p^q y dx.$$

Integration, in detail



A definite integral is the **signed area** between the curve and the x-axis – area below the axis counts as negative.

Definite integrals and area

$$\int_a^b f(x) dx = \left[F(x) \right]_a^b = F(b) - F(a),$$

the area between the curve and the x -axis from $x = a$ to $x = b$. Between two curves, integrate the **difference** – upper minus lower.

Area **below** the axis integrates to a **negative** number. If a question asks for an area that crosses the axis, split the integral at the crossing.

Using the derivative, and improper integrals

a rate of change

a derivative *is* one – and **linked rates** use the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

stationary points

where $\frac{dy}{dx} = 0$; the second derivative's sign classifies them

increasing / decreasing

where the derivative is positive / negative

tangent and normal

gradients m and $-\frac{1}{m}$ at the same point

an improper integral

has an **infinite limit**, or an endpoint where the integrand is undefined

how to handle it

replace the offending limit with a variable, integrate, then **take the limit**

For a linked rate, write down which quantity you know the rate of and which you want, then **chain the derivatives between them**. That sentence is most of the mark.

What the derivative tells you about the shape

Gradient of a curve

$f'(x)$ is the **gradient of a curve** at a point – the gradient of the tangent there, as a **chord** shortens to nothing.

Maximum and minimum

At a **maximum point** $f' = 0$ and $f'' < 0$; at a **minimum point** $f' = 0$ and $f'' > 0$.

Point of inflexion

A **point of inflexion** is where the curvature changes sign. A stationary one has $f' = 0$ and $f'' = 0$ – but $f'' = 0$ alone is not enough.

Integration

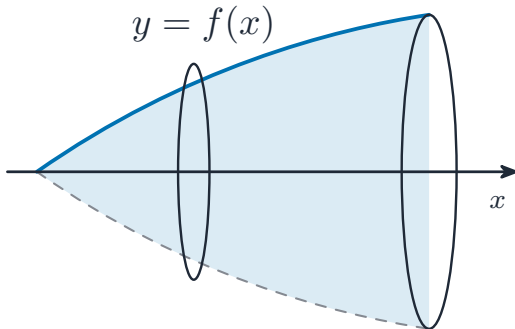
Reversing the process needs the **constant of integration** : one condition fixes it, and without it the answer is a family of curves.

Test a stationary point with the second derivative *or* with the sign of f' either side. Both are accepted; neither may be skipped.

Volume of revolution

Rotate a **region** about the x -axis – each slice is a disc of area πy^2 :

$$V = \pi \int_p^q y^2 dx.$$



Worked example – completing the square

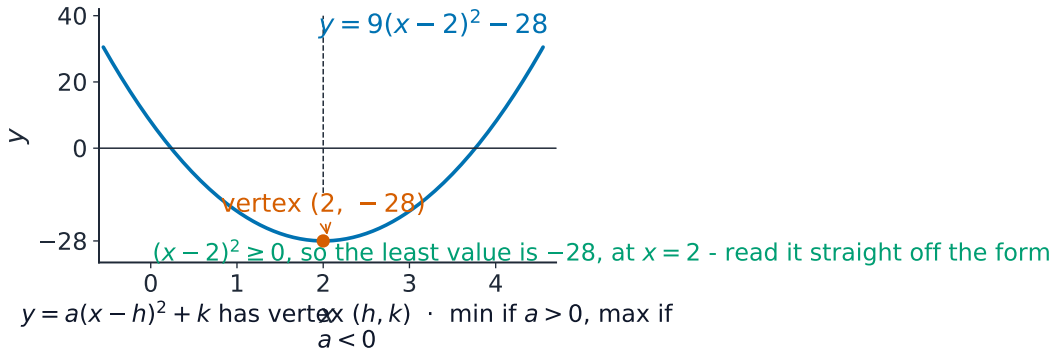
Write $9x^2 - 36x + 8$ **as** $p(x + q)^2 + r$.

Take out 9 from the first two terms, complete the square inside:

$$\begin{aligned} 9(x^2 - 4x) + 8 &= 9((x - 2)^2 - 4) + 8 \\ &= 9(x - 2)^2 - 28 \end{aligned}$$

Factor out the x^2 coefficient **before** completing the square. The form gives the minimum instantly: $(2, -28)$.

The completed square hands you the vertex



The completed square hands you the vertex: least value -28 at x equals 2

Worked example – a maximum by differentiation

$y = 4x^{1/2} - x$ has a maximum at $x = a$.

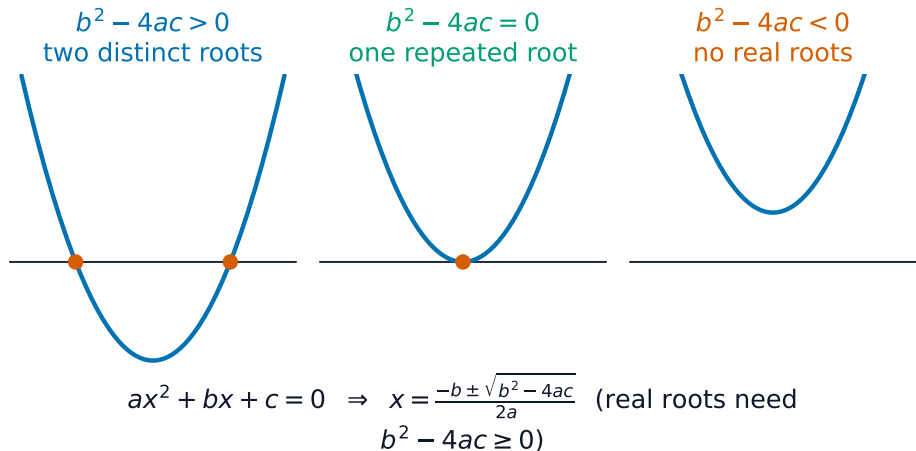
Find a .

$$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \Rightarrow x^{1/2} = 2$$

so $a = 4$ (and $y = 4 \cdot 2 - 4 = 4$)

Rewrite roots as **powers**
($\sqrt{x} = x^{1/2}$) before differentiating. Set $dy/dx = 0$ for a turning point.

The discriminant



The sign of $b^2 - 4ac$ decides how many times the parabola meets the x -axis.

Solving quadratic equations and inequalities

Three ways to solve $ax^2 + bx + c = 0$:

factorise, complete the square, or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For an **inequality**, find the roots first, then decide which side of each root makes the statement true.

Sketch the **parabola**: it is **above** the x -axis *outside* the roots and **below** it *between* them. So $(k - 4)(k - 16) > 0$ gives $k < 4$ or $k > 16$.

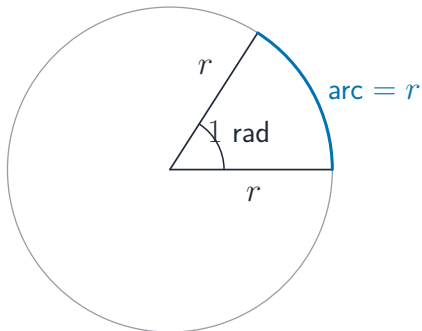
Simultaneous equations, and quadratics in disguise

One linear, one quadratic. Rearrange the **linear** one for a letter, substitute into the quadratic, and you are left with a single quadratic to solve.

Quadratic in disguise. $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2 : let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x .

The same trick returns in trigonometry, with $\sin \theta$ in place of u .

Radians



one **radian**: the angle whose arc is exactly one radius long

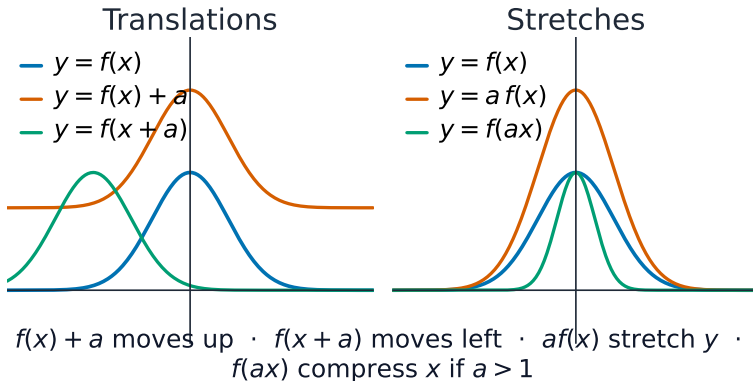
$$\pi \text{ rad} = 180^\circ, \text{ so } 1 \text{ rad} \approx 57.3^\circ$$

$$\text{degrees} \rightarrow \text{radians: } \times \frac{\pi}{180}$$

$$\text{radians} \rightarrow \text{degrees: } \times \frac{180}{\pi}$$

One radian: the angle whose arc equals the radius

Graphs of inverses and transformations



Adding to the output or input slides the curve; a multiplier stretches it.

The transformations, and stating them fully

$$y = f(x) + a$$

translation **up** by a

$$y = f(x + a)$$

translation **left** by a – the opposite of the sign

$$y = af(x)$$

stretch **parallel to the y-axis**, scale factor a

$$y = f(ax)$$

stretch parallel to the x-axis, scale factor $\frac{1}{a}$

$$y = -f(x) \text{ and } y = f(-x)$$

reflection in the x-axis, and in the y-axis

combining two

the order can matter – so state each one fully: type, direction and magnitude

An inverse function's graph is the reflection of f in the line $y = x$, which is why a function must be one-to-one before an inverse exists.

Trigonometry, continued



A Ferris wheel: a point on the rim rises and falls like a sine curve.

The exact values, and the progressions

sin

$$\sin 30^\circ = \frac{1}{2}, \quad \sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \sin 60^\circ = \frac{\sqrt{3}}{2}$$

cos

$$\cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 60^\circ = \frac{1}{2}$$

tan

$$\tan 30^\circ = \frac{1}{\sqrt{3}}, \quad \tan 45^\circ = 1, \quad \tan 60^\circ = \sqrt{3}$$

a progression

a list of terms following a rule

arithmetic

a common **difference** d : $u_n = a + (n - 1)d$, and $S_n = \frac{n}{2}(2a + (n - 1)d)$

geometric

a common **ratio** r : $u_n = ar^{n-1}$, and $S_\infty = \frac{a}{1-r}$ when $|r| < 1$

The exact values come from the half-equilateral triangle and the half-square. Reconstructing them from those two triangles beats memorising twelve fractions.

Exam tips

- Show **every line of algebra** – method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc = $r\theta$, sector area = $\frac{1}{2}r^2\theta$) and for calculus of trig functions.
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.



Topic 1 – key takeaways

1

Quadratics

discriminant decides the roots

2

Geometry

$m_1 m_2 = -1$; tangent $\perp$ radius

3

Trig

$\sin^2 + \cos^2 = 1$; sectors in radians

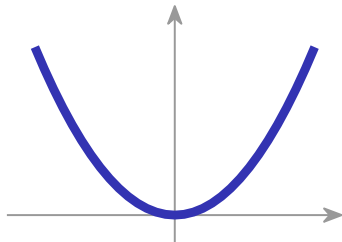
4

Calculus

$\frac{dy}{dx} = 0$ stationary; $\int y dx$ area

Check yourself

- 1 How many roots if $b^2 - 4ac < 0$?
- 2 Reflect a graph in which line to get its inverse?
- 3 Arc length of a sector (radians)?
- 4 Test for a minimum stationary point?



Answers 1. none 2. $y = x$ 3. $s = r\theta$ 4. $f''(x) > 0$