

Probability & Statistics 2

A-Level Mathematics

This handout covers Topic 6: **Probability & Statistics** 2. It adds the Poisson model, combining random variables, continuous distributions, and the ideas of estimation and testing.

The Poisson distribution



People arriving at random in a queue follow a Poisson distribution.

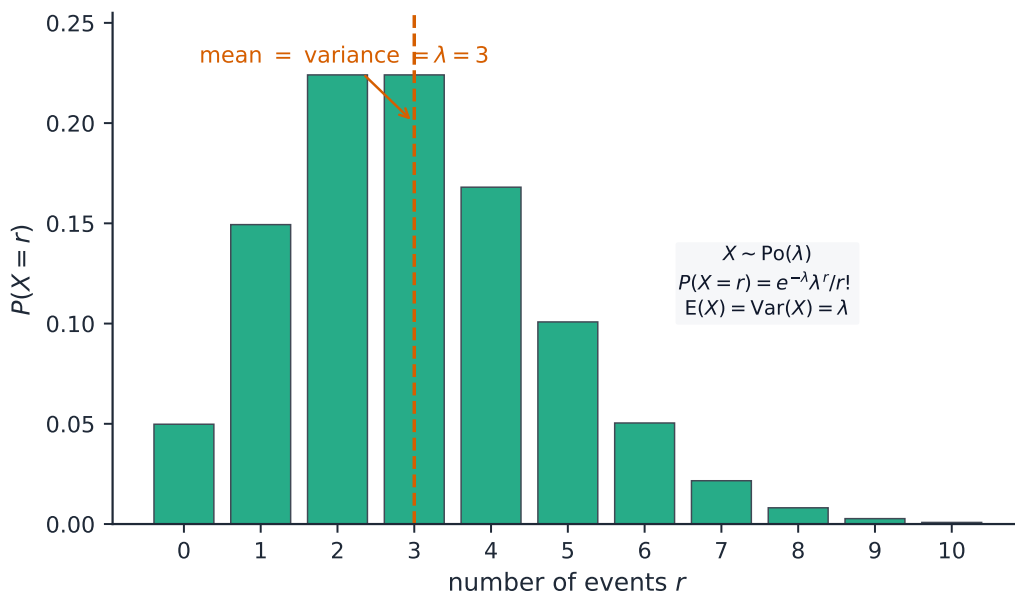
The **Poisson distribution** $X \sim \text{Po}(\lambda)$ models the number of random events in a fixed interval, when events happen at a steady average rate λ :

$$P(X = r) = e^{-\lambda} \frac{\lambda^r}{r!}.$$

For a Poisson variable the mean and the variance are both equal to λ . The Poisson distribution is a good **approximation** to the **binomial distribution** when n is large and p is small. The normal distribution (with continuity correction) approximates the Poisson when λ is large.

Worked example. $X \sim \text{Po}(3)$. Find $P(X = 2)$.

$$P(X = 2) = e^{-3} \frac{3^2}{2!} = e^{-3} \times 4.5 = 0.224.$$



The Poisson distribution $\text{Po}(3)$: for a Poisson variable the mean and variance both equal λ .

Linear combinations of random variables

When you change a variable by a linear rule, the **expectation** (mean) and **variance** follow these rules:

$$E(aX + b) = aE(X) + b, \quad \text{Var}(aX + b) = a^2 \text{Var}(X).$$

For two **independent** variables X and Y :

$$E(aX + bY) = aE(X) + bE(Y), \quad \text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y).$$

Two useful facts: if X has a **normal distribution** then so does $aX + b$; and the sum of independent Poisson variables is again Poisson.

Worked example. X has mean 5 and variance 4. Find $E(3X - 1)$ and $\text{Var}(3X - 1)$.

$$E(3X - 1) = 3(5) - 1 = 14, \quad \text{Var}(3X - 1) = 3^2 \times 4 = 36.$$

Continuous random variables

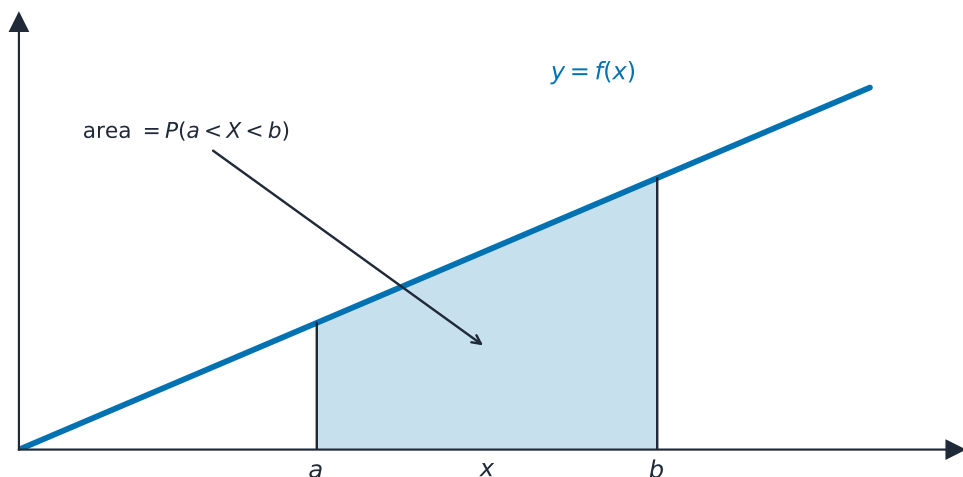
A **continuous random variable** can take any value in a range. Its probabilities come from a **probability density function** $f(x)$, with two key properties:

$$f(x) \geq 0, \quad \int_{-\infty}^{\infty} f(x) dx = 1.$$

A probability is the area under f , and the mean is found by integration:

$$P(a < X < b) = \int_a^b f(x) dx, \quad E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

The **cumulative distribution function** is $F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$; the **median** solves $F(m) = 0.5$, and other **percentiles** solve $F(x) = p$.



$$\int_{-\infty}^{\infty} f = 1 \quad \cdot \quad P(a < X < b) = \int_a^b f$$

For a continuous variable, the probability $P(a < X < b)$ is the area under $f(x)$ between a and b .

Worked example. A continuous variable has $f(x) = \frac{1}{2}x$ for $0 \leq x \leq 2$ (and 0 elsewhere). Find $E(X)$.

$$E(X) = \int_0^2 x \cdot \frac{1}{2}x dx = \int_0^2 \frac{1}{2}x^2 dx = \left[\frac{x^3}{6} \right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

The **variance** uses the same idea, $\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - (E(X))^2$. For the same $f(x) = \frac{1}{2}x$ on $[0, 2]$: $\int_0^2 x^2 \cdot \frac{1}{2}x dx = \left[\frac{x^4}{8} \right]_0^2 = 2$, so $\text{Var}(X) = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}$.

Sampling and estimation



Statistics studies a sample to learn about a whole population.

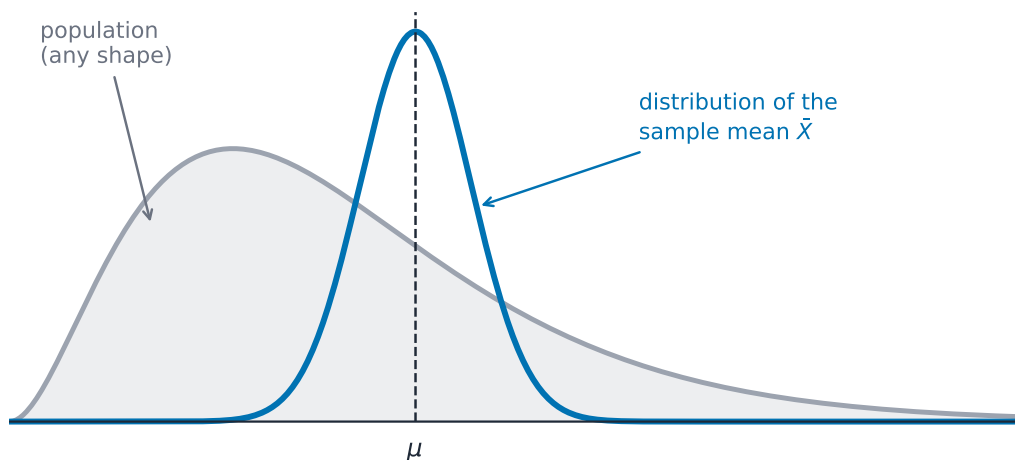
A **sample** is a small group chosen from the whole **population**. A **random sample** needs **randomness**, so that every member has a fair chance of being chosen.

Some methods are **unsatisfactory** because they are **biased**: sampling only volunteers, or the first 20 people to arrive, over-represents certain kinds of people. A genuinely random sample uses **random numbers** – number every member of the population, then draw numbers (from a table or a generator) to decide who is in the sample.

The sample mean $\bar{X}$ is itself a random variable, with

$$E(\bar{X}) = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

By the **Central Limit Theorem**, for a large sample $\bar{X}$ is approximately normal, whatever the shape of the population.

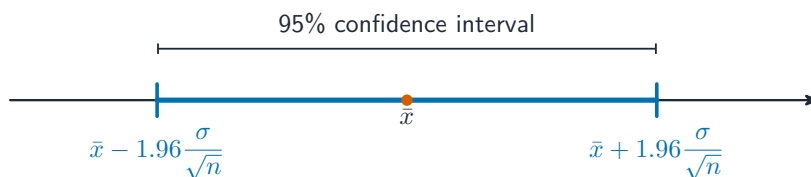


$$\text{CLT: } \bar{X} \approx N(\mu, \sigma^2/n) \text{ for large } n \cdot \text{SE} = \sigma/\sqrt{n}$$

Whatever the population's shape, the sample mean $\bar{X}$ has a narrow, near-normal distribution centred on μ .

From a sample you can find **unbiased estimates** of the population mean and variance. A **confidence interval** gives a range that probably contains the true mean. When the population is normal with known σ (or the sample is large), a 95% interval is

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}.$$



A 95% confidence interval stretches 1.96 standard errors each side of the sample mean.

You can also find a confidence interval for a **population proportion** from a large sample.

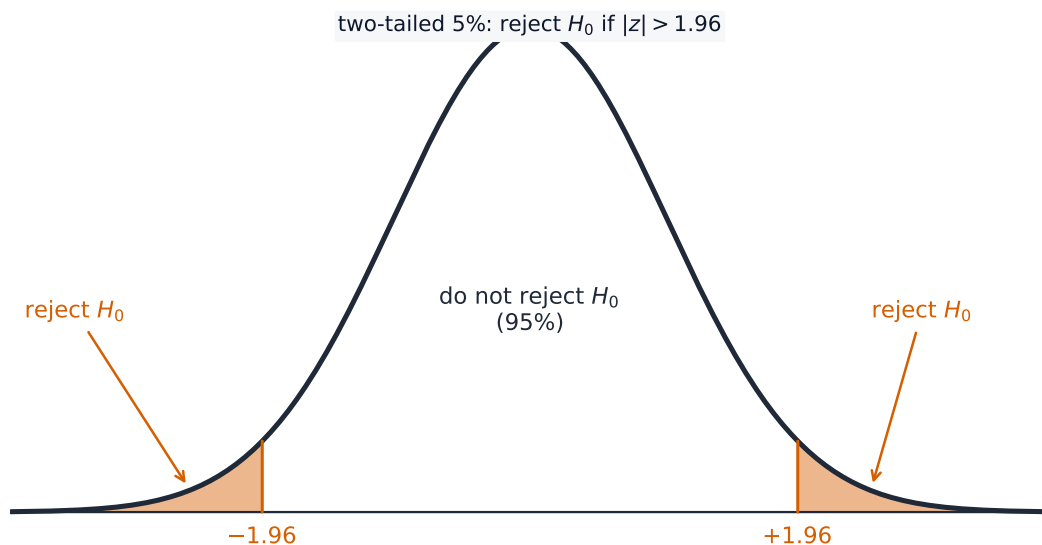
Worked example. A sample of $n = 64$ has mean $\bar{x} = 50$, from a population with $\sigma = 8$. Find a 95% confidence interval for the population mean.

$$50 \pm 1.96 \times \frac{8}{\sqrt{64}} = 50 \pm 1.96 \Rightarrow (48.0, 52.0).$$

Hypothesis tests

A **hypothesis test** uses sample data to judge a claim. You set up two statements: the **null hypothesis** H_0 (the claim being tested, usually "no change") and the **alternative hypothesis** H_1 (what you suspect instead). The test is **one-tailed** if H_1 points one way (e.g. $\mu > 50$) and **two-tailed** if it allows both ways ($\mu \neq 50$).

You fix a **significance level** (often 5%), work out a **test statistic** from the data, and see whether it lands in the **rejection region** (also called the **critical region**); if it does you reject H_0 , otherwise the statistic is in the **acceptance region**.



A two-tailed test at 5% rejects H_0 only if the test statistic falls in a shaded tail beyond ± 1.96 .

Two mistakes are possible: a **Type I error** is rejecting H_0 when it is actually true; a **Type II error** is accepting H_0 when it is actually false. You can find their probabilities from the rejection region: $P(\text{Type I}) = P(\text{statistic in the rejection region} \mid H_0)$ – this equals the significance level – and $P(\text{Type II}) = P(\text{statistic in the acceptance region} \mid H_1 \text{ true for a stated value})$, computed from the binomial, Poisson, or normal distribution.

Worked example. A population is claimed to have mean 50, with $\sigma = 8$. A sample of $n = 64$ gives $\bar{x} = 52$. Test at the 5% level whether the mean has changed.

$H_0: \mu = 50$ and $H_1: \mu \neq 50$ (two-tailed). The test statistic is

$$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{52 - 50}{8/8} = 2.$$

The critical value at 5% (two-tailed) is 1.96. Since $2 > 1.96$, you reject H_0 : there is evidence the mean has changed.

Worked example (a binomial test). A coin is claimed fair but suspected of landing heads too rarely: $H_0: p = 0.5$, $H_1: p < 0.5$. In $n = 30$ tosses you see $X = 9$ heads. Under H_0 , $X \sim B(30, 0.5)$, so the one-tailed tail probability is

$$P(X \leq 9) = \sum_{k=0}^9 \binom{30}{k} (0.5)^{30} \approx 0.021.$$

Since $0.021 < 0.05$, reject H_0 : the coin does seem biased against heads. (For large n the binomial is approximated by a normal; a **Poisson** test works the same way for rare events. And here, if the rule is "reject when $X \leq 9$ ", then $P(\text{Type I}) = P(X \leq 9 \mid p = 0.5) \approx 0.021$.)

Exam tips

- Use the **Poisson** distribution for rare, random, independent events; its mean equals its variance ($= \lambda$).
- When combining independent random variables, **variances add** (they never subtract).
- For a **hypothesis test**, state H_0 and H_1 , the significance level, the test statistic, and a conclusion **in context**.
- For a **confidence interval**, use the correct z (or t) value and interpret it in words.