Pure Mathematics 1

A-Level Mathematics

This handout covers Topic 1: **Pure Mathematics** 1. It is the algebra and calculus core of the course. Each ## section is one syllabus subtopic.

Quadratics

Under an idealised uniform deck load, a suspension-bridge cable is approximately parabolic.

A **quadratic** is an expression of the form $ax^2 + bx + c$, where $a \neq 0$. The letters a, b, c are the **coefficients** (the fixed numbers). Much of this section is about solving the **equation** $ax^2 + bx + c = 0$.

■ Completing the square

To **complete the square** means to write the quadratic in the form

$$a(x+p)^2 + q.$$

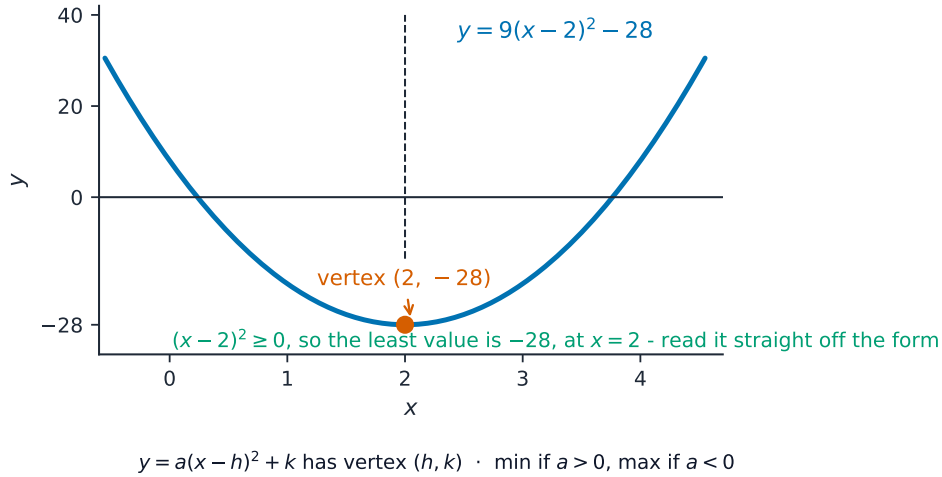
This form is useful: it shows the **vertex** (turning point) of the curve at $(-p,\ q)$, and it gives a quick way to solve the equation.

> **Worked example.** Write $9x^2 - 36x + 8$ in the form $p(x+q)^2 + r$.

Take the factor 9 out of the first two terms , then complete the square inside:

$$\begin{aligned} 9x^2 - 36x + 8 &= 9\left(x^2 - 4x\right) + 8 \\ &= 9\left((x-2)^2 - 4\right) + 8 \\ &= 9(x-2)^2 - 36 + 8 = 9(x-2)^2 - 28. \end{aligned}$$

So $p = 9$, $q = -2$, $r = -28$.



The completed square hands you the vertex: least value -28 at x equals 2

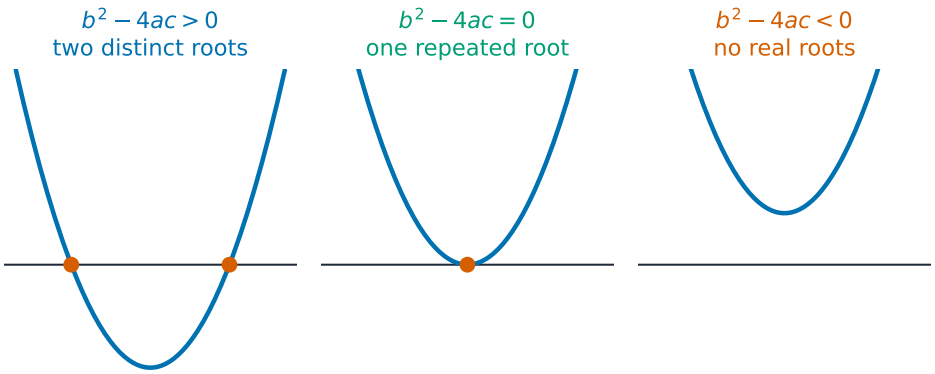
■ The discriminant

The **discriminant** of $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac.$$

It tells you how many **real roots** (real solutions) the equation $ax^2 + bx + c = 0$ has:

Discriminant	Roots
$b^2 - 4ac > 0$	two **distinct** real roots
$b^2 - 4ac = 0$	one repeated real root
$b^2 - 4ac < 0$	no real roots



$ax^2 + bx + c = 0 \;\Rightarrow\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (real roots need $b^2 - 4ac \geq 0$)

The sign of $b^2 - 4ac$ decides how many times the parabola meets the x-axis.

Worked example. Find the values of the constant k for which $3kx^2 + (k+8)x + 3 = 0$ has two distinct real roots.

Here $a = 3k$, $b = k + 8$, $c = 3$. For two distinct real roots you need $b^2 - 4ac > 0$:

$$(k+8)^2 - 4(3k)(3) > 0 \;\Rightarrow\; k^2 + 16k + 64 - 36k > 0 \;\Rightarrow\; k^2 - 20k + 64 > 0.$$

Factorise: $(k-4)(k-16) > 0$, so $k < 4$ or $k > 16$. You also need $a \neq 0$, so $k \neq 0$.

Quadratic equations and inequalities

To solve a **quadratic equation**, factorise, complete the square, or use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve a **quadratic inequality** such as $(k-4)(k-16) > 0$, find the two roots, then decide which side of each root makes the statement true. A sketch of the parabola helps: an upward-opening curve with two distinct roots is above the x-axis (positive) outside the roots and below it (negative) between them.

Worked inequality. Solve $-x^2 + 4x - 3 \geq 0$. Multiplying by -1 reverses the inequality:

$$x^2 - 4x + 3 \leq 0 \quad\Rightarrow\quad (x-1)(x-3) \leq 0.$$

The factors have opposite signs between the roots, so $1 \leq x \leq 3$. Equality is allowed at both endpoints. Testing $x = 2$ in the original expression gives -1, which satisfies the inequality; a point outside, such as $x = 0$, gives 3, which does not. Do not automatically choose the outside intervals when the original leading coefficient is negative.

For a negative leading coefficient, the completed-square constant is a maximum: $-2x^2 + 12x - 11 = -2(x-3)^2 + 7$ has maximum 7 at $x = 3$. Completing the square still works; the sign changes the interpretation.

Simultaneous equations

To solve a pair of **simultaneous equations** where one is linear and one is quadratic, use **substitution**: rearrange the linear equation for one letter, then put that into the quadratic. This gives a single quadratic to solve.

Worked substitution. Solve $x + 2y = 7$ and $x^2 + 4y^2 = 25$. The linear equation gives $x = 7 - 2y$, so

$$(7-2y)^2 + 4y^2 = 25 \quad\Rightarrow\quad 2y^2 - 7y + 6 = 0.$$

Factorising gives $(2y-3)(y-2) = 0$, hence $y = \frac{3}{2}$ or 2. Recover the matching values of x: the solutions are $(4, \frac{3}{2})$ and $(3, 2)$. Each pair gives $x + 2y = 7$ and $x^2 + 4y^2 = 25$. Keep the pairing when presenting answers.

When finding intersections, form the common equation first. For $y = x^2 - 2x + 3$ and $y = x + k$, the intersection equation is $x^2 - 3x + 3 - k = 0$ and its discriminant is $4k - 3$. It has no real intersection for $k < \frac{3}{4}$, a repeated intersection for $k = \frac{3}{4}$, and two for $k > \frac{3}{4}$. The boundary point is $(\frac{3}{2}, \frac{9}{4})$.

Equations that are quadratic in disguise

Some equations are quadratic in **some function of** x. For example $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2: let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x. You will use this idea again in trigonometry.

Here $(u-1)(u-4) = 0$, so $x^2 = 1$ or 4 and $x = \pm 1, \pm 2$. Always return from the substitute to the original variable.

Worked square-root substitution. For $6x + \sqrt{x} - 1 = 0$, let $v = \sqrt{x} \geq 0$. Then

$$6v^2 + v - 1 = (3v-1)(2v+1) = 0.$$

Only $v = \frac{1}{3}$ is allowed; $v = -\frac{1}{2}$ cannot be a square root. Thus $x = v^2 = \frac{1}{9}$, which satisfies the original equation. Squaring a rejected negative value is not a valid way to recover another solution.

Using given roots to find constants

A root is an input giving zero. Substitute each stated root into the original polynomial to obtain simultaneous conditions, or expand its root form and compare coefficients. If the roots are α and β, then

$$a(x-\alpha)(x-\beta) = ax^2 - a(\alpha+\beta)x + a\alpha\beta.$$

Worked example. Suppose $px^2 + 6x - 8$ has roots $2m$ and $-4m$, with $p, m \neq 0$. Expanding $p(x-2m)(x+4m)$ gives $px^2+2pmx-8pm^2$. Hence $pm = 3$ and $pm^2 = 1$, giving $m = \frac{1}{3}$ and $p = 9$. Both actual roots $\frac{2}{3}$ and $-\frac{4}{3}$ satisfy $9x^2 + 6x - 8 = 0$. Do not assume a parameter is positive unless the question states that condition.

Functions

A **function** is a rule that sends each input to exactly one output. Write it as $f(x)$. The **composition of functions** $fg(x)$ means apply g first, then apply f to the result.

- The **domain** is the set of allowed inputs x.
- The **range** is the set of outputs the function actually produces.

A function is **one-one** if different inputs always give different outputs. (No output is repeated.) Only a one-one function has an **inverse function** f^{-1}, which reverses the rule.

The **composition** of two functions means doing one after the other. $fg(x)$ means "do g first, then f": $fg(x) = f(g(x))$. The composite function fg exists only when the range of g lies inside the domain of f.

This condition applies to the whole stated domain of the inner function. If $f(x) = \sqrt{x}$ for $x \geq 0$ and $g(x) = x - 3$ for all real x, $fg(x) = \sqrt{x-3}$ is not defined on all of that inner domain. A new restriction $x \geq 3$ makes the formula valid there; state that change explicitly.

To find a range, check whether a boundary is attained. For $x \geq 1$, the range of $1/x$ is $0 < y \leq 1$; zero is approached but never attained. A restricted quadratic also needs its actual domain: $(x-1)^2 + 2$ for $x \geq 3$ has range $y \geq 6$, since its vertex input is excluded.

An open-endpoint condition. Let $F(x) = 1/(x-2)$ for $x > 2$ and $G(x) = 3x + 2$ for $x > a$. The range of G is $(3a + 2, \infty)$. It lies wholly in the domain of F when $3a + 2 \geq 2$, so $a \geq 0$. Equality is allowed because the lower endpoint of that range is not attained. For $a < 0$, the allowed input 0 gives the forbidden output $G(0) = 2$.

Finding an inverse

To find f^{-1}: write $y = f(x)$, make x the subject, then swap letters.

Worked example. The function $f(x) = (x+3)^2 - 12$ is defined for $x \geqslant 0$. Find $f^{-1}(x)$.

Write $y = (x+3)^2 - 12$ and solve for x:

$$(x+3)^2 = y + 12 \;\Rightarrow\; x + 3 = \sqrt{y+12} \;\Rightarrow\; x = \sqrt{y+12} - 3.$$

You take the **positive** square root because $x \geqslant 0$ means $x + 3 \geqslant 3 > 0$. So

$$f^{-1}(x) = \sqrt{x+12} - 3.$$

The original range is $[-3, \infty)$, attained from original inputs $x \geq 0$. Therefore the inverse has domain $x \geq -3$ and range $f^{-1}(x) \geq 0$. Merely requiring the radical to be real would give the larger, incorrect inverse domain $x \geq -12$.

A different restriction can require the negative square-root branch. For $h(x) = (x+1)^2 - 5$ with $x \leq -1$, solve $y + 5 = (x+1)^2$ using $x + 1 \leq 0$:

$$h^{-1}(x) = -1 - \sqrt{x+5}, \qquad x \geq -5.$$

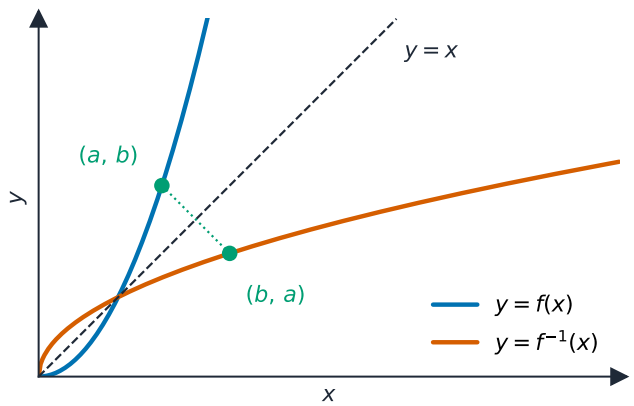
Its range is $h^{-1}(x) \leq -1$. The endpoint $(-1, -5)$ reflects to $(-5, -1)$ and the point $(-3, -1)$ reflects to $(-1, -3)$. A $\pm$ answer would not define a one-one inverse function. The inverse is also different from the reciprocal $1/h(x)$.

To invert a composite of compatible one-one functions, reverse their order: $(gf)^{-1} = f^{-1}g^{-1}$. Alternatively form the composite rule first, then invert it with the original domain restriction.

For example, $f(x) = (x-1)^2 + 2$ for $x \leq 1$ and $g(x) = 5 - x$ give $H(x) = gf(x) = 3 - (x-1)^2$, with range $H(x) \leq 3$. Solving $y = 3 - (x-1)^2$ using $x - 1 \leq 0$ gives $H^{-1}(x) = 1 - \sqrt{3-x}$ for $x \leq 3$, with range $H^{-1}(x) \leq 1$. Composing and then inverting keeps the whole branch condition visible.

Graphs of inverses and transformations

The graph of $y = f^{-1}(x)$ is the **reflection** of $y = f(x)$ in the line $y = x$.



inverse = reflection in $y = x$ · domain of f^{-1} = range of f

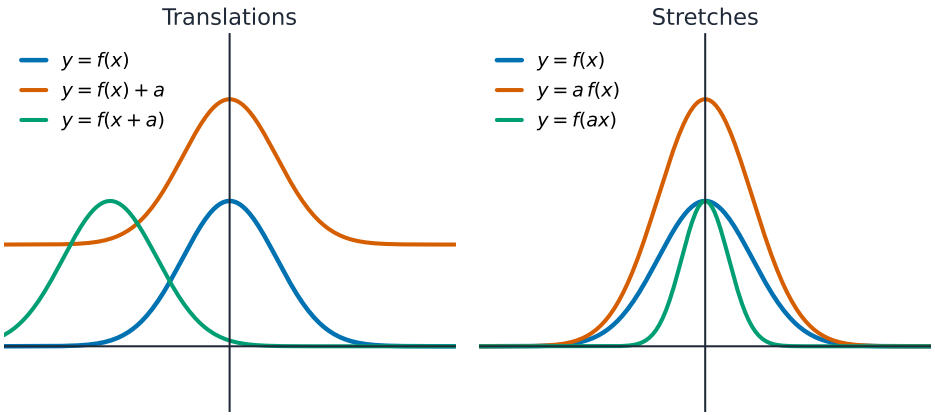
Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a).

You should know these **transformations** of $y = f(x)$: In the stretch descriptions below take $a > 0$. Negative multipliers also require a reflection : $-f(x)$ reflects in the x-axis and $f(-x)$ reflects in the y-axis.

New equation	Effect on the graph
$y = f(x) + a$	**translation** up by a
$y = f(x + a)$	translation left by a
$y = a\,f(x)$	**stretch** in the y-direction, scale factor a
$y = f(ax)$	stretch in the x-direction, scale factor $\frac{1}{a}$

When two transformations are combined, the order can matter. State each one fully (type, direction, and amount).

For $g(x) = 2f(x+3) - 1$, use two transformations: stretch vertically by factor 2, then translate by $(-3, -1)$ (left 3 and down 1 together). A point $(4, 5)$ becomes $(4, 10)$, then $(1, 9)$. Reversing these same operations gives $2(f(x+3) - 1) = 2f(x+3) - 2$, a different graph. If two transformations are required, combine the horizontal and vertical shifts into one translation. For a horizontal multiplier $f(ax)$, a point (s, t) moves to $(s/a, t)$; the scale factor is reciprocal, not a.



$f(x) + a$ moves up · $f(x + a)$ moves left · $af(x)$ stretch y · $f(ax)$ compress x if $a > 1$

Adding to the output or input slides the curve; a multiplier stretches it.

Coordinate geometry

Coordinate geometry connects positions, distances and intersections with algebra. A straight path can cross a circular boundary twice, touch it once or miss it. Solve their equations together to distinguish these cases exactly.

■ Choose and check a line equation

For two points with different x-coordinates, the **gradient** is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Useful forms of an **equation of a straight line** are

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + c = 0.$$

For example, through $(2, 3)$ and $(6, 11)$ the gradient is 2. The same line is $y - 3 = 2(x - 2)$, $y = 2x - 1$, or $2x - y - 1 = 0$. Substitute each given point to check. In $2x - 3y + 6 = 0$, rearranging gives $y = \frac{2}{3}x + 2$, so the gradient is $2/3$, not 2.

Distinct non-vertical **parallel** lines have equal gradients. For **perpendicular** lines with finite gradients, $m_1 m_2 = -1$. Horizontal lines $y = h$ have gradient zero; vertical lines $x = k$ have undefined gradient. They are perpendicular, but the product rule cannot be used for a vertical line. Through $(4, -2)$, the horizontal and vertical lines are $y = -2$ and $x = 4$.

■ Distances, midpoints and perpendicular bisectors

Pythagoras gives the distance between two points:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The **midpoint** is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

A **perpendicular bisector** passes through this midpoint at right angles to the segment. Every point on it is equally far from the two ends.

> **Worked example.** For $P(-1, 2)$ and $Q(5, 6)$, $PQ = \sqrt{6^2 + 4^2} = 2\sqrt{13}$ and $M = (2, 4)$. The segment gradient is $2/3$, so the bisector gradient is $-3/2$. Its equation is
>
> $$y - 4 = -\frac{3}{2}(x - 2) \quad \Longrightarrow \quad 3x + 2y = 14.$$

The midpoint alone is not an equation, and a perpendicular through an endpoint is not a bisector. If PQ is horizontal, its bisector is the vertical line through M; do not take a reciprocal of zero.

Read a circle in two forms

A **circle** with **centre** (a, b) and positive **radius** r has equation

$$(x-a)^2 + (y-b)^2 = r^2.$$

An expanded equation needs both squares completed:

$$\begin{aligned} x^2 + y^2 - 6x + 10y - 27 &= 0, \\ (x-3)^2 - 9 + (y+5)^2 - 25 &= 27, \\ (x-3)^2 + (y+5)^2 &= 61. \end{aligned}$$

The centre is $(3, -5)$ and radius $\sqrt{61}$. More generally, $x^2 + y^2 + 2gx + 2fy + c = 0$ gives centre $(-g, -f)$ and $r^2 = g^2 + f^2 - c$. Check that $r^2 > 0$ for a real circle of positive radius. If it is negative there are no real points; if zero there is only one point. Substitute a proposed point into the equation to test it.

Use a diameter, a semicircle or symmetry

Worked example. The points $P(1, 1)$ and $Q(7, 11)$ are the ends of a **diameter**. Its midpoint is $(4, 6)$ and its half-length is

$$r = \frac{1}{2}\sqrt{6^2 + 10^2} = \sqrt{34}.$$

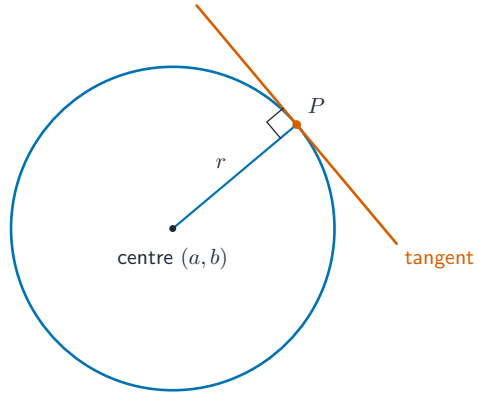
The circle is $(x-4)^2 + (y-6)^2 = 34$. A general chord's midpoint is not necessarily the centre: the diameter condition matters.

The angle formed at any other point on this circle by lines to P and Q is $90°$: it is an angle in a semicircle. For a non-diameter chord, the line from the centre to its midpoint is perpendicular to the chord and is a symmetry line. Use this to recover a reflected intersection point or a missing coordinate.

For instance, $(x-2)^2+(y-1)^2 = 5$ has diameter ends $A(0, 0)$ and $B(4, 2)$. Point $D(4, 0)$ lies on it and $\angle ADB = 90°$. The chord AD has midpoint $(2, 0)$, so its perpendicular bisector $x = 2$ passes through the centre and reflects A onto D.

Tangents: known and unknown contact points

A **tangent** is perpendicular to the radius at the contact point.



A tangent meets the radius at a right angle.

For the circle centred at $(2, 1)$ above, the radius to $B(4, 2)$ has gradient $1/2$. The tangent has gradient -2, so $y - 2 = -2(x - 4)$, or $2x + y = 10$. Use B in this equation, not the centre. At the top and bottom the tangents are horizontal, $y = 1 \pm \sqrt{5}$; at the left and right they are vertical, $x = 2 \pm \sqrt{5}$.

Worked example: unknown contact points. Find both points on this circle where the tangent gradient is -2. The radius gradient must be $1/2$, and the radius line through the centre is $y - 1 = \frac{1}{2}(x - 2)$. Put $u = x - 2$ and substitute:

$$u^2 + \frac{u^2}{4} = 5 \quad \Longrightarrow \quad u = \pm 2.$$

The contact points are $(4, 2)$ and $(0, 0)$, with tangent equations $2x + y = 10$ and $2x + y = 0$. A given tangent direction normally gives two opposite contact points. Keep exact square roots when the solutions are not integers. Radius geometry and substitution suffice; circle differentiation is not required here.

Intersections and parameter boundaries

Substitute a line into a circle to get a quadratic, solve for one coordinate and recover the other. In $(x - 2)^2 + (y - 1)^2 = 5$, the line $y = 0$ gives $(x - 2)^2 = 4$, hence the points $(0, 0)$ and $(4, 0)$.

For the family $y = k$, the same circle gives

$$(x-2)^2 = 5 - (k-1)^2.$$

There are two points when $1 - \sqrt{5} < k < 1 + \sqrt{5}$, one tangent point at each boundary, and no points outside that interval. At either boundary $x = 2$. For a general sloping line, the resulting quadratic's discriminant is positive, zero or negative in the same three cases. A repeated root is one point.

For a line and a quadratic curve, equate the two expressions for y. For $y = x^2$ and $y = 2x + k$:

$$x^2 - 2x - k = 0, \qquad \Delta = 4 + 4k.$$

Thus there are two intersections for $k > -1$, one for $k = -1$ at $(1, 1)$, and none for $k < -1$. Show the algebraic condition; a sketch alone does not prove an exact boundary. Some families have a positive discriminant for every real parameter, so they always intersect twice. Check separately if a parameter could make the quadratic coefficient zero.

■ Build a circle from axis gaps

A circle entirely in the first quadrant has centre (a, b) and radius r. Its least distances from the y-axis and x-axis are $a - r$ and $b - r$, respectively.

> **Worked example.** The gap from the y-axis is 1, the gap from the x-axis is 3, and the distance from the origin to the centre is $\sqrt{34}$. Then the centre is $(r + 1, r + 3)$, so

$$\begin{aligned}(r+1)^2 + (r+3)^2 &= 34,\\ r^2 + 4r - 12 &= 0,\\ (r+6)(r-2) &= 0.\end{aligned}$$

Reject the negative radius. Therefore $r = 2$, centre $(3, 5)$, and equation $(x-3)^2 + (y-5)^2 = 4$. The point furthest from the origin lies along the ray from the origin through the centre: its radius has gradient $5/3$, so its tangent gradient is $-3/5$.

Independent check. A different circle's least gaps from the x- and y-axes are 6 and 2, and its centre is $4\sqrt{5}$ from the origin. Find the radius, centre, circle equation and tangent gradient at its furthest point from the origin. Set up the centre coordinates before calculating.

Check after attempting: centre $(r + 2, r + 6)$ gives $(r + 2)^2 + (r + 6)^2 = 80$. Thus $(r + 10)(r - 2) = 0$, $r = 2$, centre $(4, 8)$ and equation $(x - 4)^2 + (y - 8)^2 = 4$. The radius direction has gradient 2, so the tangent gradient is $-1/2$.

Circular measure

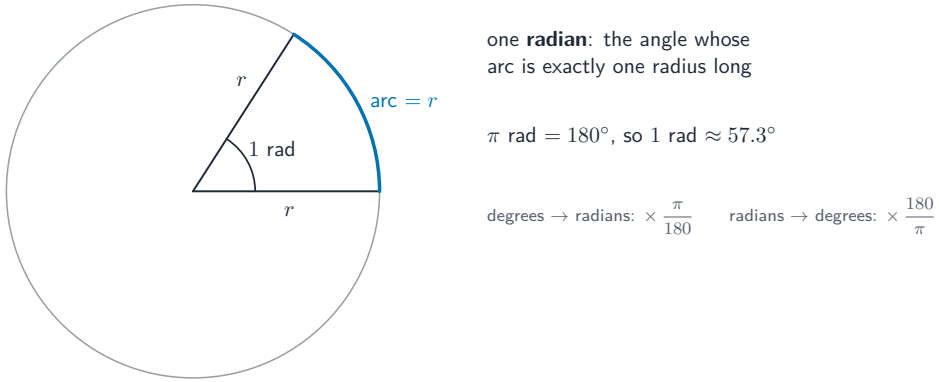
A curved boundary may contain an arc and several straight edges. Identify every piece before calculating. A sector, a triangle and a circular segment have different areas and perimeters.

■ Define a radian and convert angles

One **radian** is the angle at the centre that cuts off an **arc** equal in length to the radius. Since $\theta = s/r$, a full turn is $(2\pi r)/r = 2\pi$ radians. Therefore

$$\pi \text{ radians} = 180^\circ.$$

Multiply degrees by $\pi/180$ to get radians, or radians by $180/\pi$ to get degrees. For example, $150^\circ = 5\pi/6$ radians and $7\pi/4$ radians is 315°. Angles can exceed one turn: $450^\circ = 5\pi/2$ radians.



One radian is about 57.3°.

■ Sector formulas, perimeter and reverse problems

For a **sector** with radius r and angle θ in radians,

$$s = r\theta, \qquad A = \frac{1}{2}r^2\theta.$$

Here s is **arc length** and A is sector area. A sector's full perimeter is $s+2r$. Distinguish linear units from square units. Convert a degree angle before using these formulas, and use radian calculator mode for its sine or cosine.

If the arc length and area are given, eliminate the angle: $A = rs/2$, so $r = 2A/s$ and then $\theta = s/r$.

> **Worked example.** Arc length 8 cm and sector area 24 cm^2 give $r = 6$ cm, $\theta = 4/3$ radians and full perimeter 20 cm. The arc alone is not the perimeter.

Chords and minor circular segments

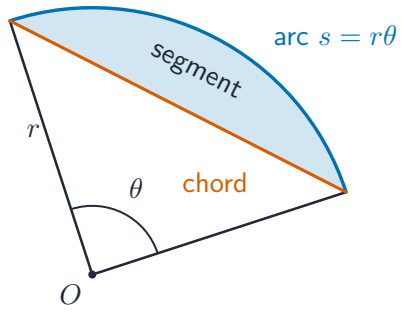
For a minor central angle $0 < \theta < \pi$, bisect the isosceles triangle formed by two radii and their chord. This gives

$$c = 2r\sin(\theta/2), \qquad A_{\text{triangle}} = \frac{1}{2}r^2\sin\theta.$$

The **circular segment** lies between the chord and arc. The minor segment is the minor sector minus the triangle:

$$A_{\text{minor}} = \frac{1}{2}r^2(\theta - \sin\theta).$$

Its perimeter is $r\theta + c$, not $r\theta + 2r$.



The chord separates the triangle from the minor segment.

For $r = 5$ cm and $\theta = 2$ radians, the arc length is 10 cm, the sector area is 25 cm^2, and the minor segment area is $12.5(2 - \sin 2) \approx 13.6$ cm^2. The minor segment perimeter is $10 + 10\sin 1 \approx 18.4$ cm.

Major segments

The major segment is the rest of the circle after removing the minor segment:

$$A_{\text{major}} = \pi r^2 - A_{\text{minor}}.$$

Its arc uses angle $2\pi - \theta$, so its perimeter is $r(2\pi - \theta) + c$.

> **Worked example.** Radius 4 cm and chord $4\sqrt{3}$ cm give $\sin(\theta/2) = \sqrt{3}/2$. The minor angle is $2\pi/3$, the triangle area is $4\sqrt{3}$, and

$$A_{\text{minor}} = \frac{16\pi}{3} - 4\sqrt{3},$$
$$A_{\text{major}} = \frac{32\pi}{3} + 4\sqrt{3}.$$

The major arc length is $16\pi/3$ cm. Add the chord for its segment perimeter. The major region includes the triangle; it is not just the major sector.

Compound boundaries: name the pieces

> **Worked example.** Let $OA = OB = 10$ cm, $\angle AOB = 1$ radian. Remove the sector of radius 4 cm, centred at O, from triangle OAB. The remaining boundary contains two straight gaps of length 6, the outer chord $AB = 20\sin(1/2)$, and an inner arc of length 4.

$$P = 6 + 6 + 20\sin(1/2) + 4 \approx 25.6 \text{ cm},$$
$$A = \frac{1}{2}(10)^2\sin 1 - \frac{1}{2}(4)^2(1) \approx 34.1 \text{ cm}^2.$$

There is no outer circular arc: AB is straight. If instead a region lies between two concentric sector arcs, its area is the difference of sector areas, but its perimeter includes both arcs and both radial gaps. For radii 9 and 5 and angle 1.4 radians, the area is 39.2 square units and perimeter is 27.6 units. Subtracting areas does not mean subtracting lengths.

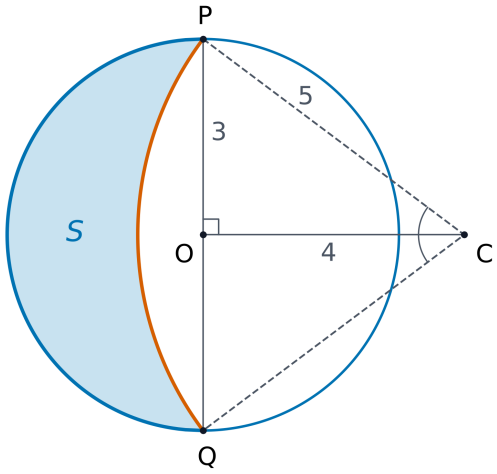
Use two conditions to find an unknown circle

If a sector has area 8θ cm^2, then $(1/2)r^2\theta = 8\theta$. Since $\theta > 0$, $r = 4$ cm. If the triangle area is 4 cm^2, then $8\sin\theta = 4$. The restriction $0 < \theta < \pi/2$ selects $\theta = \pi/6$, giving minor segment area $4\pi/3 - 4$ cm^2.

If a later part replaces the sector-area condition, start again. A chord-to-radius ratio can determine θ using $c = 2r\sin(\theta/2)$, then triangle area determines r. Do not keep a radius or angle whose original condition no longer applies.

Different circles, different radii and angles

The diagram shows a circle with centre O and radius 3, meeting a circle with centre C and radius 5. Here $OC = 4$ and PQ is the small circle's diameter, perpendicular to OC.



Only the relevant arc of the large circle is shown. Region S is inside the small circle and outside the large circle.

Let $\phi = \angle PCQ = 2\tan^{-1}(3/4) \approx 1.287$ radians. Triangle PCQ has area $\frac{1}{2}(6)(4) = 12$. The large circle's minor segment has area $\frac{25}{2}\phi - 12$. Subtract it from the small left semicircle:

$$A_S = \frac{9}{2}\pi - \frac{25}{2}\phi + 12 \approx 10.0,$$
$$P_S = 3\pi + 5\phi \approx 15.9.$$

The perimeter uses two curved pieces. The lines PQ and OC are construction lines inside the picture, not outside edges.

A shared chord can split another two-circle region into two circular segments. Sketch the required side of the chord for each circle. Calculate each sector and triangle with its own radius and central angle before combining their areas. For example, two circles of radius 6, with centres 6 apart, overlap in two equal minor segments. Equilateral triangles give each central angle $2\pi/3$. The overlap area is $2(12\pi - 9\sqrt{3}) = 24\pi - 18\sqrt{3}$ and its perimeter is the two minor arcs, 8π. The shared chord is internal.

Independent check. Replace the small radius by 4, set $OC = 3$, and keep the larger radius 5, with the same perpendicular-diameter geometry. Find the crescent area and perimeter to three significant figures.

Check after attempting: $\phi = 2\tan^{-1}(4/3)$. The large triangle still has area 12. Thus area $= 8\pi - \frac{25}{2}\phi + 12 \approx 14.0$ square units and perimeter $= 4\pi + 5\phi \approx 21.8$ units.

Trigonometry

A point on a turning wheel rises and falls. A trigonometric model describes that cycle, but its graph must have the correct height scale and timing. One calculator angle may miss another valid position in the same interval.

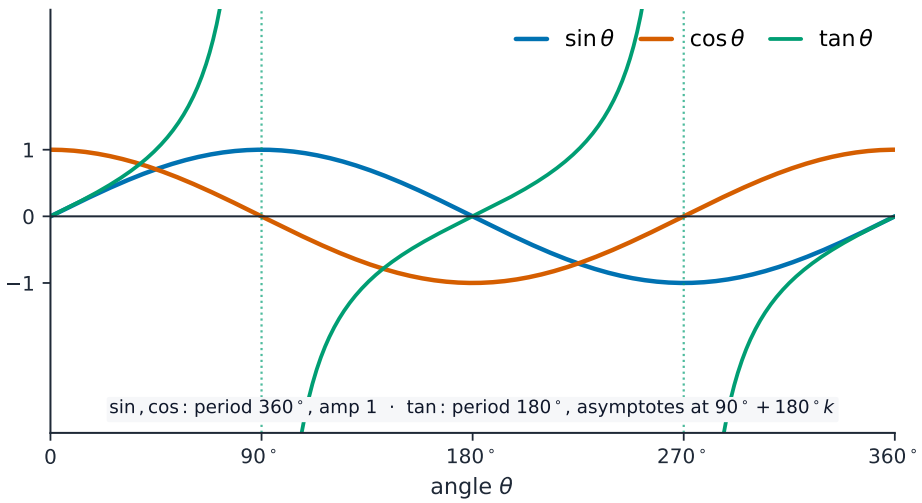
A point on a turning wheel can be modelled with a sine or cosine function.

■ Exact values and signs

On the unit circle a point has coordinates $(\cos\theta, \sin\theta)$. Signs follow the quadrant, including negative angles and turns beyond 360°. The sine and cosine graphs have range $[-1, 1]$ and period 2π radians or 360°. The tangent function repeats after π radians or 180° and is undefined where $\cos\theta = 0$.

θ	30°	45°	60°
$\sin\theta$	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$
$\cos\theta$	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\tan\theta$	$1/\sqrt{3}$	1	$\sqrt{3}$

Use the **reference angle**, the acute angle to the x-axis, to get the size. Then choose the sign. For example, $\cos 150° = -\sqrt{3}/2$, $\sin(-\pi/6) = -1/2$ and $\sin(3\pi/4) = 1/\sqrt{2}$. The notation $\sin^2\theta$ means $(\sin\theta)^2$.

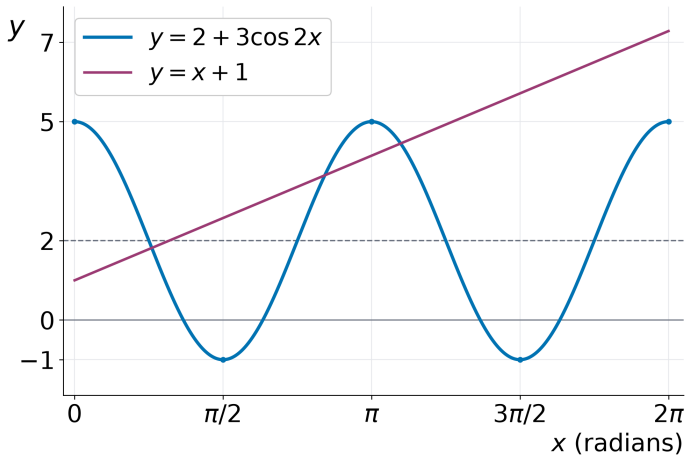


The tangent branches stay separate at their vertical asymptotes.

Transformed graphs

For $y = d + a\sin(bx + c)$ or $d + a\cos(bx + c)$, with $a, b \neq 0$, the midline is $y = d$, **amplitude** is $|a|$, and **period** is $2\pi/|b|$ radians or $360°/|b|$. Find key points by setting the angle inside the function to familiar values.

For $y = 2 + 3\cos 2x$, the range is $[-1, 5]$ and period is π. On $[0, 2\pi]$, maxima occur at $x = 0, \pi, 2\pi$ and minima at $x = \pi/2, 3\pi/2$. There are two complete cycles.



The comparison line $y = x + 1$ meets this curve three times. This counts the solutions of $2 + 3\cos 2x = x + 1$ on the displayed interval.

For $y = \tan(x + \pi/4)$, shift the base tangent graph left by $\pi/4$. In $[-\pi, \pi]$, asymptotes are $x = -3\pi/4, \pi/4$, zeros are $-\pi/4, 3\pi/4$ and the y-intercept is 1. Its period is π and it has no finite amplitude. Draw separate increasing branches; never join through an asymptote.

Principal values and inverse notation

The notations $\sin^{-1} u$, $\cos^{-1} u$ and $\tan^{-1} u$ return a **principal value**, an angle in the stated inverse range:

Inverse	Principal range in radians	In degrees
Sine	$[-\pi/2, \pi/2]$	$[-90°, 90°]$
Cosine	$[0, \pi]$	$[0°, 180°]$
Tangent	$(-\pi/2, \pi/2)$	$(-90°, 90°)$

Sine/cosine inverse require an input in $[-1, 1]$; tangent inverse accepts every real input. The superscript -1 means inverse here, not reciprocal. Thus $\sin^{-1}(\sin 150°) = 30°$, but $\sin\theta = 1/2$ has both $30°$ and $150°$ in $[0°, 360°]$.

To solve $\tan^{-1}(2x - 1) = -\pi/4$, take tangent: $2x - 1 = -1$, so $x = 0$. The given inverse output is inside the principal range.

Prove identities with valid algebra

The two core **identities** are

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}, \qquad \sin^2\theta + \cos^2\theta \equiv 1.$$

An identity holds wherever both sides are defined. State denominator restrictions, and start from one side rather than assuming the equality you must prove. For example, where $\sin\theta \neq 0$:

$$\begin{aligned}\frac{1-\cos\theta}{\sin\theta} &= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} \\ &= \frac{\sin\theta}{1+\cos\theta}.\end{aligned}$$

The condition also excludes $\cos\theta = -1$, so the cancelled factor is non-zero. Do not confuse $\sin^2\theta + \cos^2\theta$ with $(\sin\theta + \cos\theta)^2$.

A trigonometric equation asks for all angles in its stated interval.

Reduce equations to one trig function

Worked example. Solve $2\cos^2\theta - \sin\theta - 1 = 0$ on $[0, 2\pi]$. Substitute $\cos^2\theta = 1 - \sin^2\theta$:

$$2\sin^2\theta + \sin\theta - 1 = 0 \quad \Longrightarrow \quad (2\sin\theta - 1)(\sin\theta + 1) = 0.$$

Therefore $\sin\theta = 1/2$ or -1. The full answer is $\theta = \pi/6, 5\pi/6, 3\pi/2$. The principal value $-\pi/2$ is not in this interval, so use $3\pi/2$ instead. Reject sine/cosine values outside $[-1, 1]$. If solving for $\sin^2\theta$, recover both square-root signs before finding the angles.

Change the inner interval

For $\sin 2x = 1/2$ and $-\pi < x < \pi$, let $u = 2x$. Search the whole interval $-2\pi < u < 2\pi$, not just one turn. Its solutions are $u = -11\pi/6, -7\pi/6, \pi/6, 5\pi/6$. Dividing by 2 gives

$$x = -11\pi/12, -7\pi/12, \pi/12, 5\pi/12.$$

For a shifted angle such as $2x - \pi/6$, transform both interval endpoints first, then convert every inner-angle solution back to x. Check open/closed endpoints in the original equation.

Keep zero factors and exclude undefined angles

Dividing by a trig expression can lose roots. Write $\sin\theta\cos\theta = \sin\theta$ as $\sin\theta(\cos\theta - 1) = 0$, retaining the case $\sin\theta = 0$.

For a fractional equation, first state the denominator restriction. For example, solve $6\sin\theta = 1 + 2/\sin\theta$ for $-180° < \theta < 180°$. Since $\sin\theta \neq 0$, multiplication gives

$$(3\sin\theta - 2)(2\sin\theta + 1) = 0.$$

The value $2/3$ gives $41.8°, 138.2°$ and $-1/2$ gives $-150°, -30°$. All four are valid in the original equation. Keep calculator precision until the final angles.

Independent graph and equation checks

Sketch $y = 1 + \sin x$ and $y = x/2$ together for $0 \le x \le 2\pi$. Label the sine curve's five key points and count solutions of $1 + \sin x = x/2$.

Check after attempting: the curve passes through $(0, 1), (\pi/2, 2), (\pi, 1), (3\pi/2, 0), (2\pi, 1)$. There is one intersection, between $\pi/2$ and π. There the curve falls and the line rises. Before $\pi/2$ the curve is at least 1 and the line below 1; after π the curve is at most 1 and the line above 1.

Also solve $\sin(2x - \pi/6) = -1/2$ on $[-\pi, \pi]$. The inner interval is $[-13\pi/6, 11\pi/6]$, giving $x = -\pi, -\pi/3, 0, 2\pi/3, \pi$. Both endpoints are valid. A correct list of properties

does not replace an actual graph, and one principal inverse value does not replace all equation solutions.

Series

The binomial expansion

For a positive integer n, the **binomial expansion** is

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + b^n,$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$ is a **binomial coefficient** .

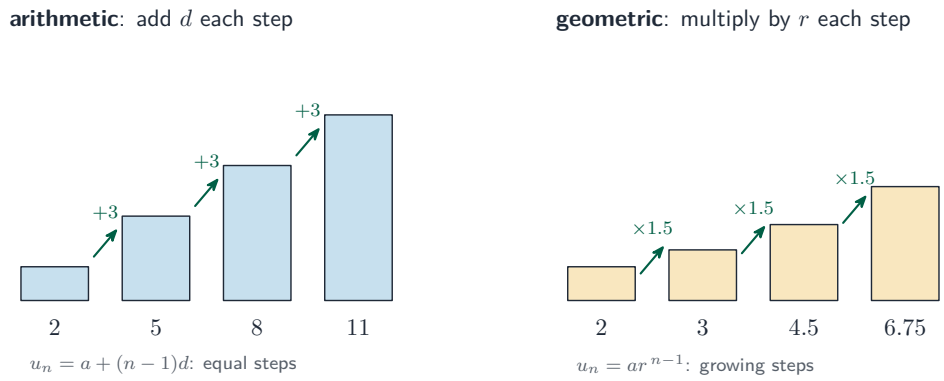
Worked example. Find the first three terms, in ascending powers of x, of $(2-px)^5$.

$$(2-px)^5 = 2^5 + \binom{5}{1}2^4(-px) + \binom{5}{2}2^3(-px)^2 + \cdots = 32 - 80px + 80p^2x^2 + \cdots$$

Arithmetic and geometric progressions

A **progression** (sequence) is a list of terms following a rule.

- An **arithmetic progression** (AP) adds a fixed **common difference** d each step. The nth **term** is $u_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}\big(2a + (n-1)d\big)$.
- A **geometric progression** (GP) multiplies by a fixed **common ratio** r each step. The nth term is $u_n = ar^{n-1}$, and $S_n = \dfrac{a(1-r^n)}{1-r}$.



An AP climbs in equal steps; a GP's steps grow by the same ratio

For a nonzero first term, a geometric series is **convergent** – its partial sums **converge**

(settle to a limit) – exactly when $|r| < 1$. Then it has a **sum to infinity**

$$S_\infty = \frac{a}{1-r}.$$

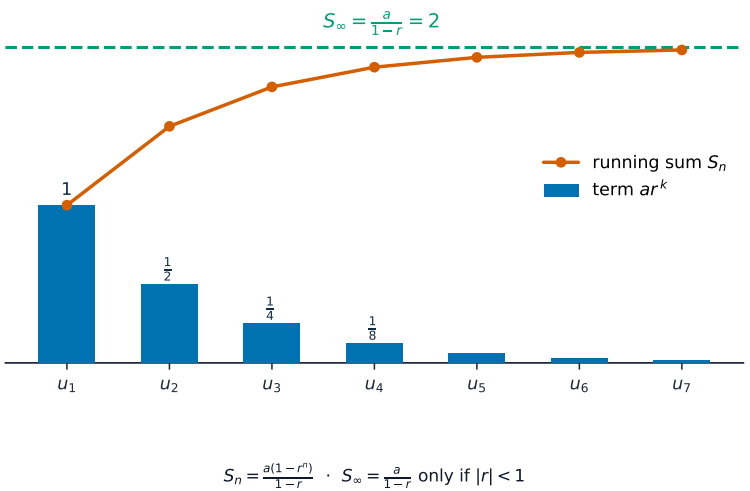
> **Worked example.** The third term of a GP is 18 and the sum of the first three terms is 26. The common ratio is negative. Find the sum to infinity.

From $ar^2 = 18$ you get $a = \dfrac{18}{r^2}$. Put this into $a(1+r+r^2) = 26$:

$$18(1+r+r^2) = 26r^2 \;\Rightarrow\; 8r^2 - 18r - 18 = 0 \;\Rightarrow\; (4r+3)(r-3) = 0.$$

The ratio is negative, so $r = -\frac{3}{4}$ and $a = \dfrac{18}{(3/4)^2} = 32$. Then

$$S_\infty = \frac{32}{1-(-\frac{3}{4})} = \frac{32}{\frac{7}{4}} = \frac{128}{7}.$$



Convergent GP: bars shrink toward zero while the running sum approaches S infinity = a/(1-r)

■ Select a binomial term or a coefficient in a product

The term numbered $k+1$ in $(a+b)^n$ is $\binom{n}{k}a^{n-k}b^k$, with $k = 0, \ldots, n$ and $0! = 1$. The power n is a positive integer here. Keep the sign inside the powered second part.

> **Worked example.** In $(x^2 + 2/x)^6$, the general term is $\binom{6}{k}2^k x^{12-3k}$. The constant term needs $12 - 3k = 0$, so $k = 4$ and the answer is $15 \times 16 = 240$.

For the coefficient of x^2 in $(1+3x)(2-x)^4$, only two pairings contribute: the constant 1 times $24x^2$, and $3x$ times $-32x$. The coefficient is $24 - 96 = -72$. When negative powers occur, list every possible pairing before selecting the terms.

■ Find an unknown AP and select its terms

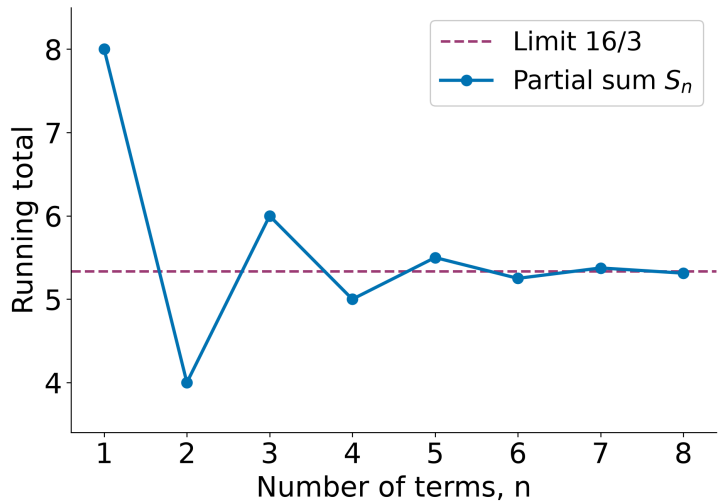
If $u_3 = 13$ and $u_8 = 33$, then $a + 2d = 13$ and $a + 7d = 33$. Subtraction gives $d = 4$ and $a = 5$. Hence $u_{10} = 41$ and $S_{10} = \frac{10}{2}(5+41) = 230$.

For $a = 4, d = 3$, the terms strictly between 20 and 41 satisfy $20 < 3n+1 < 41$. Their positions are 7 through 13: there are $13 - 7 + 1 = 7$ terms, from 22 to 40. The sum is $\frac{7}{2}(22+40) = 217$, also $S_{13} - S_6$. Check whether the endpoints are included in the question.

■ Connect progressions and check convergence

Three consecutive AP terms p, q, s satisfy $2q = p + s$. Three consecutive GP terms satisfy $q^2 = ps$. If p, q, s are GP terms while $p, q, -2s$ are AP terms, substitute $q = pr, s = pr^2$ into $2q = p - 2s$. For $p \neq 0$, this gives $2r^2 + 2r - 1 = 0$. Solve, then apply any sign or convergence restrictions.

For a finite GP with $r = 1$, use $S_n = na$; the fraction formula has a zero denominator. For a non-zero first term, an infinite geometric series has a finite sum exactly when $|r| < 1$. A negative ratio can converge: signs alternate and the partial sums approach a limit from both sides. For $r = -1$ they keep oscillating instead.



For $a = 8, r = -1/2$, the limiting sum is 16/3.

After the first n terms are removed, the remaining sum is

$$S_\infty - S_n = \frac{ar^n}{1-r}.$$

> **Worked example.** A second term of -12 and infinite sum 16 give $ar = -12$ and $a = 16(1-r)$. Thus $4r^2 - 4r - 3 = 0$. Reject $r = 3/2$ because it cannot converge, leaving $r = -1/2, a = 24$. After three terms the remaining infinite sum is $24(-1/2)^3/(3/2) = -2$. A negative tail is valid: the finite total was above the limiting sum.

Differentiation

Differentiation finds the **gradient of a curve** at each point. The gradient is the **limit** of the gradients of shorter and shorter **chords**, called the **derivative** .

■ The rules

Write the derivative as $f'(x)$ or $\dfrac{dy}{dx}$. The basic rule is

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad \text{for any rational } n.$$

Differentiate sums term by term, and use the **chain rule** for a function inside a function:

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

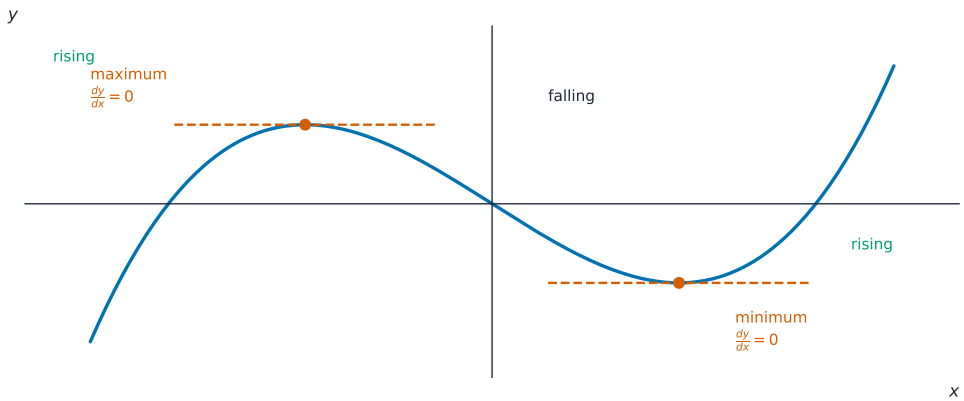
Differentiating again gives the **second derivative** $f''(x)$ or $\dfrac{d^2y}{dx^2}$.

■ Using the derivative

- **Tangent and normal** . The gradient of the curve at a point is the gradient of the **tangent** there. The **normal** is perpendicular to the tangent, so its gradient is $-\dfrac{1}{(\text{tangent gradient})}$ when the tangent gradient is non-zero. A horizontal tangent has a vertical normal.
- **Increasing or decreasing.** The function is an **increasing function** where $\dfrac{dy}{dx} > 0$, and a **decreasing function** where $\dfrac{dy}{dx} < 0$.
- **Rate of change** . A derivative is a **rate of change**. Linked rates use the chain rule, e.g. $\dfrac{dy}{dt} = \dfrac{dy}{dx} \cdot \dfrac{dx}{dt}$.

■ Stationary points

A **stationary point** is where $\dfrac{dy}{dx} = 0$. Test its nature with the second derivative: $f''(x) > 0$ gives a **minimum point** , and $f''(x) < 0$ gives a **maximum point** . When $f''(x) = 0$ the test is inconclusive. Check the sign of the first derivative on both sides: negative to positive proves a minimum, and positive to negative proves a maximum. For example, $y = x^4$ has a minimum at zero despite both derivatives being zero there.



stationary points: $\frac{dy}{dx} = 0$ · use $\frac{d^2y}{dx^2}$ or first-derivative sign to classify

At a maximum or a minimum the tangent is flat, so $\frac{dy}{dx} = 0$.

> **Worked example.** The curve $y = 4x^{1/2} - x$ has a maximum point at $x = a$. Find a.
>
> $$\frac{dy}{dx} = 2x^{-1/2} - 1 = 0 \;\Rightarrow\; \frac{2}{\sqrt{x}} = 1 \;\Rightarrow\; \sqrt{x} = 2 \;\Rightarrow\; x = 4.$$
>
> So $a = 4$.

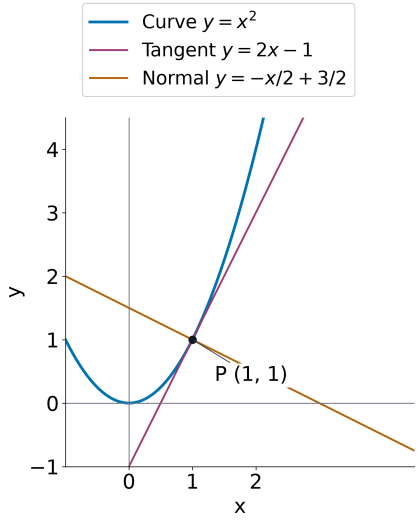
■ Chords, rational powers and a nested root

For $y = x^3$, the gradient of the chord from $x = 2$ to $x = 2 + h$ is $[(2+h)^3 - 8]/h = 12 + 6h + h^2$ for $h \neq 0$. As $h \to 0$, this tends to 12, the tangent gradient. This is an informal explanation of the derivative; do not substitute zero into the original quotient.

For $x > 0$, $f = 3\sqrt{x} + 2/x$ gives $f' = \frac{3}{2}x^{-1/2} - 2x^{-2}$ and $f'' = -\frac{3}{4}x^{-3/2} + 4x^{-3}$. For a nested function $y = \sqrt{2x^3 + 5}$, use the chain rule to obtain $dy/dx = 3x^2/\sqrt{2x^3+5}$ where the root's argument is positive.

Write the tangent and the normal

Find the point as well as the gradient. On $y = x^2$ at $(1, 1)$, $dy/dx = 2$, giving tangent $y - 1 = 2(x - 1)$ and normal $y - 1 = -\frac{1}{2}(x - 1)$. If a point (p, q) has gradient zero, its tangent is $y = q$ and its normal is $x = p$.



The normal is perpendicular to the tangent. Equal axis scales preserve the visible right angle.

From stationary points to a sketch

For $f = x^3 - 6x^2 + 9x + 2$, $f' = 3(x - 1)(x - 3)$, so stationary points have $x = 1, 3$. Substitute into the original function: $(1, 6)$ is a maximum because $f''(1) = -6$; $(3, 2)$ is a minimum because $f''(3) = 6$. The derivative is positive for $x < 1$ and $x > 3$, and negative for $1 < x < 3$. A sketch also needs intercepts and any domain restrictions.

Connected rates and optimisation

If $A = \pi r^2$, then $dA/dt = (dA/dr)(dr/dt) = 2\pi r\, dr/dt$. At $r = 5$ cm and $dr/dt = 0.3$ cm/s, $dA/dt = 3\pi$ cm^2/s. Start with a negative rate for a decreasing quantity. Derivatives with respect to a length and with respect to time are different rates.

For a rectangle beside a wall, 60 m of fence covers two widths x and a length y. The constraint is $2x + y = 60$. Therefore $A = x(60 - 2x)$ with $0 < x < 30$. Solve $A' = 60 - 4x = 0$, giving $x = 15, y = 30$. Since $A'' = -4 < 0$, the maximum area is 450 m^2. Check the physical domain and its boundaries before accepting an optimum.

Integration

Integration is the reverse of differentiation. Reversing the power rule gives

$$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + C \quad (n \neq -1,\ a \neq 0).$$

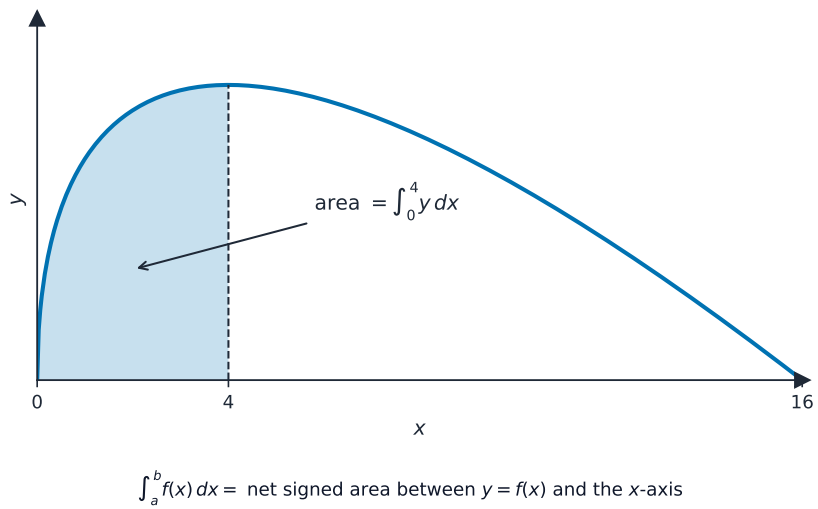
The $+C$ is the **constant of integration**. If you know one point on the curve, substitute it to find C.

Definite integrals and area

A **definite integral** has limits and gives a number:

$$\int_p^q f(x)\, dx = \big[F(x)\big]_p^q = F(q) - F(p).$$

For $p < q$, a definite integral counts contributions above the x-axis positively and those below negatively. It equals geometric area only when the curve is nonnegative throughout. For total area, split at the roots and add the magnitudes of each integral. For the area between two curves, integrate (top curve − bottom curve), splitting wherever their order changes. Reversing the limits reverses the sign of the integral.



The definite integral $\int_0^4 y\, dx$ is the shaded area under the curve.

Worked example. The curve $y = 4x^{1/2} - x$ meets the x-axis again at $x = 16$. Find the area between the curve and the x-axis from $x = 0$ to $x = 4$.

$$\int_0^4 \left(4x^{1/2} - x\right) dx = \left[\frac{8}{3}x^{3/2} - \frac{x^2}{2}\right]_0^4 = \frac{8}{3}(8) - \frac{16}{2} = \frac{64}{3} - 8 = \frac{40}{3}.$$

An **improper integral** has an infinite limit or an endpoint where the integrand is undefined; evaluate it as a **limit**. For example,

$$\int_1^{\infty} x^{-2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(1-\frac{1}{b}\right) = 1,$$

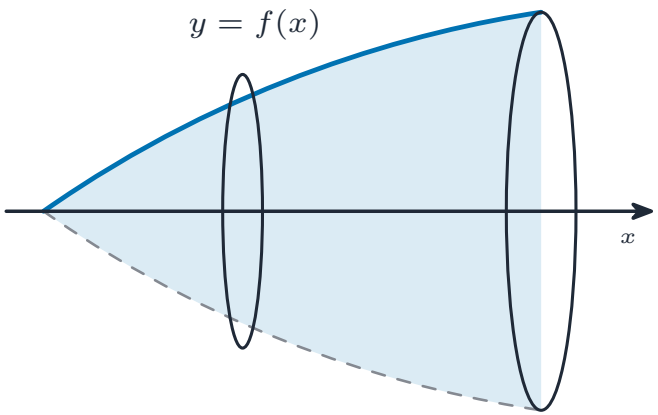
and $\displaystyle\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0^+}\left[2x^{1/2}\right]_a^1 = 2$, which is finite even though the integrand blows up at $x = 0$.

■ Volume of revolution

When a region is turned all the way around an axis it sweeps out a solid. The **volume of revolution** about the x-axis is

$$V = \pi\int_p^q y^2\,dx,$$

and about the y-axis it is $V = \pi\displaystyle\int x^2\,dy$. For example, the region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$, turned about the x-axis, has volume $\pi\displaystyle\int_0^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_0^4 = 8\pi$.



rotate the region around the x-axis to sweep a solid

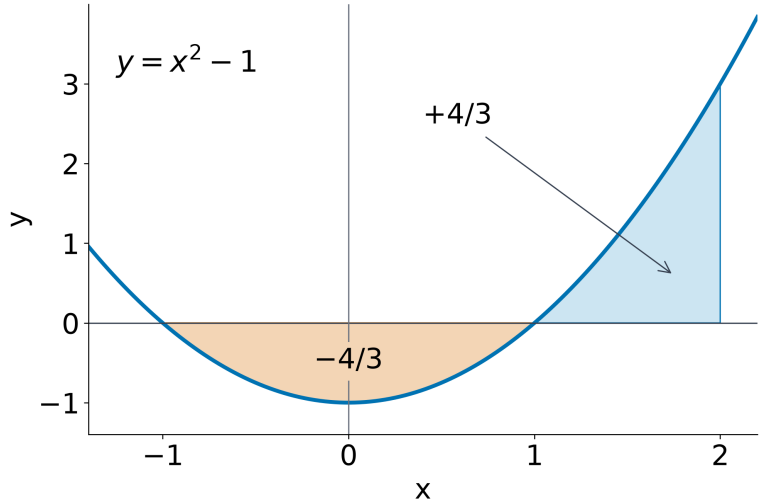
Rotating the region under a curve around the x-axis sweeps out a solid; each thin slice is a disc of area πy^2, so $V = \pi\int y^2\,dx$

■ A curve from two conditions

If $y'' = 6x$ and $(1, 4)$ is a stationary point, integrate once to $y' = 3x^2 + A$. The gradient is zero there, so $A = -3$. Integrate again to $y = x^3 - 3x + B$; using the point gives $B = 6$. Two integrations need two separately determined constants. Check the result by differentiating.

■ Signed total and geometric area

For $y = x^2 - 1$ from $x = -1$ to 2, the signed integral is zero. Split at $x = 1$: the contribution on $[-1, 1]$ is $-4/3$, and the contribution on $[1, 2]$ is $4/3$. The geometric area is $4/3 + 4/3 = 8/3$ square units.



The integral can vanish even when the total geometric area is positive.

An improper integral is finite only if its limiting value exists as a real number. For example, $\int_0^1 x^{-2}dx = \lim_{a\to 0^+}(1/a - 1)$ has no finite value. The infinity symbol is a bound described by a limit, not a number to substitute.

■ A gap between curves or a tangent and a curve

The curves $y = 2x$ and $y = x^2$ intersect at $x = 0, 2$. The line is above the parabola between them, so the enclosed area is $\int_0^2 (2x - x^2)dx = 4/3$ square units. If the upper and lower curves exchange places, split the integral at that intersection.

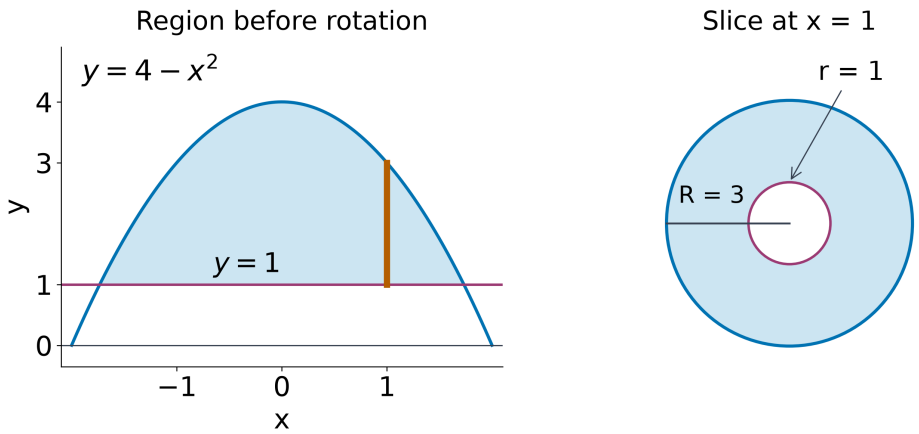
For $y = 2x - x^2$, the tangent at $(1, 1)$ is $y = 1$. On $0 \le x \le 1$, the area under the curve is $2/3$ and the rectangle beneath the tangent has area 1. The gap is $1 - 2/3 = 1/3$. Find a tangent's equation and relevant intercepts before selecting the bounds in a compound area question.

■ Rotation with a hole and rotation about either axis

A region need not touch the axis of rotation. A slice then has outer radius R and inner radius r; its cross-sectional area is $\pi(R^2 - r^2)$, not $\pi(R - r)^2$.

Between $y = 4 - x^2$ and $y = 1$, intersections occur at $x = \pm\sqrt{3}$. Rotating about the x-axis gives

$$V = \pi \int_{-\sqrt{3}}^{\sqrt{3}} [(4 - x^2)^2 - 1]dx = \frac{88\pi\sqrt{3}}{5}.$$



At $x = 1$ the outer radius is 3 and the inner radius is 1.

About the y-axis, use horizontal slices and express radii in terms of y: $V = \pi \int (R(y)^2 - r(y)^2)dy$. The region $0 \le x \le \sqrt{y}$, $0 \le y \le 4$, produces $V = \pi \int_0^4 y\,dy = 8\pi$. Bounds must be y values when integrating with respect to y. Include any inner radius, and state cubic units.

Exam tips

- Show **every line of algebra** – method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc $= r\theta$, sector area $= \frac{1}{2}r^2\theta$).
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.