2.6 Numerical solution of equations

Name: _____ Class: _____ Date: _____ **Total: 12 marks**

KEY WORDS iterative formula • sign change • iteration • root • converge

Objective

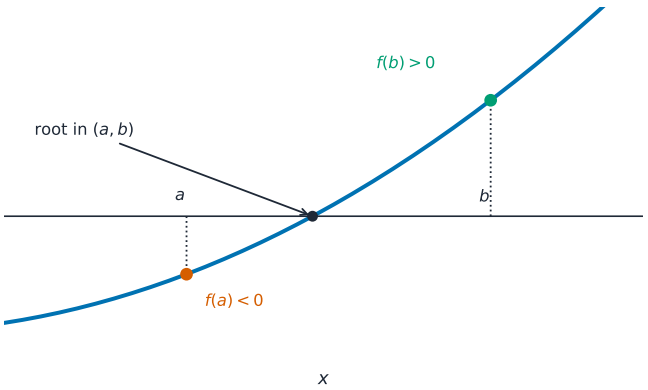
Build the skills to answer exam questions on the **numerical solution of equations** – locating a **root** by a **sign change**, and finding it by **iteration**.

You must be able to:

- use a sign change to show a root lies in an interval
- carry out an iteration $x_{n+1} = F(x_n)$ until it converges
- show a given equation rearranges into a stated iterative form

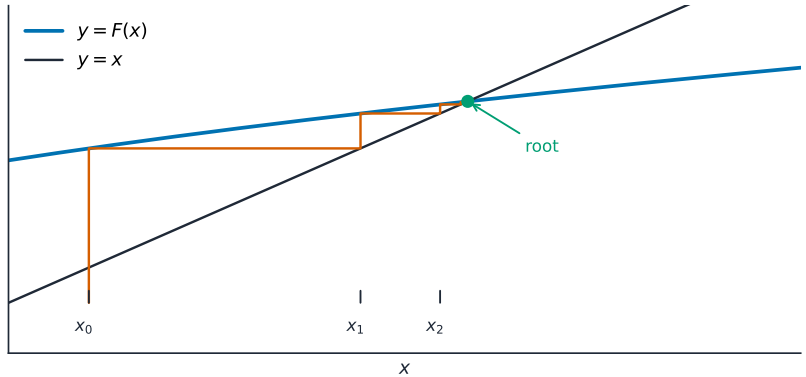
1 Key knowledge and worked examples

■ Sign change & iteration



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

If $f(a)$ and $f(b)$ have opposite signs, a root is trapped between a and b



$x_{n+1} = g(x_n)$ · converges near root if $|g'(x)| < 1$

Each step: up to $y = F(x)$, across to $y = x$; the staircase converges to the root

Iterate $x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, … $\to \beta = -1.26$ (3 s.f.).

■ Key knowledge: A valid sign-change argument

Opposite signs guarantee a root only when the function is continuous on the whole interval. A discontinuity can also create a sign change.

■ Worked example: A valid sign-change argument

For $f(x) = x^3 + x - 3$, $f(1) = -1$ and $f(2) = 7$. A polynomial is continuous, so a root lies in $(1, 2)$. In contrast, $g(x) = 1/x$ changes sign between -1 and 1 but has no root: it is undefined at 0.

2 Practice

2.1 $f(x) = x^3 - 5x + 1$. Evaluate $f(0)$ and $f(1)$, and state what they tell you. [2]

2.2 State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$. [1]

3 Exam-style questions

3.1 A root of $f(x) = 0$ lies between $x = 2$ and $x = 3$ if: [1]

- ☐ **A** $f(2)$ and $f(3)$ are both positive
- **B** $f(2)$ and $f(3)$ have opposite signs
- **C** $f(2) = f(3)$
- **D** $f'(2) = 0$

3.2 The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1.

(a) Verify this by a sign change. [2]

(b) Show that the equation can be written as $x = \dfrac{x^3 + 2}{6}$, and use the iteration $x_{n+1} = \dfrac{x_n^3 + 2}{6}$ with $x_0 = 0.3$ to find α to 2 d.p. [4]

3.3 Explain why a sign change between a and b does **not** guarantee exactly one root in that interval. [2]