

2.5 Integration (standard forms, trapezium rule)

Name: _____ Class: _____ Date: _____

Total: 34 marks**KEY WORDS** trapezium rule • integration • modulus

Objective

Build the skills to answer exam questions on **integration** – integrating e^{ax+b} , $\frac{1}{ax+b}$ and trig functions, powers of $\sin/\cos$ via identities, and the **trapezium rule**.

You must be able to:

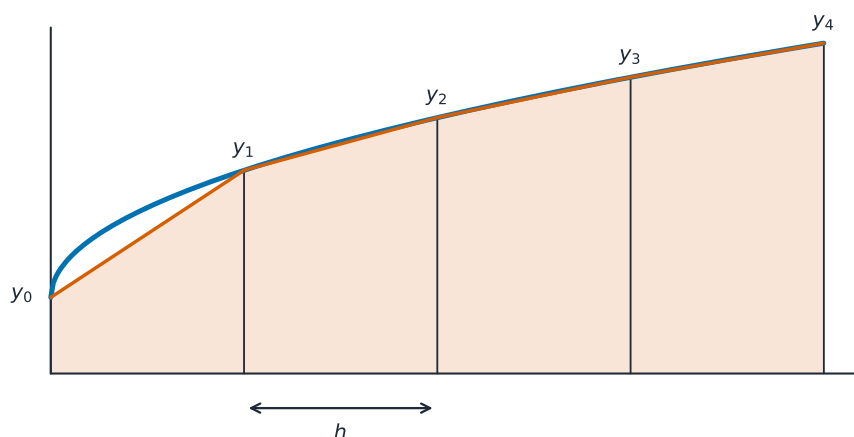
- integrate e^{ax+b} , $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$
- integrate $\sin^2/\cos^2$ using a double-angle identity
- estimate a definite integral with the trapezium rule

1 Key knowledge and worked examples

■ Powers of $\sin$ & the trapezium rule

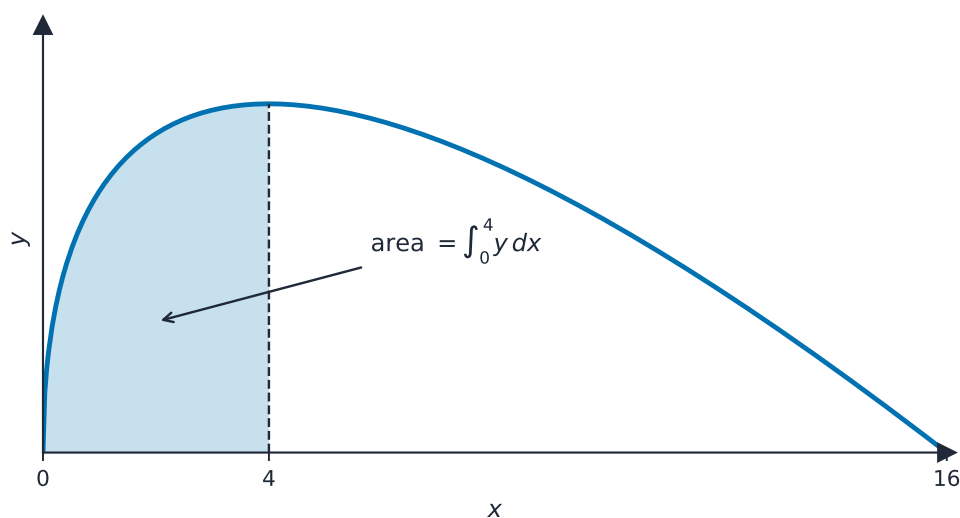
$\int 6 \sin^2 x \, dx$ – use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$:

$$\int (3 - 3 \cos 2x) \, dx = 3x - \frac{3}{2} \sin 2x + C.$$



$$\int_a^b y \, dx \approx \frac{h}{2}(y_0 + 2(y_1 + \dots + y_{n-1}) + y_n)$$

Each strip is a trapezium; their areas estimate

$$\int_a^b y \, dx \approx \frac{h}{2}[y_0 + y_n + 2(y_1 + \dots + y_{n-1})]$$


$$\int_a^b f(x) \, dx = \text{net signed area between } y = f(x) \text{ and the } x\text{-axis}$$

The trapezium rule estimates this area under the curve

■ Choosing the method, and the standard results

Look at the integrand and pick the first method that fits:

1. **A standard form**, with a **linear** function of x inside. Integrate as usual and **divide by the coefficient of x** :

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C \quad \int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln |ax+b| + C$$

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + C \quad \int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + C$$

2. **A fraction whose numerator is the derivative of its denominator** $\rightarrow \ln$:

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

3. **A substitution**, when a function and (a multiple of) its derivative both appear. Change the **limits** too if the integral is definite, and there is then no need to substitute back.
4. **Integration by parts**, for a product: $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$. Choose u as the factor that gets **simpler** when differentiated – a power of x , or $\ln x$.
5. **Partial fractions**, when the integrand is a proper algebraic fraction with a factorisable denominator.
6. **A trigonometric identity first**, when a power of a trig function appears: $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ turns an unintegrable square into two standard forms.

Worked example (parts, and why the choice of u matters). Find $\int x e^{2x} dx$.

1. Take $u = x$ (it differentiates to 1) and $\frac{dv}{dx} = e^{2x}$, so $v = \frac{1}{2}e^{2x}$.
2. $\int x e^{2x} dx = \frac{1}{2}x e^{2x} - \int \frac{1}{2}e^{2x} dx$.
3. $= \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + C$.

Choosing $u = e^{2x}$ instead gives an integral **harder** than the one you started with – that is the check that you picked the wrong way round.

$\triangle$ **Every indefinite integral needs $+C$** , and every $\ln$ needs the **modulus**. Both are marks, and both are dropped more often than any actual method.

■ Key knowledge: A trapezium-rule estimate

With n equal strips, $h = (b - a)/n$. The end ordinates appear once and every interior ordinate appears twice.

■ Worked example: A trapezium-rule estimate

Estimate $\int_0^2 x^2 dx$ with two strips. The ordinates are 0, 1, 4 and $h = 1$. The estimate is $A \approx \frac{h}{2}[y_0 + y_2 + 2y_1] = \frac{1}{2}[0 + 4 + 2(1)] = 3$. The exact integral is $8/3$; the convex curve makes this trapezium estimate too large.

2 Practice

2.1 Find $\int e^{2x+1} dx$. [2]

2.2 Find $\int \frac{1}{2x+3} dx$. [2]

2.3 Find $\int \frac{6x}{3x^2+1} dx$. [2]

2.4 Find $\int \sin(3x-1) dx$. [2]

2.5 Use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to find $\int \cos^2 x dx$. [3]

3 Exam-style questions

3.1 $\int \cos 4x dx$ equals: [1]

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A $4 \sin 4x + C$

B $\frac{1}{4} \sin 4x + C$

C $-\frac{1}{4} \sin 4x + C$

D $-4 \sin 4x + C$

3.2 Find $\int_0^{\pi/4} 4 \cos^2 x \, dx$, giving an exact answer. [4]

3.3 Use the trapezium rule with **four** strips to estimate $\int_0^2 \sqrt{1+x^3} \, dx$, giving your answer to 3 s.f. [4]

3.4 (a) Find $\int x \ln x \, dx$, stating which factor you take as u and why. [4]

(b) Use the substitution $u = 1 + x^2$ to evaluate $\int_0^2 \frac{x}{\sqrt{1+x^2}} dx$, giving your answer in exact form. [5]

(c) Express $\frac{5x+1}{(x+2)(x-1)}$ in partial fractions, and hence find $\int \frac{5x+1}{(x+2)(x-1)} dx$. [5]