6.3 Continuous random variables

Name: _____ Class: _____ Date: _____ **Total: 17 marks**

KEY WORDS probability density function • continuous random variable • percentiles • median • cumulative distribution function

Objective

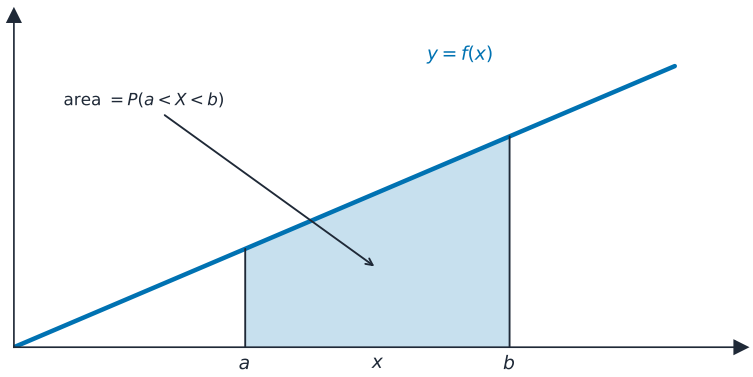
Build the skills to answer exam questions on **continuous random variables** – the **probability density function** $f(x)$, finding probabilities and $E(X)$, and the **cumulative distribution function** $F(x)$.

You must be able to:

- use $\int f(x)\,dx = 1$ to find a constant; find $P(a < X < b) = \int_a^b f\,dx$
- find $E(X) = \int x f(x)\,dx$ (and $\mathrm{Var}(X)$)
- find $F(x)$ and use it for the median/percentiles

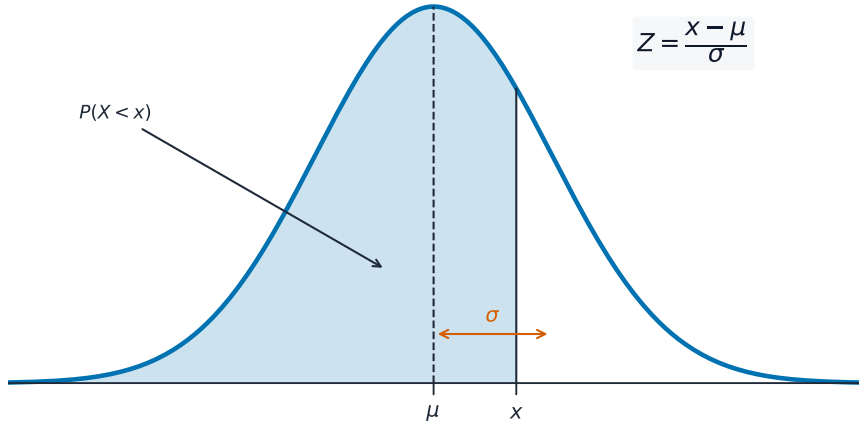
1 Key knowledge and worked examples

Density and expectation



$\int_{-\infty}^{\infty} f = 1 \quad \cdot \quad P(a < X < b) = \int_a^b f$

$P(a < X < b)$ is the area under $f(x)$ between a and b; the total area is 1



The normal distribution is a continuous random variable

$f(x) = \frac{1}{2}x$ **for** $0 \leqslant x \leqslant 2$**:** $E(X) = \displaystyle\int_0^2 x \cdot \tfrac{1}{2}x\,dx = \left[\tfrac{x^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$.

Median m**:** solve $F(m) = 0.5$.

Key knowledge: Normalising a density

A density must be non-negative and have total integral one. Probabilities are areas, not density heights.

Worked example: Normalising a density

Let $f(x) = kx$ for $0 \le x \le 2$, and zero elsewhere. Normalisation gives $\int_0^2 kx\,dx = 2k = 1$, so $k = 1/2$. Then $P(1 < X < 2) = \int_1^2 x/2\,dx = 3/4$. Endpoints have zero individual probability in this continuous model.

2 Practice

2.1 $f(x) = kx$ for $0 \leqslant x \leqslant 2$. Show that $k = \frac{1}{2}$. [2]

2.2 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

3 Exam-style questions

3.1 For a valid pdf, $\int_{-\infty}^{\infty} f(x)\,dx$ must equal: [1]

- **A** 0
- **B** 0.5
- **C** 1
- **D** ∞

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1-x^2)$ for $-1 \leqslant x \leqslant 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^{1} f(x)\,dx = 1$. [3]

(b) Find $P(X > 0.5)$. [3]

3.3 A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leqslant x \leqslant 2$.

(a) Find $E(X)$. [3]

(b) Find the median m. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\int_0^2 kx\,dx = \left[\frac{kx^2}{2}\right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}$

2.2

- $\int_0^1 \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$

3.1 C

- The total area under a pdf is 1

3.2

(a)

- $\int_{-1}^1 \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^1 = \frac{3}{4}\left(\frac{2}{3} - (-\frac{2}{3})\right) = \frac{3}{4}\cdot\frac{4}{3} = 1$

(b)

- $\int_{0.5}^1 \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{0.5}^1 = \frac{3}{4}\left(\frac{2}{3} - (0.5 - 0.0417)\right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156$

3.3

(a)

- $E(X) = \int_0^2 x \cdot \frac{3}{8}x^2\,dx = \frac{3}{8}\left[\frac{x^4}{4}\right]_0^2 = \frac{3}{8}\cdot 4 = \frac{3}{2}$

(b)

- $F(m) = \int_0^m \frac{3}{8}x^2\,dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59$