

6.2 Linear combinations of random variables

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS expectation • variance

Objective

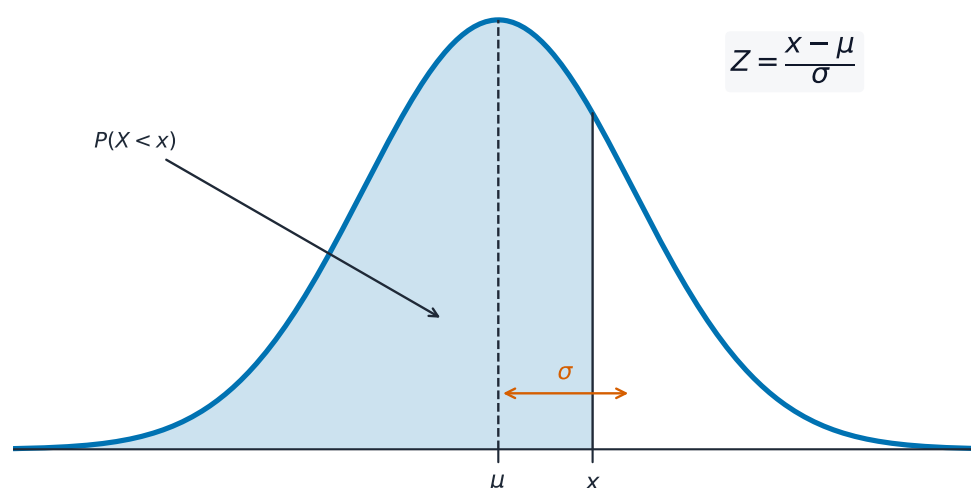
Build the skills to answer exam questions on **linear combinations of random variables** – the expectation and variance rules, and combining independent normal (or Poisson) variables.

You must be able to:

- use $E(aX + b) = aE(X) + b$, $\text{Var}(aX + b) = a^2\text{Var}(X)$
- use $E(aX + bY) = aE(X) + bE(Y)$, $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$
- recognise that a linear combination of independent normals is normal

1 Key knowledge and worked examples

■ Mean and variance rules



If X is normal then so is $aX + b$; variances add for independent variables (means always add)

X has mean 5, variance 4: $E(3X - 1) = 3(5) - 1 = 14$; $\text{Var}(3X - 1) = 3^2(4) = 36$.

Watch the signs: $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y)$ (variances **add**).

■ **Key knowledge: Subtracting independent variables**

For independent variables, variances add even when the variables are subtracted. Coefficients are squared in a variance.

■ **Worked example: Subtracting independent variables**

If $E(X) = 10$, $E(Y) = 4$, $\text{Var}(X) = 9$ and $\text{Var}(Y) = 1$, for independent X, Y , let $W = 2X - 3Y$. Then $E(W) = 2(10) - 3(4) = 8$ and $\text{Var}(W) = 2^2(9) + (-3)^2(1) = 45$. Subtracting the second variance would be incorrect.

2 Practice

2.1 X has mean 10 and variance 9. Find $E(2X + 5)$. [2]

2.2 For independent X, Y with $\text{Var}(X) = 4$, $\text{Var}(Y) = 9$, find $\text{Var}(X + Y)$. [1]

3 Exam-style questions

3.1 X has variance 5. Then $\text{Var}(4X)$ is: [1]

- ☐ A 20
☐ B 80
☐ C 9
☐ D 5

3.2 Independent variables have $X \sim N(20, 3^2)$ and $Y \sim N(15, 4^2)$.

(a) Find the distribution of $X - Y$. [3]

(b) Find $P(X - Y > 2)$. [3]

3.3 The mass of a tin is $X \sim N(500, 4^2)$ g. Four tins are packed in a box of mass $N(100, 2^2)$ g, all independent. Find the mean and standard deviation of the total mass of a packed box. [4]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $E(2X + 5) = 2(10) + 5 = 25$

2.2

- $\text{Var}(X + Y) = 4 + 9 = 13$

3.1 B

- $\text{Var}(4X) = 4^2 \times 5 = 80$

3.2

(a)

- mean = $20 - 15 = 5$; variance = $3^2 + 4^2 = 25$; so $X - Y \sim N(5, 25)$

(b)

- $Z = \frac{2 - 5}{5} = -0.6$; $P(X - Y > 2) = P(Z > -0.6) = \Phi(0.6) = 0.7257$

3.3

- total $T = X_1 + X_2 + X_3 + X_4 + B$; mean = $4(500) + 100 = 2100$ g; variance = $4(4^2) + 2^2 = 64 + 4 = 68$, so SD = $\sqrt{68} = 8.25$ g