

3.8 Differential equations

Name: _____ Class: _____ Date: _____

Total: 15 marks

KEY WORDS general solution • initial condition • particular solution
 • differential equations • separable

Objective

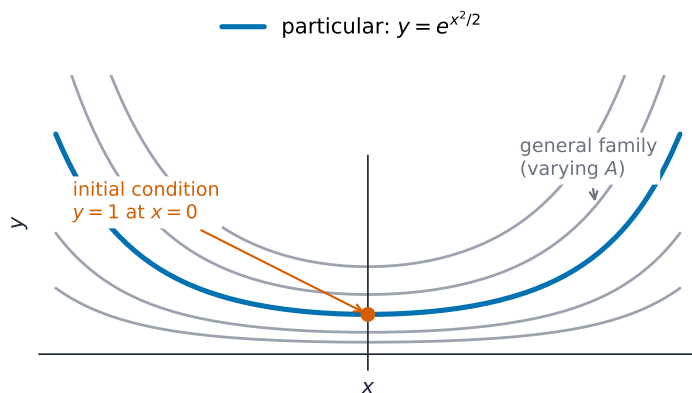
Build the skills to answer exam questions on **differential equations** – solving a **separable** first-order equation, finding the **general** and **particular solution**, and setting up a model.

You must be able to:

- separate variables and integrate both sides
- use an initial condition to find the constant
- form and solve a differential equation from a rate description

1 Key knowledge and worked examples

■ Separate and solve



The constant gives a family of curves; an initial condition picks one particular solution

Solve $\frac{dy}{dx} = xy$, $y = 1$ at $x = 0$.

$$\int \frac{1}{y} dy = \int x dx \Rightarrow \ln y = \frac{1}{2}x^2 + c \Rightarrow y = Ae^{x^2/2}; \quad y(0) = 1 \Rightarrow A = 1, \text{ so } y = e^{x^2/2}.$$

■ Key knowledge: An initial condition after separation

Integrate both sides, include a constant, then use the initial condition. An initial condition selects one curve from the solution family.

■ Worked example: An initial condition after separation

Solve $dy/dx = 2xy$ with $y(0) = 3$. Separating gives $\int dy/y = \int 2x dx$, hence $\ln |y| = x^2 + C$. The positive initial value selects $y = Ae^{x^2}$ and $A = 3$. Thus $y = 3e^{x^2}$; differentiating gives $y' = 6xe^{x^2} = 2xy$.

2 Practice

2.1 Separate the variables for $\frac{dy}{dx} = \frac{y}{x}$ (write each side ready to integrate). [1]

2.2 Solve $\frac{dy}{dx} = 3y$, giving the general solution. [2]

3 Exam-style questions

3.1 The general solution of $\frac{dy}{dx} = ky$ is: [1]

- ☐ A $y = kx + c$
☐ B $y = Ae^{kx}$
☐ C $y = Ax^k$
☐ D $y = \frac{1}{2}kx^2 + c$

3.2 Solve $\frac{dy}{dx} = 2x(y + 1)$, given $y = 0$ when $x = 0$. Give y explicitly. [5]

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\frac{dh}{dt} = -k\sqrt{h}$. Initially $h = H$.

(a) Solve to show $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$. [4]

(b) Find, in terms of H and k , the time for the tank to empty. [2]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\frac{1}{y} dy = \frac{1}{x} dx$

2.2

- $\ln y = 3x + c \Rightarrow y = Ae^{3x}$

3.1 B

- $\frac{dy}{dx} = ky$ gives $y = Ae^{kx}$

3.2

- $\int \frac{1}{y+1} dy = \int 2x dx; \ln(y+1) = x^2 + c; y(0) = 0 \Rightarrow c = 0; y+1 = e^{x^2}; y = e^{x^2} - 1$

3.3

(a)

- $\int h^{-1/2} dh = \int -k dt; 2\sqrt{h} = -kt + c; h = H \text{ at } t = 0 \Rightarrow c = 2\sqrt{H}; \sqrt{h} = \sqrt{H} - \frac{1}{2}kt$

(b)

- empty when $h = 0$: $0 = \sqrt{H} - \frac{1}{2}kt \Rightarrow t = \frac{2\sqrt{H}}{k}$