3.6 Numerical solution of equations

Name: _____ Class: _____ Date: _____ **Total: 13 marks**

KEY WORDS iteration • converges • sign change • numerical solution of equations

Objective

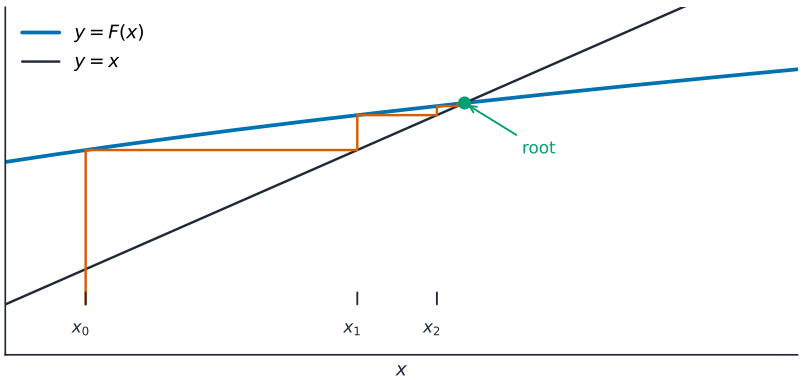
Build the skills to answer exam questions on the **numerical solution of equations** in Pure 3 – locating a root by a **sign change** and finding it by **iteration**, often after some algebra/calculus.

You must be able to:

- show a root lies in an interval by a sign change
- show an equation rearranges into a given iterative form
- iterate $x_{n+1} = F(x_n)$ to the required accuracy

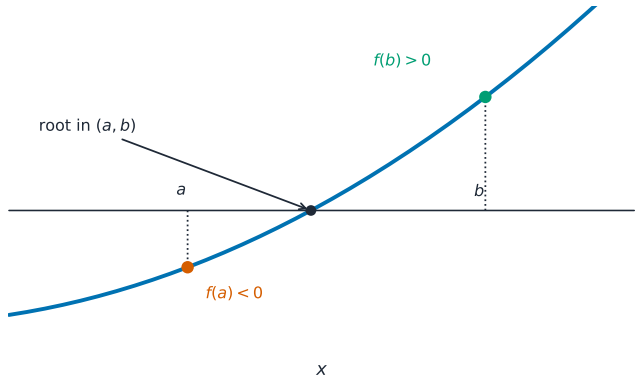
1 Key knowledge and worked examples

■ Iteration to a root



$x_{n+1} = g(x_n)$ · converges near root if $|g'(x)| < 1$

Each step: up to $y = F(x)$, across to $y = x$ – the iteration converges to the root



if f continuous on $[a, b]$ and $f(a)f(b) < 0$, then a root lies in (a, b)

Locate the root first by a sign change

$x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, … $\to$ root ≈ -1.26 (3 s.f.).

■ Key knowledge: Checking an iteration before rounding

Keep guard digits in the iteration. Two successive rounded values are evidence of numerical stability, not a proof that every starting value converges.

■ Worked example: Checking an iteration before rounding

Use $x_{n+1} = (x_n + 2/x_n)/2$ with $x_0 = 1$ to solve $x^2 = 2$. The iterates are $x_1 = 1.5$, $x_2 = 1.4166667$, $x_3 = 1.4142157$ and $x_4 = 1.4142136$. Both final values round to 1.414 at three decimal places. Substitution gives a small residual $x_4^2 - 2$.

2 Practice

2.1 $f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$. [2]

2.2 Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration. [1]

3 Exam-style questions

3.1 An iteration $x_{n+1} = F(x_n)$ from a suitable start **converges** to a root if the values: [1]

- ☐ **A** grow without limit
- **B** settle towards a fixed number
- **C** alternate between two values forever
- **D** stay equal to x_0

3.2 The equation $x^3 = 4x + 1$ has a root α near $x = 2$.

(a) Verify there is a root between $x = 2$ and $x = 2.2$. [2]

(b) Show the equation can be written $x = \sqrt[3]{4x+1}$ and use $x_{n+1} = \sqrt[3]{4x_n + 1}$, $x_0 = 2.1$, to find α to 2 d.p. [4]

3.3 By sketching $y = e^{-x}$ and $y = x$ on the same axes, state the number of roots of $e^{-x} = x$, and give an interval of width 0.5 containing the root. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $f(1) = e - 3 = -0.282$, $f(2) = e^2 - 6 = 1.389$; opposite signs
- a root lies between 1 and 2

2.2

- e.g. $x = \sqrt[3]{5 - x}$

3.1 B

- Convergence means the iterates settle towards a fixed value

3.2

(a)

- $f(x) = x^3 - 4x - 1$: $f(2) = -1 < 0$, $f(2.2) = 10.648 - 8.8 - 1 = 0.848 > 0$; sign change
- root between

(b)

- $x^3 = 4x + 1 \Rightarrow x = \sqrt[3]{4x + 1}$; $x_1 = \sqrt[3]{9.4} = 2.110$, $x_2 = 2.113$, $x_3 = 2.114$; $\alpha = 2.11$

3.3

- the decreasing curve $y = e^{-x}$ meets the line $y = x$ exactly **once**, so there is 1 root; $g(x) = e^{-x} - x$ has $g(0.5) = 0.607 - 0.5 > 0$ and $g(1) = 0.368 - 1 < 0$, so the root lies between 0.5 and 1