3.5 Integration (by parts, substitution, partial fractions)

Name: _____ Class: _____ Date: _____ **Total: 16 marks**

KEY WORDS substitution • integration • integration by parts • partial fractions • rational function

Objective

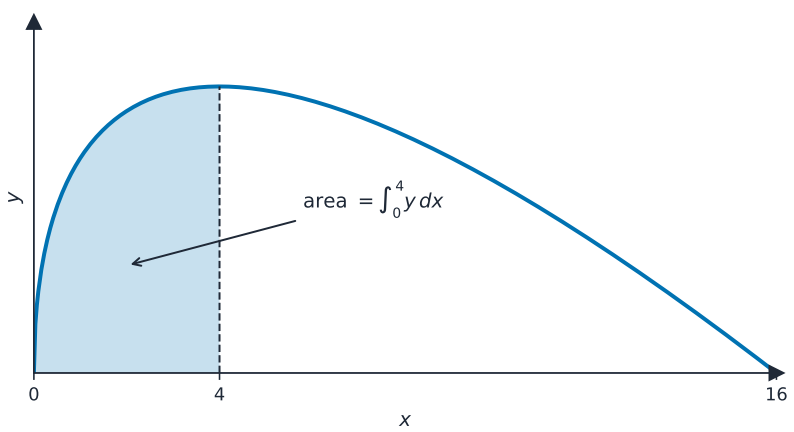
Build the skills to answer exam questions on **integration** in Pure 3 – the standard integral $\dfrac{1}{x^2+a^2}$, **integration by parts**, **substitution**, partial fractions, and the $\dfrac{kf'(x)}{f(x)}$ pattern.

You must be able to:

- recognise $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ and $\int \frac{kf'(x)}{f(x)}\,dx = k\ln|f(x)| + C$
- integrate by parts and by a given substitution
- integrate a rational function via partial fractions

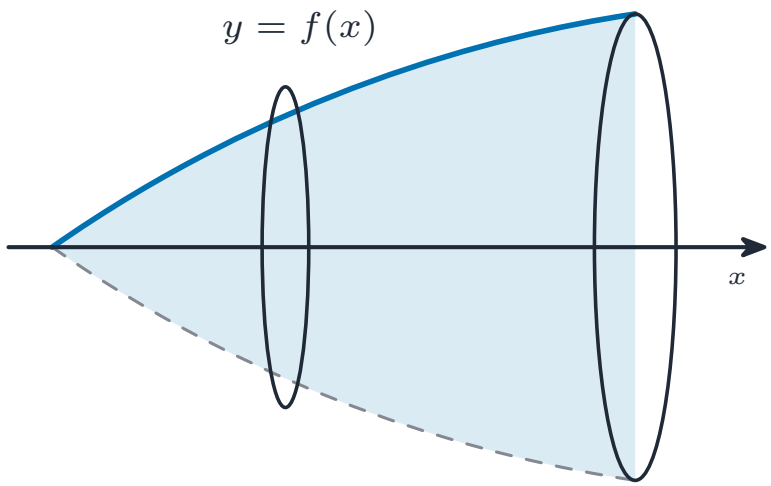
1 Key knowledge and worked examples

■ By parts & the log pattern



$\int_a^b f(x)\,dx =$ net signed area between $y = f(x)$ and the x-axis

A definite integral is still an area; Pure 3 adds the methods to evaluate harder ones



rotate the region around the x-axis to sweep a solid

An application of integration: a volume of revolution

$\int x \cos x \, dx$ – by parts ($u = x$, $dv = \cos x \, dx$): $= x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$.

$\int \frac{2x}{x^2+1} \, dx = \ln(x^2+1) + C$ (top is the derivative of the bottom).

■ **Key knowledge: Definite integration by parts**

Choose the factor that simplifies on differentiation as u. Evaluate the boundary term and the remaining integral with the same limits.

■ **Worked example: Definite integration by parts**

Find $I = \int_0^1 xe^x \, dx$. Take $u = x$ and $dv = e^x dx$, so $du = dx$ and $v = e^x$. Then $I = [xe^x]_0^1 - \int_0^1 e^x dx = e - (e-1) = 1$. A derivative check of $(x-1)e^x$ recovers xe^x.

2 Practice

2.1 Find $\int \frac{1}{x^2+9} \, dx$. [2]

2.2 Find $\displaystyle\int \frac{3x^2}{x^3+1}\,dx$. [2]

3 Exam-style questions

3.1 $\displaystyle\int xe^x\,dx$ equals: [1]

- ☐ **A** $xe^x - e^x + C$
- **B** $xe^x + e^x + C$
- **C** $\frac{1}{2}x^2e^x + C$
- **D** $e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\displaystyle\int_0^2 x\sqrt{1+x^2}\,dx$. [5]

3.3 (a) Express $\dfrac{4}{(x)(x+2)}$ in partial fractions. [3]

(b) Hence find $\displaystyle\int_1^3 \frac{4}{x(x+2)}\,dx$, giving an exact answer. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\frac{1}{3}\tan^{-1}\dfrac{x}{3} + C$

2.2

- $\ln|x^3+1| + C$

3.1 A

- By parts: $xe^x - \int e^x\,dx = xe^x - e^x + C$

3.2

- $u = 1 + x^2 \Rightarrow du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$; limits $x = 0 \to u = 1$, $x = 2 \to u = 5$; $\displaystyle\int_1^5 \tfrac{1}{2}\sqrt{u}\,du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^5 = \tfrac{1}{3}(5^{3/2} - 1); = \tfrac{1}{3}(11.180 - 1) = 3.39$

3.3

(a)

- $4 = A(x+2) + Bx$; $x = 0 \Rightarrow A = 2$; $x = -2 \Rightarrow B = -2$; $\dfrac{2}{x} - \dfrac{2}{x+2}$

(b)

- $\displaystyle\int_1^3 \left(\frac{2}{x} - \frac{2}{x+2}\right) dx = \Big[2\ln x - 2\ln(x+2)\Big]_1^3 = 2\ln\frac{x}{x+2}\Big|_1^3; = 2\left(\ln\tfrac{3}{5} - \ln\tfrac{1}{3}\right) = 2\ln\tfrac{9}{5}$