2.3 Trigonometry (reciprocal, compound, double angle, R-formula)

Name: _____ Class: _____ Date: _____ **Total: 12 marks**

KEY WORDS r-formula • compound • double angle • further trigonometry

Objective

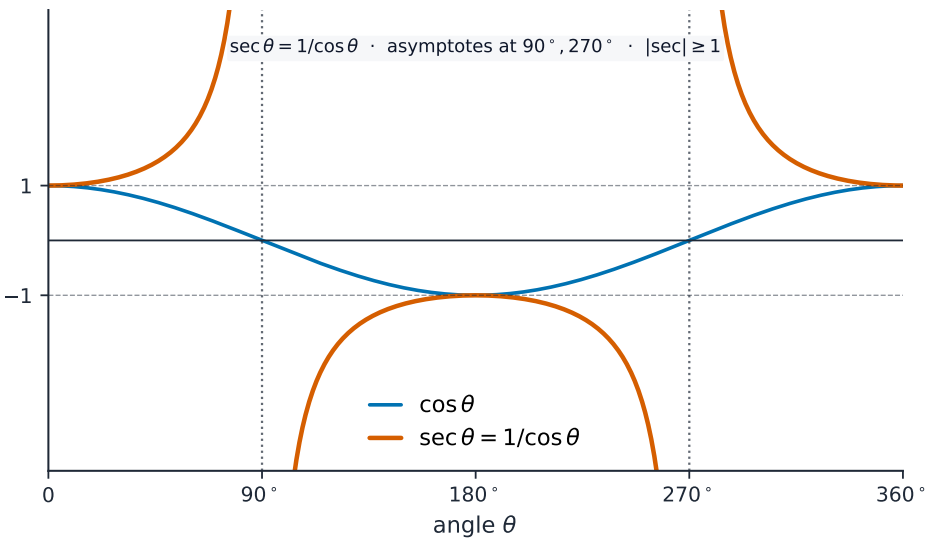
Build the skills to answer exam questions on **further trigonometry** – the reciprocal functions, the identities $\sec^2\theta \equiv 1 + \tan^2\theta$, **compound** and **double angle** formulae, and the **R-formula**.

You must be able to:

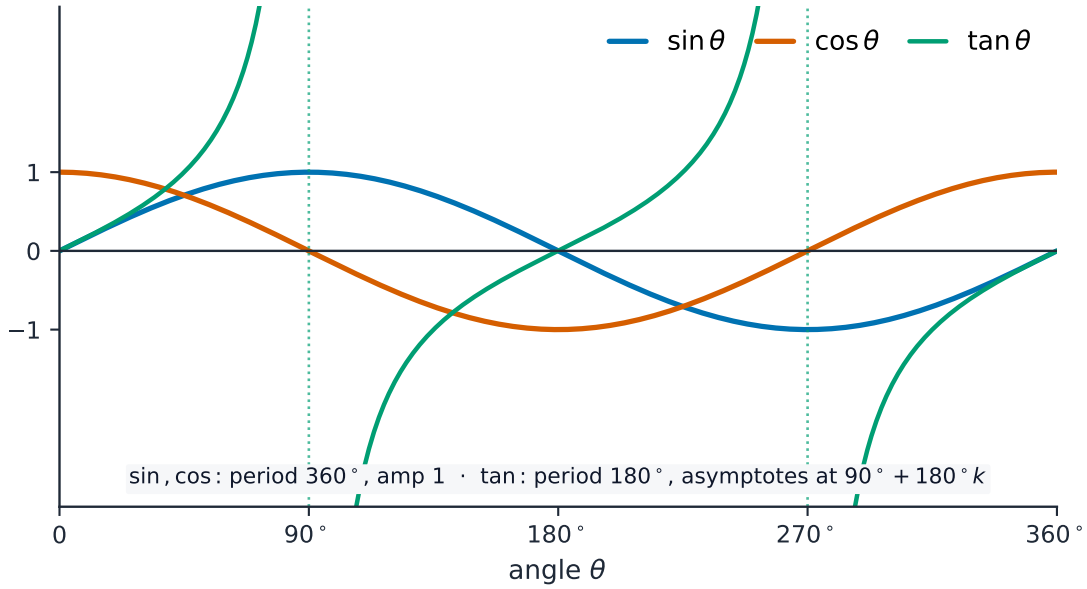
- use $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\csc^2\theta \equiv 1 + \cot^2\theta$
- apply compound- and double-angle formulae
- write $a\sin\theta + b\cos\theta$ as $R\sin(\theta + \alpha)$ and solve

1 Key knowledge and worked examples

Reciprocal identity & R-formula



$\sec\theta = 1/\cos\theta$ *rises to an asymptote wherever* $\cos\theta = 0$



The underlying sine and cosine graphs

Solve $2\tan^2\theta + 3\sec\theta = 18$ **for** $-180° < \theta < 180°$**.** Use $\tan^2\theta = \sec^2\theta - 1$: $2\sec^2\theta + 3\sec\theta - 20 = 0 \Rightarrow (2\sec\theta - 5)(\sec\theta + 4) = 0$. $\cos\theta = \frac{2}{5}$ or $-\frac{1}{4} \Rightarrow \theta = \pm 66.4°,\ \pm 104.5°$.

R-formula: $a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$, $R = \sqrt{a^2 + b^2}$, $\tan\alpha = \frac{b}{a}$.

■ **Key knowledge: Amplitude and phase**

Match sine and cosine coefficients when using the R-formula. Choose the phase angle in the correct quadrant, rather than relying only on its tangent.

■ **Worked example: Amplitude and phase**

Write $3\sin x + 4\cos x = R\sin(x + \alpha)$. Expansion gives $R\cos\alpha = 3$ and $R\sin\alpha = 4$. Thus $R = \sqrt{3^2 + 4^2} = 5$ and $\alpha = \tan^{-1}(4/3) \approx 53.1°$. The maximum value is 5, reached when $x + \alpha = 90°$ modulo $360°$.

2 Practice

2.1 Write $\sec^2\theta$ in terms of $\tan\theta$. [1]

2.2 Use a double-angle formula to write $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$. [1]

3 Exam-style questions

3.1 Expressed as $R\sin(\theta + \alpha)$, the value of R for $3\sin\theta + 4\cos\theta$ is: [1]

- **A** 5
- **B** 7
- **C** $\sqrt{7}$
- **D** 12

3.2 Express $5\sin\theta - 12\cos\theta$ in the form $R\sin(\theta - \alpha)$ with $R > 0$ and $0° < \alpha < 90°$. [3]

3.3 (a) Show that $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} \equiv 2\sec^2\theta$. [3]

(b) Hence solve $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 8$ for $0° \leqslant \theta \leqslant 180°$. [3]

Solutions

Each answer shows the steps, then the **result**.

2.1

- $\sec^2\theta = 1 + \tan^2\theta$

2.2

- $\sin 2\theta = 2\sin\theta\cos\theta$

3.1 A

- $R = \sqrt{3^2 + 4^2} = 5$

3.2

- $R = \sqrt{5^2 + 12^2} = 13$; $\tan\alpha = \frac{12}{5} \Rightarrow \alpha = 67.4°$; $13\sin(\theta - 67.4°)$

3.3

(a)

- common denominator: $\dfrac{(1+\sin\theta)+(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)} = \dfrac{2}{1-\sin^2\theta} = \dfrac{2}{\cos^2\theta} = 2\sec^2\theta$

(b)

- $2\sec^2\theta = 8 \Rightarrow \sec^2\theta = 4 \Rightarrow \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$; $\theta = 60°$ or $120°$