

Exercise walk-through

6.3

Continuous random variables

A-Level Mathematics

Continuous random variables

A density function

$$f(x) \geq 0, \int f = 1.$$

$$P(a < X < b) = \int_a^b f, \quad E(X) = \int xf.$$

Median

solves $F(m) = 0.5$.

The two functions of a continuous variable

Probability density function

The **probability density function** *area* beneath it, and the total area is 1.

$f(x)$: probability is the

Cumulative distribution function

The **cumulative distribution function** – the integral of f , and differentiating F gives f back.

$F(x) = P(X \leq x)$

Expectation

The **expectation** non-zero.

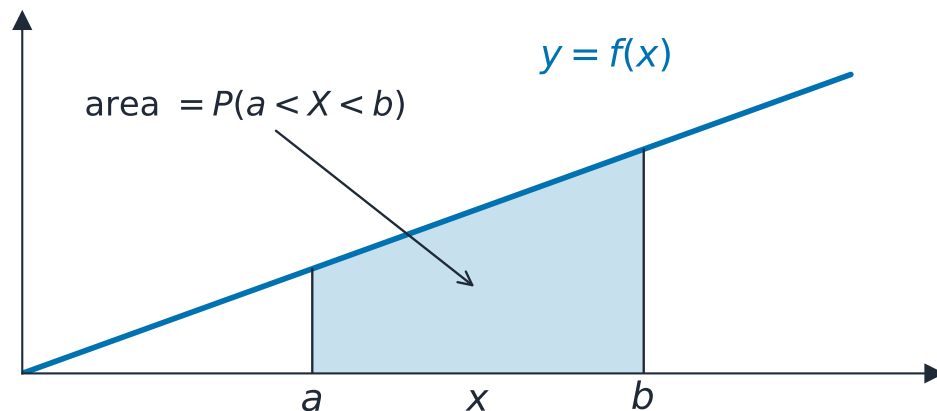
is $E(X) = \int xf(x) dx$ over the range where f is

The normal distribution

The **normal distribution** is the case whose density is that symmetric bell – and the one every table is written for.

Medians and **percentiles** come from F , means and variances from f . Knowing which one a question needs saves most of the work.

Continuous random variables, in detail



$$\int_{-\infty}^{\infty} f = 1 \quad .$$

$$P(a < X < b) = \int_a^b f$$

A **density function** is never negative and encloses **unit area**: $f(x) \geq 0$ and $\int f = 1$. Probability is an **area**, so $P(X = a) = 0$ for any single value.

Sampling and estimation

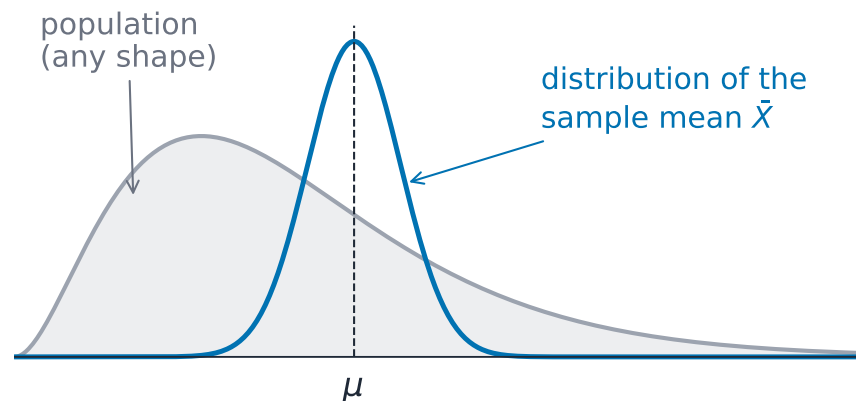
$E(\bar{X}) = \mu$, $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$. A 95% **confidence interval** :

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}.$$

$$n = 64, \bar{x} = 50, \sigma = 8$$

$$50 \pm 1.96 \Rightarrow (48.0, 52.0)$$

Sampling and estimation, in detail



CLT: $\bar{X} \approx N(\mu, \sigma^2/n)$ for large n · $SE = \sigma/\sqrt{n}$

CLT : $\bar{X}$ near-normal

$$E(\bar{X}) = \mu, \text{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

Exercises 2.1·2.2

2.1 $f(x) = kx$ for $0 \leq x \leq 2$. Show that $k = \frac{1}{2}$. [2]

2.2 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

Exercises 2.1·2.2

2.1 $f(x) = kx$ for $0 \leq x \leq 2$. Show that $k = \frac{1}{2}$. [2]

2.1

- $$\int_0^2 kx \, dx = \left[\frac{kx^2}{2} \right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}$$

2.2 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

2.2

- $$\int_0^1 \frac{1}{2}x \, dx = \left[\frac{x^2}{4} \right]_0^1 = \frac{1}{4}$$

Exercise 3.1

3.1 For a valid pdf, $\int_{-\infty}^{\infty} f(x) dx$ must equal:

[1]

☐

A 0

B 0.5

C 1

D ∞

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[1]

☐**A** 0**B** 0.5**C** 1**D** ∞

3.1 C

- The total area under a pdf is 1

Exercise 3.2 (a)

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^1 f(x) dx = 1$. [3]

Exercise 3.2 (a)

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^1 f(x) dx = 1$. [3]

(a)

$$\bullet \int_{-1}^1 \frac{3}{4}(1 - x^2) dx = \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{3}{4} \left(\frac{2}{3} - \left(-\frac{2}{3} \right) \right) = \frac{3}{4} \cdot \frac{4}{3} = 1$$

Exercise 3.2 (b)

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \leq x \leq 1$ (and 0 elsewhere).

(b) Find $P(X > 0.5)$.

[3]

Exercise 3.2 (b)

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1-x^2)$ for $-1 \leq x \leq 1$ (and 0 elsewhere).

(b) Find $P(X > 0.5)$. [3]

(b)

$$\bullet \int_{0.5}^1 \frac{3}{4}(1-x^2) dx = \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{0.5}^1 = \frac{3}{4} \left(\frac{2}{3} - (0.5 - 0.0417) \right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156$$

Exercise 3.3 (a)

3.3 A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$.

(a) Find $E(X)$.

[3]

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(a)

- $$E(X) = \int_0^2 x \cdot \frac{3}{8}x^2 dx = \frac{3}{8} \left[\frac{x^4}{4} \right]_0^2 = \frac{3}{8} \cdot 4 = \frac{3}{2}$$

Exercise 3.3 (b)

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(b) Find the median m .

[3]

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(b)

- $$F(m) = \int_0^m \frac{3}{8}x^2 dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59$$