

Exercise walk-through

3.8

Differential equations

A-Level Mathematics

Differential equations

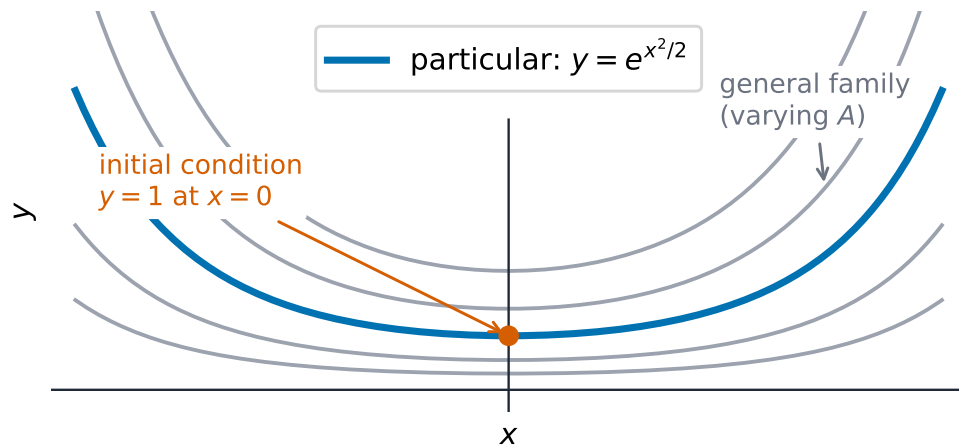
Separable : split y and x , then integrate both sides.

$$\frac{dy}{dx} = xy, \quad y = 1 \text{ at } x = 0$$

$$\ln y = \frac{1}{2}x^2 + c \Rightarrow y = e^{x^2/2}$$

A condition fixes C (the **particular solution**).

Differential equations, in detail



general: family with A · IC → particular solution

$\frac{dy}{dx} = xy$ with $y = 1$ at $x = 0$ gives $\ln y = \frac{1}{2}x^2 + c$, so $y = e^{x^2/2}$. A condition fixes C (the **particular solution**).

Separating the variables

a differential equation

links a quantity to its **rate of change**

separable

when it can be written $\frac{dy}{dx} = f(x)g(y)$

the method

$\int \frac{dy}{g(y)} = \int f(x) dx$ – all the y 's one side, all the x 's the other

the general solution

includes an arbitrary constant $+C$

the initial condition

one given pair of values fixes C , giving the particular solution

the modelling step

read the sentence into an equation: “proportional to” means $= k(\dots)$

Do not forget $+C$ before you apply the condition. Applying it afterwards to an equation with no constant leaves nothing to determine.

Exercises 2.1·2.2

2.1 Separate the variables for $\frac{dy}{dx} = \frac{y}{x}$ (write each side ready to integrate). [1]

2.2 Solve $\frac{dy}{dx} = 3y$, giving the general solution. [2]

Exercises 2.1·2.2

2.1 Separate the variables for $\frac{dy}{dx} = \frac{y}{x}$ (write each side ready to integrate). [1]

2.1

- $\frac{1}{y} dy = \frac{1}{x} dx$

2.2 Solve $\frac{dy}{dx} = 3y$, giving the general solution. [2]

2.2

- $\ln y = 3x + c \Rightarrow y = Ae^{3x}$

Exercises 3.1·3.2

3.1 The general solution of $\frac{dy}{dx} = ky$ is:

[1]

☐

A $y = kx + c$

B $y = Ae^{kx}$

C $y = Ax^k$

D $y = \frac{1}{2}kx^2 + c$

3.2 Solve $\frac{dy}{dx} = 2x(y + 1)$, given $y = 0$ when $x = 0$. Give y explicitly.

[5]

Exercises 3.1·3.2

3.1 The general solution of $\frac{dy}{dx} = ky$ is: [1]

- ☐ **A** $y = kx + c$
B $y = Ae^{kx}$
C $y = Ax^k$
D $y = \frac{1}{2}kx^2 + c$

3.1 B

- $\frac{dy}{dx} = ky$ gives $y = Ae^{kx}$

3.2 Solve $\frac{dy}{dx} = 2x(y + 1)$, given $y = 0$ when $x = 0$. Give y explicitly. [5]

3.2

- $\int \frac{1}{y+1} dy = \int 2x dx$; $\ln(y+1) = x^2 + c$; $y(0) = 0 \Rightarrow c = 0$; $y+1 = e^{x^2}$; $y = e^{x^2} - 1$

Exercise 3.3 (a)

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\frac{dh}{dt} = -k\sqrt{h}$.
Initially $h = H$.

(a) Solve to show $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$. [4]

Exercise 3.3 (a)

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\frac{dh}{dt} = -k\sqrt{h}$. Initially $h = H$.

(a) Solve to show $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$. [4]

(a)

- $\int h^{-1/2} dh = \int -k dt; 2\sqrt{h} = -kt + c; h = H \text{ at } t = 0 \Rightarrow c = 2\sqrt{H}; \sqrt{h} = \sqrt{H} - \frac{1}{2}kt$

Exercise 3.3 (b)

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\frac{dh}{dt} = -k\sqrt{h}$.
Initially $h = H$.

(b) Find, in terms of H and k , the time for the tank to empty. [2]

Exercise 3.3 (b)

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\frac{dh}{dt} = -k\sqrt{h}$. Initially $h = H$.

(b) Find, in terms of H and k , the time for the tank to empty. [2]

(b)

- empty when $h = 0$: $0 = \sqrt{H} - \frac{1}{2}kt \Rightarrow t = \frac{2\sqrt{H}}{k}$