

Exercise walk-through

3.6

Numerical solution of equations

A-Level Mathematics

Differential equations

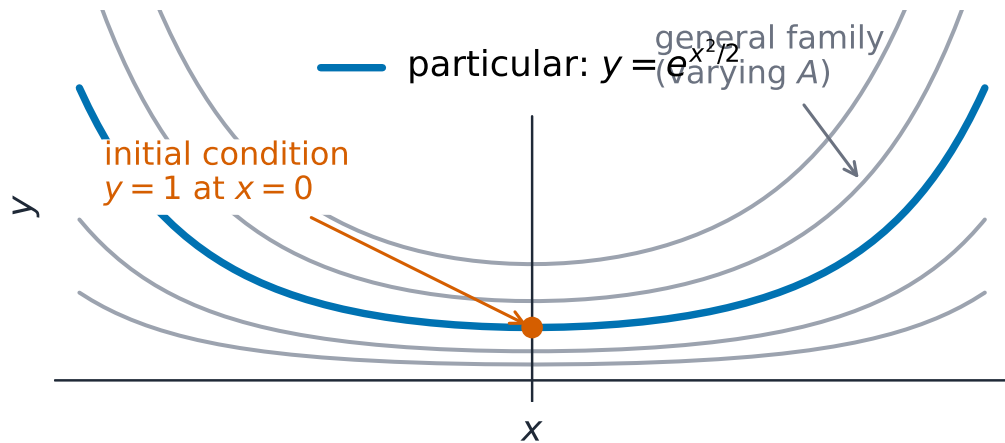
Separable : split y and x , then integrate both sides.

$$\frac{dy}{dx} = xy, \quad y = 1 \text{ at } x = 0$$

$$\ln y = \frac{1}{2}x^2 + c \Rightarrow y = e^{x^2/2}$$

A condition fixes C (the **particular solution**).

Differential equations, in detail



general: family with A · IC → particular solution

$\frac{dy}{dx} = xy$ with $y = 1$ at $x = 0$ gives $\ln y = \frac{1}{2}x^2 + c$, so $y = e^{x^2/2}$. A condition fixes C (the **particular solution**).

Separating the variables

a differential equation

links a quantity to its **rate of change**

separable

when it can be written $\frac{dy}{dx} = f(x)g(y)$

the method

$\int \frac{dy}{g(y)} = \int f(x) dx$ – all the y 's one side, all the x 's the other

the general solution

includes an arbitrary constant $+C$

the initial condition

one given pair of values fixes C , giving the particular solution

the modelling step

read the sentence into an equation: “proportional to” means $= k(\dots)$

Do not forget $+C$ before you apply the condition. Applying it afterwards to an equation with no constant leaves nothing to determine.

Exercises 2.1·2.2

2.1 $f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$. [2]

2.2 Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration. [1]

Exercises 2.1·2.2

2.1 $f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$. [2]

2.1

- $f(1) = e - 3 = -0.282$, $f(2) = e^2 - 6 = 1.389$; opposite signs
 - a root lies between 1 and 2
-

2.2 Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration. [1]

2.2

- e.g. $x = \sqrt[3]{5 - x}$

Exercise 3.1

3.1 An iteration $x_{n+1} = F(x_n)$ from a suitable start **converges** to a root if the values:[1]

☐

A grow without limit

B settle towards a fixed number

C alternate between two values forever

D stay equal to x_0

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3.1 B

- Convergence means the iterates settle towards a fixed value

Exercise 3.2 (a)

3.2 The equation $x^3 = 4x + 1$ has a root α near $x = 2$.

(a) Verify there is a root between $x = 2$ and $x = 2.2$.

[2]

Exercise 3.2 (a)

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(a) Verify there is a root between $x = 2$ and $x = 2.2$. [2]

(a)

- $f(x) = x^3 - 4x - 1$: $f(2) = -1 < 0$, $f(2.2) = 10.648 - 8.8 - 1 = 0.848 > 0$; sign change
- root between

Exercise 3.2 (b)

3.2 The equation $x^3 = 4x + 1$ has a root α near $x = 2$.

(b) Show the equation can be written $x = \sqrt[3]{4x + 1}$ and use $x_{n+1} = \sqrt[3]{4x_n + 1}$, $x_0 = 2.1$, to find α to 2 d.p. [4]

Exercise 3.2 (b)

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(b)

- $x^3 = 4x + 1 \Rightarrow x = \sqrt[3]{4x + 1}$; $x_1 = \sqrt[3]{9.4} = 2.110$, $x_2 = 2.113$, $x_3 = 2.114$; $\alpha = 2.11$

Exercise 3.3

3.3 By sketching $y = e^{-x}$ and $y = x$ on the same axes, state the number of roots of $e^{-x} = x$, and give an interval of width 0.5 containing the root. [3]

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3.3

- the decreasing curve $y = e^{-x}$ meets the line $y = x$ exactly **once**, so there is 1 root; $g(x) = e^{-x} - x$ has $g(0.5) = 0.607 - 0.5 > 0$ and $g(1) = 0.368 - 1 < 0$, so the root lies between 0.5 and 1