

Exercise walk-through

3.5

Integration

A-Level Mathematics

Partial fractions and integration

Split a proper rational function into simpler pieces, then integrate term by term:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Repeated factors need a squared denominator; a quadratic factor that will not factor stays as a linear numerator over it.

$\int \frac{1}{x} dx = \ln |x| + C$. Partial fractions turn many rational integrals into logs.

Partial fractions: check the degree first

the precondition

the top must be of **lower degree** than the bottom

if it is not

“top-heavy”: **divide** the polynomial first, giving a **quotient** plus a proper remainder

distinct linear factors

$$\frac{A}{ax + b} + \frac{B}{cx + d}$$

a repeated factor

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

an irreducible quadratic

$$\frac{Ax + B}{x^2 + c} - \text{a linear numerator over a quadratic}$$

why bother

each piece integrates to a logarithm or a power, which the original fraction does not

Skipping the degree check is the commonest error, and it produces an answer that cannot be right because the shapes do not match.

The binomial expansion for a rational power

The **binomial expansion** negative.

also works when the power is a fraction or is

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$.

The three new trig functions are the **reciprocals** of the familiar ones: the **secant**

$$\sec \theta = \frac{1}{\cos \theta}.$$

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves).

One derivative is added – the **inverse tangent** :

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}.$$

Exercises 2.1·2.2

2.1 Find $\int \frac{1}{x^2 + 9} dx$. [2]

2.2 Find $\int \frac{3x^2}{x^3 + 1} dx$. [2]

Exercises 2.1·2.2

2.1 Find $\int \frac{1}{x^2 + 9} dx$. [2]

2.1

- $\frac{1}{3} \tan^{-1} \frac{x}{3} + C$

2.2 Find $\int \frac{3x^2}{x^3 + 1} dx$. [2]

2.2

- $\ln |x^3 + 1| + C$

Exercises 3.1·3.2

3.1 $\int xe^x dx$ equals:

[1]

☐

A $xe^x - e^x + C$

B $xe^x + e^x + C$

C $\frac{1}{2}x^2e^x + C$

D $e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\int_0^2 x\sqrt{1+x^2} dx$.

[5]

Exercises 3.1·3.2

3.1 $\int x e^x dx$ equals:

[1]

☐

A $x e^x - e^x + C$

B $x e^x + e^x + C$

C $\frac{1}{2} x^2 e^x + C$

D $e^x + C$

3.1 A

- By parts: $x e^x - \int e^x dx = x e^x - e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\int_0^2 x \sqrt{1 + x^2} dx$.

[5]

3.2

- $u = 1 + x^2 \Rightarrow du = 2x dx$, so $x dx = \frac{1}{2} du$; limits $x = 0 \rightarrow u = 1$, $x = 2 \rightarrow u = 5$;
 $\int_1^5 \frac{1}{2} \sqrt{u} du = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_1^5 = \frac{1}{3} (5^{3/2} - 1) = \frac{1}{3} (11.180 - 1) = 3.39$

Exercise 3.3

3.3 (a) Express $\frac{4}{(x)(x+2)}$ in partial fractions. [3]

(b) Hence find $\int_1^3 \frac{4}{x(x+2)} dx$, giving an exact answer. [3]

Exercise 3.3

3.3 (a) Express $\frac{4}{(x)(x+2)}$ in partial fractions. [3]

(b) Hence find $\int_1^3 \frac{4}{x(x+2)} dx$, giving an exact answer. [3]

3.3

(a)

- $4 = A(x+2) + Bx; x=0 \Rightarrow A=2; x=-2 \Rightarrow B=-2; \frac{2}{x} - \frac{2}{x+2}$

(b)

- $\int_1^3 \left(\frac{2}{x} - \frac{2}{x+2} \right) dx = \left[2 \ln x - 2 \ln(x+2) \right]_1^3 = 2 \ln \frac{x}{x+2} \Big|_1^3 = 2 \left(\ln \frac{3}{5} - \ln \frac{1}{3} \right) = 2 \ln \frac{9}{5}$