

Exercise walk-through

2.6

Numerical solution of equations

A-Level Mathematics

What we will cover

- 1 Modulus & polynomials
- 2 Logarithms & exponentials
- 3 Trigonometry
- 4 Differentiation
- 5 Integration & numerical methods
- 6 Recap



Finding a root when the equation cannot be solved

a root

a solution of $f(x) = 0$ – and many equations have no exact form for one

the sign change rule

if $f(a)$ and $f(b)$ have opposite signs and f is continuous, a root lies between them

iteration

rearrange to $x = F(x)$, then use the **iterative formula**

$$x_{n+1} =$$

starting

begin from a value near the root, found by the sign change

converge

the sequence may converge to the root – or diverge, depending on the rearrangement chosen

the check

a sign change on a small interval either side confirms the root to the required accuracy

Different rearrangements of the *same* equation converge at different speeds, or not at all. The exam gives you the rearrangement to use.

Exam tips

- Use the **laws of logarithms** to solve equations; remember $\ln$ and e^x are inverses.
- For **numerical methods**, show a **sign change** to locate a root, set out the iteration clearly, and give the answer to the stated accuracy.
- Learn the **chain, product and quotient rules** and identify which the function needs.
- Solve modulus equations $|f(x)| = g(x)$ by considering both the positive and negative cases, and sketch to check.

Exercises 2.1·2.2

2.1 $f(x) = x^3 - 5x + 1$. Evaluate $f(0)$ and $f(1)$, and state what they tell you. [2]

2.2 State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$. [1]

Exercises 2.1·2.2

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2.1

- $f(0) = 1$, $f(1) = 1 - 5 + 1 = -3$; opposite signs, so a root lies between 0 and 1
-

2.2 State the iterative formula you would use, written as $x_{n+1} = F(x_n)$, for the equation $x = \cos x$. [1]

2.2

- $x_{n+1} = \cos x_n$

Exercise 3.1

3.1 A root of $f(x) = 0$ lies between $x = 2$ and $x = 3$ if:

[1]

☐

A $f(2)$ and $f(3)$ are both positive

B $f(2)$ and $f(3)$ have opposite signs

C $f(2) = f(3)$

D $f'(2) = 0$

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3.1 B

- Opposite signs (with a continuous graph) trap a root between them

Exercise 3.2 (a)

3.2 The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1.

(a) Verify this by a sign change.

[2]

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(a) Verify this by a sign change.

[2]

(a)

- $f(0) = 2 > 0$, $f(1) = 1 - 6 + 2 = -3 < 0$; opposite signs
- root between 0 and 1

Exercise 3.2 (b)

3.2 The equation $x^3 - 6x + 2 = 0$ has a root α between 0 and 1.

(b) Show that the equation can be written as $x = \frac{x^3 + 2}{6}$, and use the iteration $x_{n+1} = \frac{x_n^3 + 2}{6}$ with $x_0 = 0.3$ to find α to 2 d.p. [4]

Exercise 3.2 (b)

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(b)

- $x^3 - 6x + 2 = 0 \Rightarrow 6x = x^3 + 2 \Rightarrow x = \frac{x^3 + 2}{6}; x_1 = \frac{0.027 + 2}{6} = 0.3378, x_2 = 0.3397, x_3 = 0.3398; \alpha = 0.34$

Exercise 3.3

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3.3

- there could be several roots between a and b – the graph may cross the axis more than once (an odd number of times)