

Exercise walk-through

2.5

Integration

A-Level Mathematics

What we will cover

- 1 Modulus & polynomials
- 2 Logarithms & exponentials
- 3 Trigonometry
- 4 Differentiation
- 5 Integration & numerical methods
- 6 Recap



Four techniques worth naming

Product rule

The **product rule** composition.

: $(uv)' = u'v + uv'$ – for a product, never for a

Integration by substitution

Integration by substitution

variable *and* the limits, or convert back at the end.

reverses the chain rule: change the

Natural logarithm

The **natural logarithm** c.

is the inverse of e^x , and $\int \frac{1}{x} dx = \ln |x| +$

Remainder theorem

The **remainder theorem** $(x - a)$ is $f(a)$ – so a remainder of zero is the factor theorem.

: the remainder when $f(x)$ is divided by

Indices and **trigonometric identities** simplify first, integrate second. A polynomial's **degree**

are what make an integral doable: is its highest power.

Finding a root when the equation cannot be solved

a root

a solution of $f(x) = 0$ – and many equations have no exact form for one

the sign change rule

if $f(a)$ and $f(b)$ have opposite signs and f is continuous, a root lies between them

iteration

rearrange to $x = F(x)$, then use the **iterative formula**

$$x_{n+1} =$$

starting

begin from a value near the root, found by the sign change

converge

the sequence may converge to the root – or diverge, depending on the rearrangement chosen

the check

a sign change on a small interval either side confirms the root to the required accuracy

Different rearrangements of the *same* equation converge at different speeds, or not at all. The exam gives you the rearrangement to use.

Exercises 2.1·2.3

2.1 Find $\int e^{2x+1} dx$. [2]

2.2 Find $\int \frac{1}{2x+3} dx$. [2]

2.3 Find $\int \frac{6x}{3x^2+1} dx$. [2]

Exercises 2.1·2.3

2.1 Find $\int e^{2x+1} dx$.

[2]

2.1

- $\frac{1}{2}e^{2x+1} + C$

2.2 Find $\int \frac{1}{2x+3} dx$.

[2]

2.2

- $\frac{1}{2} \ln |2x+3| + C$

2.3 Find $\int \frac{6x}{3x^2+1} dx$.

[2]

2.3

- The numerator is the derivative of the denominator; $= \ln |3x^2+1| + C$

Exercises 2.4·2.5

2.4 Find $\int \sin(3x - 1) dx$. [2]

2.5 Use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to find $\int \cos^2 x dx$. [3]

Exercises 2.4·2.5

2.4 Find $\int \sin(3x - 1) dx$. [2]

2.4

- $-\frac{1}{3} \cos(3x - 1) + C$

2.5 Use the identity $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ to find $\int \cos^2 x dx$. [3]

2.5

- $\int \frac{1}{2}(1 + \cos 2x) dx = \frac{1}{2}x + \frac{1}{4} \sin 2x + C$

Exercises 3.1·3.2

3.1 $\int \cos 4x \, dx$ equals:

[1]

☐

A $4 \sin 4x + C$

B $\frac{1}{4} \sin 4x + C$

C $-\frac{1}{4} \sin 4x + C$

D $-4 \sin 4x + C$

3.2 Find $\int_0^{\pi/4} 4 \cos^2 x \, dx$, giving an exact answer.

[4]

Exercises 3.1·3.2

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3.1 B

- $\int \cos 4x \, dx = \frac{1}{4} \sin 4x + C$

3.2 Find $\int_0^{\pi/4} 4 \cos^2 x \, dx$, giving an exact answer.

[4]

3.2

- $4 \cos^2 x = 2(1 + \cos 2x); \int_0^{\pi/4} (2 + 2 \cos 2x) \, dx = [2x + \sin 2x]_0^{\pi/4} = \left(\frac{\pi}{2} + 1\right) - 0 = \frac{\pi}{2} + 1$

Exercise 3.3

3.3 Use the trapezium rule with **four** strips to estimate $\int_0^2 \sqrt{1+x^3} dx$, giving your answer to 3 s.f. [4]

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3.3

- $h = 0.5$; y at $x = 0, 0.5, 1, 1.5, 2$: $1, \sqrt{1.125} = 1.0607, \sqrt{2} = 1.4142, \sqrt{4.375} = 2.0917, \sqrt{9} = 3$; $\int \approx \frac{0.5}{2}[1 + 3 + 2(1.0607 + 1.4142 + 2.0917)]$; $= 0.25[4 + 2(4.5666)] = 0.25(13.133) = 3.28$ (3 s.f.)

Setup for exercise 3.4

3.4 (a) Find $\int x \ln x \, dx$, stating which factor you take as u and why. [4]

Exercise 3.4

- (b) Use the substitution $u = 1 + x^2$ to evaluate $\int_0^2 \frac{x}{\sqrt{1+x^2}} dx$, giving your answer in exact form. [5]

- (c) Express $\frac{5x+1}{(x+2)(x-1)}$ in partial fractions, and hence find $\int \frac{5x+1}{(x+2)(x-1)} dx$. [5]

Exercise 3.4

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(c) Express $\frac{5x+1}{(x+2)(x-1)}$ in partial fractions, and hence find $\int \frac{5x+1}{(x+2)(x-1)} dx$. [5]

3.4

(a)

Full solution 3.4

3.4

(a)

- Take $u = \ln x$, because it differentiates to the simpler $\frac{1}{x}$, and $\frac{dv}{dx} = x$ so $v = \frac{1}{2}x^2$;
$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx; = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

(b)

- $u = 1 + x^2$ so $\frac{du}{dx} = 2x$ and $x \, dx = \frac{1}{2} du$; the limits become $u = 1$ and $u = 5$;
$$\int_1^5 \frac{1}{2}u^{-1/2} \, du = [u^{1/2}]_1^5; = \sqrt{5} - 1$$

(c)

- $\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$, so $5x+1 = A(x-1) + B(x+2)$; $x = 1$ gives $B = 2$,
 $x = -2$ gives $A = 3$; $\int \frac{3}{x+2} + \frac{2}{x-1} \, dx; = 3 \ln|x+2| + 2 \ln|x-1| + C$