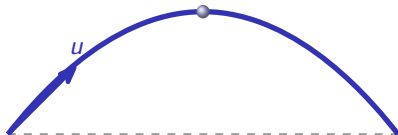


Further Mechanics

A-Level Further Mathematics ■ Topic 3

A-Level Further Mathematics



What we will cover

- 1 Projectiles
- 2 Rigid-body equilibrium
- 3 Circular motion
- 4 Hooke's law & elastic energy
- 5 Variable force & restitution
- 6 Recap



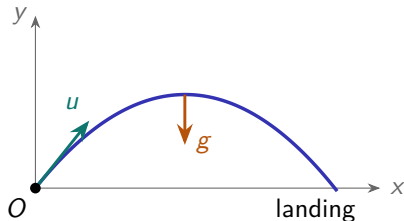
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Projectiles and equilibrium

Model the flight before calculating

A ball is thrown across a court. Predict its position, speed and landing point.

- Model a **projectile** 抛射体 as a **particle** under constant gravity.
- **Ignore air resistance**: less accurate for a shuttlecock or in strong wind.
- Origin at launch: rightwards x , upwards y .



Take $g = 10 \text{ m s}^{-2}$ in these examples. State the model's limitations.

Resolve velocity and share one clock

For speed u at angle α above horizontal, $u_x = u \cos \alpha$, $u_y = u \sin \alpha$.

$$\begin{aligned}x &= u_x t, & y &= u_y t - \frac{1}{2} g t^2, \\v_x &= u_x, & v_y &= u_y - g t.\end{aligned}$$

Worked: $u = 20 \text{ m s}^{-1}$, $\tan \alpha = 4/3$

A 3–4–5 triangle gives $u_x = 12$, $u_y = 16$ in metres per second.

Horizontal acceleration is zero; vertical acceleration is $-g$. Both directions use the same time.

Worked: position is different from velocity

Use $u_x = 12$, $u_y = 16$, $g = 10$ at $t = 1.5$ s.

$$\begin{aligned}x &= 12(1.5) = 18 \text{ m}, & y &= 16(1.5) - 5(1.5)^2 = 12.75 \text{ m}, \\v_x &= 12, & v_y &= 16 - 10(1.5) = 1.\end{aligned}$$

Speed $= \sqrt{12^2 + 1^2} = \sqrt{145} \text{ m s}^{-1}$. Direction $= \tan^{-1}(1/12) = 4.76^\circ$ above horizontal.

Use velocity components for speed and direction; use position components for location.

Worked: a direction fixes signed velocity

The same ball later travels at 45° to the downward vertical.

$$v_y = -v_x = -12,$$

$$t = 2.8 \text{ s},$$

$$16 - 10t = -12,$$

$$(x, y) = (33.6, 5.6) \text{ m}.$$

Its speed is $12\sqrt{2} \text{ m s}^{-1}$. For angle β below horizontal: $v_y = -v_x \tan \beta$. For angle γ to downward vertical: $v_y = -v_x \cot \gamma$.

Read the reference direction and the sign before applying tangent.

Worked: one height can give two times

At height 10 m, the same ball satisfies

$$16t - 5t^2 = 10 \quad \Rightarrow \quad t = \frac{8 \pm \sqrt{14}}{5} \text{ s.}$$

Both times are positive. The corresponding positions are

$$x = \frac{96 \pm 12\sqrt{14}}{5} \text{ m.}$$

The earlier vertical velocity is $+2\sqrt{14}$; the later is $-2\sqrt{14}$. Both speeds equal $\sqrt{144 + 56} = 10\sqrt{2} \text{ m s}^{-1}$.

Keep both roots until “ascending”, “descending” or another event selects one.

Derive the greatest height from vertical velocity

For an upward launch, $v_y = u_y - gt = 0$ at the apex. Hence

$$t_{\text{top}} = u_y/g, \quad H = \frac{u_y^2}{2g} = \frac{u^2 \sin^2 \alpha}{2g}.$$

Worked: $u = 20 \text{ m s}^{-1}$, $\alpha = 30^\circ$, $g = 10$

$u_y = 10$, so the apex occurs at 1 s, 5 m above launch. Horizontal speed remains $10\sqrt{3} \text{ m s}^{-1}$.

Only the vertical component vanishes at the top. A purely vertical throw is the zero-speed exception.

Derive range on the launch-height plane

At the launch height, $0 = u_y t - gt^2/2 = t(u_y - gt/2)$. Discard the launch root $t = 0$; for $u_y > 0$,

$$T = \frac{2u_y}{g}, \quad R = u_x T = \frac{u^2 \sin 2\alpha}{g}.$$

For $u = 20 \text{ m s}^{-1}$, $\alpha = 30^\circ$, $g = 10$:

$$T = 2 \text{ s}, \quad R = 20\sqrt{3} \text{ m} \approx 34.6 \text{ m}.$$

This range formula requires landing at the launch height.

A maximum-range claim needs its assumptions

Keep launch speed u and gravity g fixed. Ignore drag and land at launch height.

$$R = \frac{u^2}{g} \sin 2\alpha \leq \frac{u^2}{g}.$$

For upward launches the maximum occurs at $\alpha = 45^\circ$. Complementary angles give equal ranges, but different heights and times. For 30° and 60° , the higher launch lasts $\sqrt{3}$ times as long and reaches three times the height.

Do not apply the 45° conclusion to unequal landing heights without a new calculation.

Worked: land below a horizontal launch

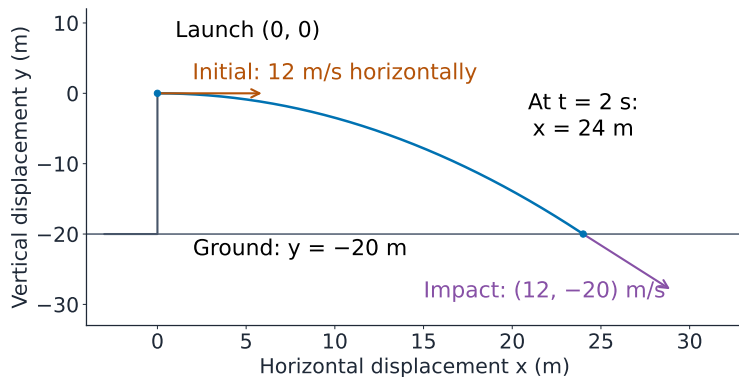
A ball leaves a 20 m platform horizontally at 12 m s^{-1} .

$$\begin{aligned}y_{\text{ground}} &= -20, & -5t^2 &= -20, \\t &= 2 \text{ s}, & x &= 24 \text{ m}.\end{aligned}$$

At impact $v_x = 12$, $v_y = -20$, so speed is $4\sqrt{34} \text{ m s}^{-1}$ and direction is $\tan^{-1}(20/12) \approx 59.0^\circ$ below horizontal.

Ground is $y = -20$, because the origin is at launch. Reject the negative time.

Coordinates and the impact velocity



The ground lies below the launch origin. The purple arrow represents impact velocity, not continued flight through the ground.

Derive the Cartesian equation by eliminating time

If $u \cos \alpha \neq 0$, the horizontal equation gives $t = x/(u \cos \alpha)$. Substitute this expression into the vertical equation:

$$\begin{aligned} y &= u \sin \alpha \frac{x}{u \cos \alpha} - \frac{g}{2} \left(\frac{x}{u \cos \alpha} \right)^2 \\ &= x \tan \alpha - \frac{gx^2}{2u^2 \cos^2 \alpha}. \end{aligned}$$

This is the **Cartesian equation** 直角坐标方程 of the **trajectory** 轨迹, a parabola for non-vertical launch.

The mathematical curve extends beyond the physical flight. Stop at the first impact.

Worked: recover speed from a direction and target

Let $\tan \alpha = 3/2$, and require the path to pass through $(10, 10)$ m. With $T = \tan \alpha$ and $\sec^2 \alpha = 1 + T^2$,

$$u^2 = \frac{gX^2(1 + T^2)}{2(XT - Y)} = \frac{10(100)(13/4)}{2(15 - 10)} = 325.$$

Thus $u = 5\sqrt{13} \text{ m s}^{-1}$, $u_x = 10$, $u_y = 15$, and

$$y = \frac{3}{2}x - \frac{x^2}{20}.$$

For a target to the right, finite positive speed requires $XT > Y$.

Worked: a fixed speed can give two launch angles

Fix $u = 20\sqrt{2} \text{ m s}^{-1}$ and target the wall top $(40, 20) \text{ m}$. Set $T = \tan \alpha$. Then

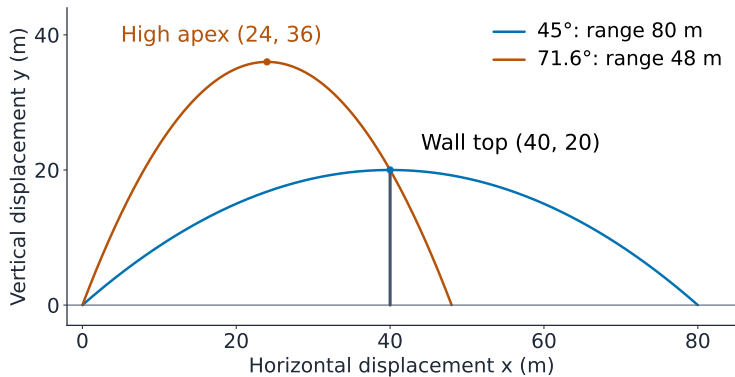
$$20 = 40T - 10(1 + T^2),$$

$$0 = T^2 - 4T + 3 = (T - 1)(T - 3).$$

Both roots are positive: $\alpha = 45^\circ$ or $\tan^{-1} 3 \approx 71.6^\circ$.

Substitute both roots back into the target equation. Do not discard the higher angle.

The higher flight can have the shorter range



Both paths pass through (40,20). Their ranges are 80 m and 48 m; their vertices are (40,20) and (24,36).

Worked: clearance gives an angular interval

To reach or pass above the same wall top,

$$40T - 10(1 + T^2) \geq 20 \quad \Rightarrow \quad 1 \leq T \leq 3.$$

The same-height range is

$$R = \frac{160T}{1 + T^2}, \quad R' = \frac{160(1 - T^2)}{(1 + T^2)^2}.$$

It decreases for $1 \leq T \leq 3$, so the minimum range is 48 m at $T = 3$.

If touching the wall top is forbidden, exclude the endpoints: 48 m becomes an unattained lower bound.

Worked: a low barrier gives two algebraic branches

Return to $y = 3x/2 - x^2/20$. After passing $(10, 10)$, the ball meets a barrier of height $25/4$ m at $x = D$.

$$y(D) \leq 25/4 \quad \Rightarrow \quad (D - 5)(D - 25) \geq 0,$$

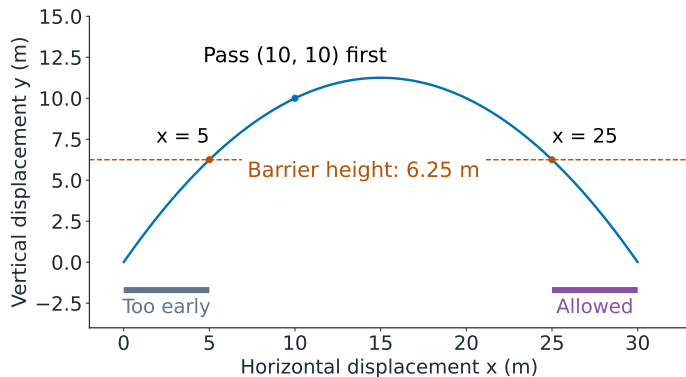
so $D \leq 5$ or $D \geq 25$.

The ground interval is $0 \leq D \leq 30$. Passing $x = 10$ first removes the early branch.

$$\boxed{25 \leq D \leq 30} \quad (\text{boundary contact allowed}).$$

Combine the inequality, the ground interval and the stated event order.

Event order removes a tempting early answer



A height inequality alone leaves an early interval. Passing $(10, 10)$ first selects only $25 \leq D \leq 30$.

Worked: two observed points determine the launch

An unknown launch passes through $(10, 15)$ and $(20, 20)$ in metres. Write $y = Ax - Bx^2$, with $A = \tan \alpha$, $B = g/(2u_x^2) > 0$.

$$10A - 100B = 15,$$

$$20A - 400B = 20,$$

$$B = 1/20,$$

$$A = 2.$$

With $g = 10$, $u_x = 10$, $u_y = 20$, $u = 10\sqrt{5} \text{ m s}^{-1}$ and $\alpha = \tan^{-1} 2$. Range 40 m; greatest height 20 m.

Recover components before taking the speed magnitude.

Retrieval: calculate first, then select the event

Use $u_x = 10$, $u_y = 15$, $g = 10$.

- 1 Find both times and positions at height 10 m. Which is descending?
- 2 Find the time and position when the velocity is 45° below horizontal.
- 3 Explain why y/x does not give the direction of velocity.

Choose equations from the event; check signed components and physical time.

Retrieval answers: height roots and velocity direction

- 1 $15t - 5t^2 = 10$ gives $t = 1, 2$ s and positions $(10, 10)$, $(20, 10)$ m. Vertical velocities are $+5, -5$: the later point is descending.
- 2 $15 - 10t = -10$ gives $t = 2.5$ s and $(x, y) = (25, 6.25)$ m.
- 3 v_y/v_x gives the slope of the tangent. The ratio y/x gives the direction from the origin to the position, a different line.

A correct algebraic answer still needs the event's direction and timing restrictions.

A rigid-body model keeps the force positions

A door opens more easily at its handle than beside its hinge: the **same force**, a different **line of action**.

- A **rigid body** 刚体 keeps its shape and the distances between its parts.
- Here the forces are **coplanar forces** 共面力: their lines lie in one plane.
- A particle model alone cannot predict turning or toppling.



Keep both the force and its position in the model.

Measure the perpendicular arm of a force

moment = force $\times$ perpendicular distance to its line of action.

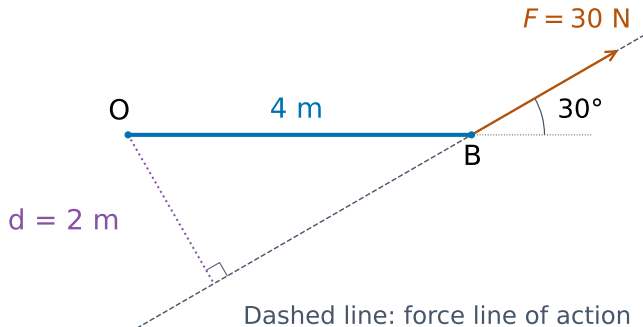
Choose anticlockwise positive. An upward 20 N force acting 0.6 m to the right of the pivot gives +12 N m.

Worked: 30 N at 30° , applied 4 m along a horizontal rod

Use its perpendicular component: $(30 \sin 30^\circ)(4) = 60 \text{ N m}$. Or use its perpendicular arm: $30(4 \sin 30^\circ) = 60 \text{ N m}$.

Apply one sine factor. A force through the pivot has zero moment, not zero force.

The force line determines the moment arm



Use the force with its perpendicular arm, or its perpendicular component with the rod length.

Zero resultant alone does not give equilibrium

Two opposite parallel 10 N forces act on lines 0.4 m apart. Their resultant force is zero, but the moment magnitude is 4 N m. For equilibrium under coplanar forces, require

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_O = 0.$$

These conditions are also sufficient in this model.

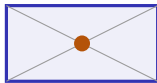
Zero force means zero translational acceleration. Static equilibrium also needs the body initially at rest.

Gravity acts at the centre of mass

In a uniform gravitational field, replace the distributed weight by $F_W = mg$ through the **centre of mass** 质心.



rod: midpoint



rectangle: diagonals



frame: in the hole

- Uniform rod: its midpoint. Uniform rectangle: intersection of its diagonals.
- The centre lies on every symmetry axis of the mass distribution.
- A symmetric shape with unequal density need not have a symmetric mass distribution.

A centre of mass may lie in empty space, such as the hole in a frame.

Worked: the centroid of a triangular lamina

A uniform triangle's centre is two-thirds along a median from its vertex. For vertices $(0, 0)$, $(6, 0)$, $(0, 3)$, the opposite midpoint from the origin is $(3, 1.5)$.

$$\frac{2}{3}(3, 1.5) = (2, 1) \text{ m.}$$

Equivalently, average the three vertex coordinates for a uniform triangular lamina:

$$\bar{x} = (0 + 6 + 0)/3 = 2, \quad \bar{y} = (0 + 0 + 3)/3 = 1.$$

Use the same origin for every component. The MF19 result is given; its proof is not required.

MF19 positions need the correct reference point

uniform body	centre position on symmetry axis
solid cone or pyramid	$3h/4$ from vertex; $h/4$ from base
solid hemisphere	$3r/8$ from base-circle centre
thin hemispherical shell	$r/2$ from base-circle centre

For a cone of height 8 cm, the distance from its base is 2 cm. For radius 4 cm, a solid hemisphere gives 1.5 cm; a shell gives 2 cm.

A shell and a solid use different mass distributions. Convert positions to one signed axis.

MF19 distinguishes an arc from a sector plate

Let the angle be 2α , with half-angle α in radians.

$$\text{wire arc: } d = \frac{r \sin \alpha}{\alpha},$$

$$\text{sector lamina: } d = \frac{2r \sin \alpha}{3\alpha}.$$

A semicircle has $\alpha = \pi/2$, giving $2r/\pi$ for the arc and $4r/(3\pi)$ for the plate. Both positions are measured from the circle centre along the symmetry axis.

Use length for a uniform wire, area for a uniform plate and volume for a uniform solid.

Replace each part by an equivalent particle

For component masses m_i at their own centres (x_i, y_i) ,

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

Two particles: 2 kg **at** $x = 0$, 3 kg **at** $x = 10$ m

$\bar{x} = (2(0) + 3(10))/5 = 6$ m. A plain average would give the wrong answer.

Use area or volume ratios only when the relevant densities are equal.

Weight every position by its component mass before adding.

Worked: add a rectangle and a triangle

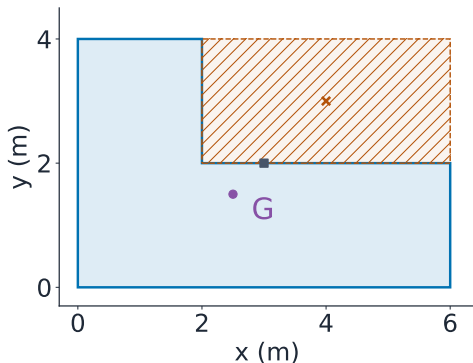
Join a rectangle $0 \leq x \leq 4$, $0 \leq y \leq 2$ to a triangle with vertices $(4, 0)$, $(4, 2)$, $(7, 0)$. Lengths are metres; thickness and density are equal.

part	area	centre
rectangle	8	$(2, 1)$
triangle	3	$(5, 2/3)$

$$\bar{x} = \frac{8(2) + 3(5)}{11} = \frac{31}{11}, \quad \bar{y} = \frac{8(1) + 3(2/3)}{11} = \frac{10}{11}.$$

Find each component centre before forming the weighted sum.

A removed corner contributes negative area and moments



Full rectangle
Area 24 m^2 ;
centre $(3, 2)$

Removed rectangle
Area 8 m^2 ;
centre $(4, 3)$

Remaining plate
Area 16 m^2 ;
 $G = (2.5, 1.5)$

Full rectangle minus the upper-right corner leaves the L-shaped plate.

Worked: subtract a hole from an L-shaped plate

The full 6×4 rectangle has area 24 and centre $(3, 2)$. The removed 4×2 corner has area 8 and centre $(4, 3)$.

$$\bar{x} = \frac{24(3) - 8(4)}{24 - 8} = 2.5 \text{ m},$$

$$\bar{y} = \frac{24(2) - 8(3)}{24 - 8} = 1.5 \text{ m}.$$

The same subtraction applies to both coordinate moments and the total area.

Do not subtract the hole from the denominator while leaving its moments in the numerator.

Worked: a wire frame uses length and line density

Two rods extend from the origin: 6 m horizontally and 4 m vertically. Their centres are $(3, 0)$ and $(0, 2)$.

same line density: $(\bar{x}, \bar{y}) = (18/10, 8/10) = (1.8, 0.8)$ m,

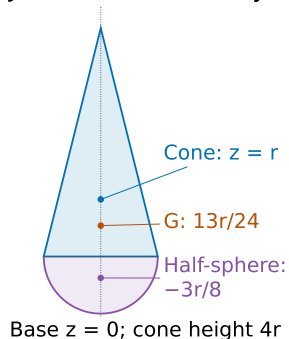
vertical density doubled: $(\bar{x}, \bar{y}) = (18/14, 16/14) = (9/7, 8/7)$ m.

The mass weights change from $(6, 4)$ to $(6, 8)$.

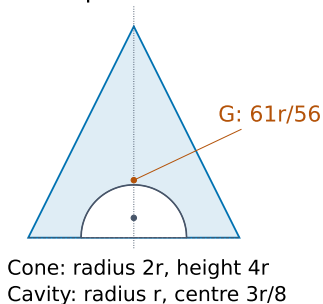
The empty area between rods has no mass. A density change alters the weights in the average.

Solids need volume weights and a common signed axis

Joined, same density



Hemisphere removed



Axial sections: the hemisphere is below the common base when joined, above it when removed.

Worked: join a cone and hemisphere at their bases

Cone radius r , height $4r$; hemisphere radius r . Both have the same density. Put $z = 0$ at the common base and take positive z towards the cone vertex.

part	volume	z
cone	$4\pi r^3/3$	r
hemisphere	$2\pi r^3/3$	$-3r/8$

$$\bar{z} = \frac{2(r) + 1(-3r/8)}{3} = \frac{13r}{24}.$$

The common volume factor cancels; the negative position does not.

Worked: remove a hemispherical cavity

A cone has radius $2r$, height $4r$. A hemispherical cavity of radius r has its base in the same plane and curves upwards into the cone.

part	volume weight	height above base
full cone	16	r
removed hemisphere	2	$3r/8$

$$\bar{z} = \frac{16(r) - 2(3r/8)}{16 - 2} = \frac{61r}{56} > r.$$

Removing material near the base raises the centre of mass.

Worked: a suspended plate puts G below its support

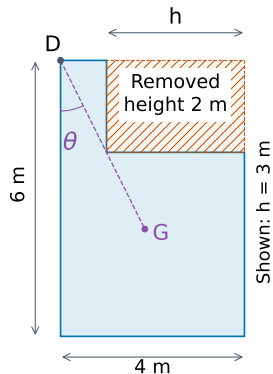
The L-shaped plate has $G = (2.5, 1.5)$ m. Suspend it from $D = (0, 4)$. The support and weight must act on the same vertical line, so DG becomes vertical.

$$G - D = (2.5, -2.5), \quad \tan \theta = \frac{2.5}{4 - 1.5} = 1.$$

The original downward edge direction makes angle 45° with the hanging vertical.

For stable rest, G is below D. The centre-above-support orientation is unstable.

Keep the suspension point inside the surviving plate



Original coordinates, with $h = 3 \text{ m}$ shown. The line DG becomes vertical when the plate hangs.

Worked: an algebraic root can delete the support

From a 4×6 plate remove an upper-right rectangle of width h , height 2, with $0 < h < 4$. Suspend from $D = (0, 6)$.

$$\bar{x} = \frac{h^2 - 8h + 48}{24 - 2h}, \quad \bar{y} = \frac{72 - 10h}{24 - 2h}.$$

The condition $\tan \theta = 1/2$ gives

$$\frac{\bar{x}}{6 - \bar{y}} = \frac{1}{2} \quad \Rightarrow \quad (h - 3)(h - 4) = 0.$$

Only $h = 3$ survives. At $h = 4$, D has been removed with the whole upper strip.

Check the geometric domain after solving the equation.

Contact forces have directions and limits

- Weight acts through G; a taut light string pulls along the string.
- A smooth surface supplies only a normal reaction, perpendicular to the surface.
- A hinge can supply two reaction components. A simple contact can push, with $F_N \geq 0$.
- Rough contact has friction opposing impending relative sliding:

$$|F_f| \leq \mu F_N.$$

Use equality only at limiting friction. Static friction adjusts to the amount actually required.

Worked: find the beam force and its pivot reaction

A uniform horizontal 4 m beam weighs 100 N. It is pivoted at A, with an upward vertical force F at B.

$$\text{moments about A: } 4F = 100(2),$$

$$F = 50 \text{ N},$$

$$\text{vertical balance: } R_A + F = 100,$$

$$R_A = 50 \text{ N upwards.}$$

Taking moments about A removes R_A from that equation because its arm is zero.

A zero moment contribution does not make the reaction force zero.

Worked: end reactions on a loaded beam

A uniform 6 m beam weighs 120 N and rests on end supports A and B. A 60 N load is 5 m from A.

$$R_A + R_B = 180,$$

$$6R_B = 120(3) + 60(5) = 660,$$

$$R_B = 110 \text{ N}, \quad R_A = 70 \text{ N}.$$

Both reactions are positive, consistent with supports that can only push.

A negative simple-support reaction means the assumed contact model has failed.

Worked: an angled rod with a horizontal string

A uniform 4 m rod weighs 100 N and is hinged at A. It makes 30° above horizontal; a horizontal string at its far end pulls towards the wall.

$$F_T(4 \sin 30^\circ) = 100(2 \cos 30^\circ) \quad \Rightarrow \quad F_T = 50\sqrt{3} \text{ N.}$$

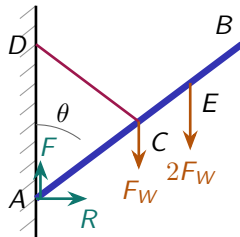
The hinge reaction is $50\sqrt{3}$ N rightwards and 100 N upwards.

$$|F_R| = \sqrt{(50\sqrt{3})^2 + 100^2} = 50\sqrt{7} \text{ N.}$$

Horizontal and vertical forces have different perpendicular arms. Add reaction components by Pythagoras.

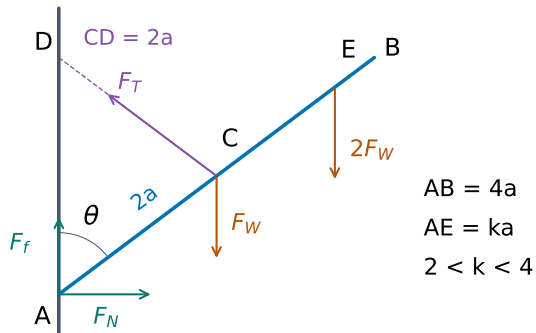
Rough-wall rod: use the geometry before resolving

- Rod $AB = 4a$, weight F_W at midpoint C ; end A on a **rough wall**, $\mu = \frac{1}{4}$.
- String $CD = 2a$ to wall point D , so $AC = CD$.
- $\tan \theta = \frac{4}{3}$ (rod to wall); particle $2F_W$ at $AE = ka$.
- Rod is **about to slip down**.



ACD is isosceles: rod and string make the same angle with the wall. Friction is upwards.

The rough-wall rod has five external force contributions



Weight at C, particle weight at E, tension at C and two contact components at A.

Rough-wall rod: solve the two force balances

With $\sin \theta = 4/5$, $\cos \theta = 3/5$, impending downward slip gives $F_f = F_N/4$.

$$\begin{aligned}F_N &= \frac{4}{5}F_T, \\ \frac{3}{5}F_T + \frac{1}{4}F_N &= 3F_W, \\ \frac{4}{5}F_T &= 3F_W.\end{aligned}$$

Hence $F_T = 15F_W/4$, $F_N = 3F_W$ and $F_f = 3F_W/4$.

The friction limit uses the normal force, not the total weight.

Rough-wall rod: moments determine the particle position

Take moments about A to remove both contact components:

$$F_T(2a \sin 2\theta) = F_W(2a \sin \theta) + 2F_W(ka \sin \theta).$$

Substitute $F_T = 15F_W/4$, $\sin 2\theta = 24/25$, $\sin \theta = 4/5$:

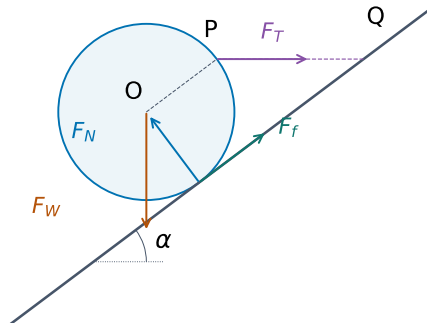
$$\frac{36}{5} = \frac{8}{5}(1 + k) \quad \Rightarrow \quad k = \frac{7}{2}.$$

The result obeys $2 < k < 4$: E lies between C and B.

Include the extra particle weight in both the force balance and its own moment.

An offset horizontal string also turns a ring

OP parallel to the plane



The normal force and weight have zero moment about O ; friction and tension need not.

Worked: ring moments and force balance along a slope

A ring has weight F_W and radius r on a plane with $\tan \alpha = 3/4$. The string pulls horizontally at P, with OP parallel to the plane and pointing uphill.

$$\text{moments about O: } F_f r = F_T r \sin \alpha,$$

$$\text{along plane: } F_f + F_T \cos \alpha = F_W \sin \alpha.$$

Since $\sin \alpha = 3/5$, $\cos \alpha = 4/5$,

$$F_T = \frac{3}{7} F_W, \quad F_f = \frac{9}{35} F_W.$$

Being stationary does not remove the need for a moment equation.

Worked: normal balance fixes the limiting coefficient

The ring's horizontal tension also pushes into the plane:

$$F_N = F_W \cos \alpha + F_T \sin \alpha = \frac{4}{5} F_W + \frac{9}{35} F_W = \frac{37}{35} F_W.$$

At limiting equilibrium,

$$\mu = \frac{F_f}{F_N} = \frac{9}{37}.$$

For a larger coefficient the same required friction can act below its maximum.

Do not replace the normal force by $F_W \cos \alpha$ when another force has a normal component.

Toppling under gravity uses the actual support edge

With weight and contact forces alone, the vertical through G must meet the base of support. At the toppling limit the contact resultant reaches an edge and the weight line passes through it.

The L-shaped plate has $\bar{x} = 2.5$ m

On its full 6 m lower edge, the projection is within the base. Turn it over onto its 2 m short top edge: the projection is outside that base, so it topples.

An extra applied force changes the moment balance; weight inside the base is then insufficient.

A pushed block has a moving normal resultant

A uniform rectangular block has weight F_W , base half-width b and a horizontal push P at height h .

$$\begin{aligned}F_N &= F_W, & F_f &= P, \\F_N \delta &= Ph, & \delta &= Ph/F_W.\end{aligned}$$

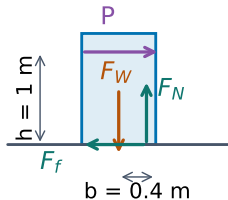
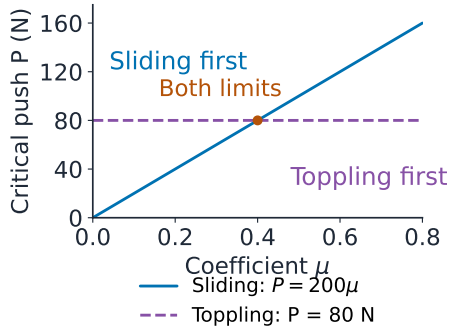
The normal resultant moves towards the leading edge. At $\delta = b$,

$$P_{\text{topple}} = F_W b/h, \quad P_{\text{slide}} = \mu F_W.$$

Find both thresholds. They are generally different.

The smaller push threshold selects the first motion

Weight 200 N; $P = 60$ N shown



The orange intersection gives simultaneous limits; on either side one threshold is reached first.

Worked: compare sliding and toppling

For $F_W = 200 \text{ N}$, $b = 0.4 \text{ m}$, $h = 1 \text{ m}$,

$$P_{\text{topple}} = 80 \text{ N}.$$

μ	P_{slide}	first motion
0.3	60 N	sliding
0.5	100 N	toppling

The limits coincide at $\mu = b/h = 0.4$.

These are predictions of the first loss of static equilibrium, not the later motion.

Inclined bodies also have two critical limits

Let the base half-width be b and the centre be a perpendicular height c above the base centre.

$$\tan \alpha_{\text{topple}} = b/c, \quad \tan \alpha_{\text{slide}} = \mu.$$

For $b = 0.4 \text{ m}$, $c = 0.6 \text{ m}$, $\mu = 0.5$, sliding occurs at 26.6° , before toppling at 33.7° .

For the cone with a hemispherical cavity, $c = 61r/56$, $b = 2r$:

$$\tan \alpha_{\text{topple}} = 112/61.$$

The last toppling result assumes enough friction to prevent an earlier slide.

Retrieval: model the mass, then check the contact

- 1 A uniform 6 cm square loses its upper-right 4 cm square, whose centre is $(4, 4)$. Find the remaining centre from the original lower-left origin.
- 2 A uniform rectangular block weighs 300 N, with $b = 0.3$ m, $h = 0.9$ m, $\mu = 0.2$. Find the first push threshold.
- 3 Why is zero resultant force alone insufficient for a rigid body?

State the mass model and compare both possible losses of equilibrium.

Retrieval answers: holes, thresholds and moments

- 1 Full area 36 at (3, 3); removed area 16 at (4, 4). Both coordinates are $(108 - 64)/20 = 11/5$ cm. This centre lies in the hole, which is possible.
- 2 Sliding requires 60 N; toppling requires 100 N. Sliding begins first.
- 3 Opposite parallel forces on different lines have zero resultant and a non-zero moment. Both force and moment balances are required.

Check physical geometry and contact limits after the algebra.

2

Circular motion

A turn needs a change in velocity

The string pulls the ball inwards; the ball pulls your hand outwards. Those forces act on **different objects**.

Choose the ball as the system

Draw the string's force on the ball. Do not put the force on your hand into the ball's free-body diagram.

In a ground-fixed inertial frame, use real interactions. Inertia is not an extra outward force; a rotating-frame centrifugal force is not needed.

Angular speed comes from arc length

An angle θ in **radians** 弧度 sweeps distance $s = r\theta$. At fixed radius,

$$\text{linear speed} = r \times \text{angular speed}, \quad v = r\omega.$$

Two points on one rotating wheel

They share the same angular speed. A point twice as far from the axis travels twice as far each turn and has twice the linear speed.

Convert degrees to radians before using $v = r\omega$.

Worked: convert a revolution rate

The **period** 周期 τ is time per turn; **frequency** 频率 f is turns per second.

$$\omega = 2\pi f = \frac{2\pi}{\tau}.$$

30 revolutions per minute; orbit radius 0.4 m

$$f = \frac{30}{60} = \frac{1}{2}, \quad \omega = \pi \text{ rad/s}, \quad v = 0.4\pi \text{ m/s}, \quad \tau = 2 \text{ s}.$$

Revolutions per minute and radians per second are different units.

Inward acceleration is a radial component

Velocity is tangent to the circle. The **centripetal acceleration** 向心加速度 is the inward radial component:

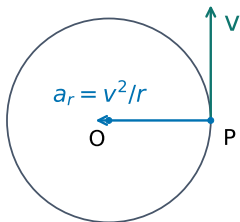
$$a_r = \frac{v^2}{r} = r\omega^2.$$

At constant speed this is the whole acceleration. When speed changes, a **tangential component** is also present.

In a vertical circle under gravity, the full acceleration need not point exactly towards the centre. A proof of the radial formula is not required.

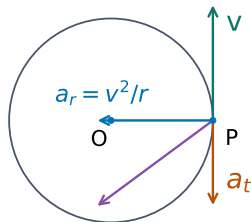
Changing speed adds a tangential component

Constant speed



Total acceleration is radial.

Changing speed



Total a

The radial component changes direction; the tangential component changes speed.

Worked: check both acceleration formulae

For $r = 2\text{ m}$ and constant $\omega = 3\text{ rad/s}$,

$$v = r\omega = 6\text{ m/s}, \quad a_r = r\omega^2 = 18\text{ m/s}^2 = \frac{6^2}{2}\text{ m/s}^2.$$

What if the radius doubles?

At fixed v , v^2/r halves. At fixed ω , $r\omega^2$ doubles.

State what remains fixed when comparing circular motions.

Centripetal names the resultant, not an extra force

Choose inwards as positive and resolve actual forces:

$$\sum F_{\text{inward}} = \frac{mv^2}{r} = mr\omega^2.$$

A **centripetal force** 向心力 is the net inward force required by the motion. It may come from tension, gravity, friction and normal-force components.

For a steady horizontal circle, vertical and tangential accelerations are zero. Radial acceleration is not zero. A horizontal path alone does not guarantee constant speed.

Worked: a friction limit on a rotating disc

Only friction acts horizontally. First find the normal reaction:

$$N = mg, \quad mr\omega^2 = F_f \leq \mu N.$$

For $r = 1.2 \text{ m}$, $\mu = 0.3$ and $g = 10 \text{ m/s}^2$,

$$\omega^2 \leq \frac{\mu g}{r} = 2.5, \quad \omega_{\max} = \sqrt{2.5} \text{ rad/s}.$$

The maximum revolution rate is $\frac{60\sqrt{2.5}}{2\pi} \approx 15.1$ per minute.

Below the limit, static friction adjusts to the required value.

Worked: an inclined string changes the reaction

A 2 kg particle has orbit radius 1.5 m. Tension is 10 N, with $\sin \alpha = 3/5$, $\cos \alpha = 4/5$ from vertical. Let $g = 10$ and $\mu = 1/4$.

$$N = 20 - 10 \left(\frac{4}{5} \right) = 12 \text{ N}, \quad F_{f,\max} = \frac{1}{4}(12) = 3 \text{ N}.$$

At impending outward slipping, friction acts inwards:

$$10 \left(\frac{3}{5} \right) + 3 = 2(1.5)\omega^2 \quad \Rightarrow \quad \omega = \sqrt{3} \text{ rad/s}.$$

The upward tension component reduces the normal reaction and its friction limit.

A friction direction can reverse

Keep that tension and geometry fixed. Signed inward friction must satisfy

$$F_f = 3\omega^2 - 6, \quad |F_f| \leq 3.$$

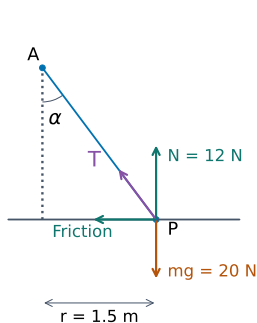
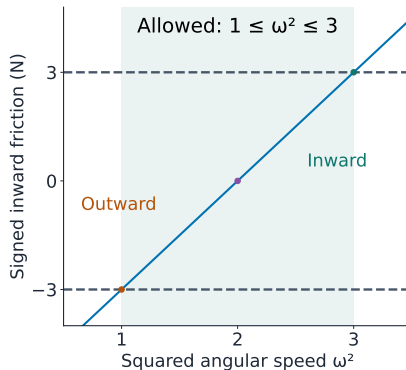
Hence

$$-3 \leq 3\omega^2 - 6 \leq 3 \quad \Rightarrow \quad 1 \leq \omega^2 \leq 3.$$

Below $\omega^2 = 2$, friction acts outwards; above it, friction acts inwards.

Individual real forces may point outwards. Their radial resultant must still be inwards.

The disc has two static-friction boundaries



The upper and lower limits correspond to opposite impending slip directions.

Resolve a conical pendulum vertically and inwards

For string length L and angle θ from vertical, the orbit radius is $r = L \sin \theta$.

$$T \cos \theta = mg, \quad T \sin \theta = m(L \sin \theta) \omega^2.$$

For a non-zero orbit, divide and simplify:

$$\omega^2 = \frac{g}{L \cos \theta}.$$

The string length is a sloping distance; the orbit radius is horizontal.

Worked: find the orbit before its speed

Let $L = 2.5 \text{ m}$, $m = 0.4 \text{ kg}$, $\cos \theta = 4/5$ and $g = 10 \text{ m/s}^2$.

$$r = 2.5 \left(\frac{3}{5} \right) = 1.5 \text{ m}, \quad T = \frac{4}{4/5} = 5 \text{ N}.$$

$$\omega^2 = \frac{10}{2.5(4/5)} = 5, \quad v = 1.5\sqrt{5} \text{ m/s}, \quad \tau = \frac{2\pi}{\sqrt{5}} \text{ s}.$$

Using L in place of r would give the wrong linear speed.

Worked: a second particle fixes the tension

A string of total length $9a$ passes through a smooth ring R . Mass $2m$ hangs at A ; mass m moves in a horizontal circle centred at A .

$$T = 2mg, \quad T \cos \theta = mg \quad \Rightarrow \quad \cos \theta = \frac{1}{2}.$$

If $RB = L$, geometry gives $RA = L/2$ and $L + L/2 = 9a$, so $L = 6a$.

$$\omega^2 = \frac{g}{L \cos \theta} = \frac{g}{3a}.$$

Use equilibrium of the hanging mass and the total-length constraint.

A bowl radius and an orbit radius are different

On the smooth inside of a hemisphere of radius a , let its normal make angle θ with upward vertical.

$$r = a \sin \theta, \quad N \cos \theta = mg, \quad N \sin \theta = m(a \sin \theta) \omega^2.$$

Therefore

$$\omega^2 = \frac{g}{a \cos \theta}.$$

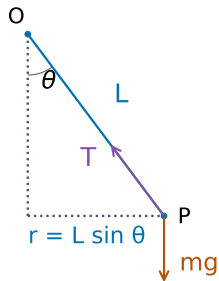
If cosine changes from $4/5$ to $5/13$, then

$$\frac{\omega_1}{\omega_2} = \sqrt{\frac{5/13}{4/5}} = \frac{5}{2\sqrt{13}}.$$

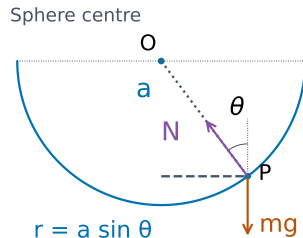
Take the square root of a squared-speed ratio.

The orbit radius comes from the geometry

Conical pendulum



Inside a bowl



A conical pendulum uses $L \sin \theta$; a bowl uses $a \sin \theta$.

A cone needs its own radius relation

A smooth cone has vertex downwards and semi-vertical angle α . At height h ,

$$r = h \tan \alpha.$$

With gravity and contact alone,

$$N \sin \alpha = mg, \quad N \cos \alpha = mr\omega^2.$$

Combining gives

$$\omega^2 = \frac{g \cot^2 \alpha}{h}.$$

Resolve the normal upwards and inwards; do not copy a bowl's geometry.

Worked: add a string through the cone vertex

The moving mass m is linked through a smooth vertex hole to a hanging mass m . Let $\tan \alpha = 4/3$, so $r = 4h/3$.

$$T = mg, \quad N \left(\frac{4}{5} \right) = mg + T \left(\frac{3}{5} \right) \Rightarrow N = 2mg.$$

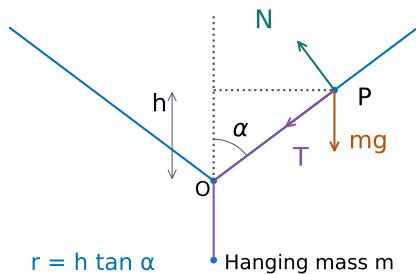
Radially,

$$N \left(\frac{3}{5} \right) + T \left(\frac{4}{5} \right) = mr\omega^2 \Rightarrow \omega^2 = \frac{3g}{2h}.$$

The tension pulls downwards as well as inwards, so it increases the normal reaction.

Cone contact and string tension have different directions

Normal: up and in; tension: down and in.



Use the hanging mass to find tension before resolving forces on the orbiting mass.

Measure vertical-circle height from a clear reference

Take θ from the **downward vertical** and measure height above the bottom:

$$x = r \sin \theta, \quad h = r(1 - \cos \theta).$$

position	bottom	side	top
θ	0	$\pi/2$	π
height	0	r	$2r$

An angle from upward vertical gives a different height expression. Mark the reference direction.

Energy gives speed at a height, not the constraint force

A light inextensible string or fixed smooth surface exerts a force perpendicular to velocity, so it does no work while the particle follows the circle.

$$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgr(1 - \cos \theta),$$

$$v^2 = u^2 - 2gr(1 - \cos \theta).$$

Here u is bottom speed. Resistance is neglected.

A non-negative v^2 is necessary. It does not prove that tension or contact is physically possible.

The inward weight component changes sign

For a string pulling towards the centre,

$$T - mg \cos \theta = \frac{mv^2}{r}.$$

Combine this with energy:

$$\frac{T}{m} = \frac{u^2}{r} - 2g + 3g \cos \theta.$$

Thus

$$T_{\text{bottom}} = \frac{mu^2}{r} + mg, \quad T_{\text{top}} = m \left(\frac{u^2}{r} - 5g \right).$$

The top and bottom equations use different speeds.

Worked: speed and tension from a bottom launch

Let $u^2 = 6gr$. At $\cos \theta = 1/2$,

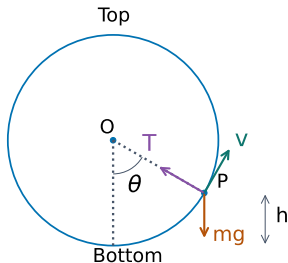
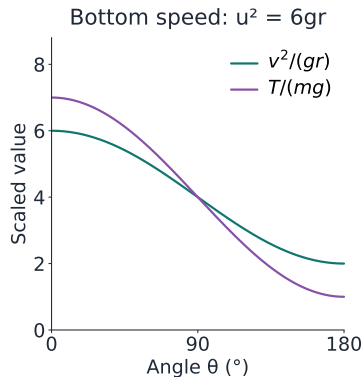
$$v^2 = 6gr - 2gr \left(1 - \frac{1}{2}\right) = 5gr,$$

$$T = \frac{m(5gr)}{r} + \frac{1}{2}mg = \frac{11}{2}mg.$$

Bottom tension is $7mg$; top tension is mg .

The particle is fastest at the bottom, where tension is also greatest.

Energy and tension vary differently around the loop



The plotted angle starts at downward vertical. Both curves use $u^2 = 6gr$.

A complete loop needs enough speed at the top

A string can pull but cannot push. The top is the least-tension position:

$$T_{\text{top}} + mg = \frac{mv_{\text{top}}^2}{r}, \quad T_{\text{top}} \geq 0 \quad \Rightarrow \quad v_{\text{top}}^2 \geq gr.$$

Energy over the $2r$ rise gives

$$u^2 = v_{\text{top}}^2 + 4gr \geq 5gr.$$

Equality is the limiting complete loop with zero tension at the top. The weaker energy-only test $u^2 \geq 4gr$ misses an earlier slack point.

Worked: the starting point need not be the bottom

Let the starting radius make angle ϕ from **upward vertical**. The climb to the top is $r(1 - \cos \phi)$.

$$v^2 \geq gr + 2gr(1 - \cos \phi) = gr(3 - 2 \cos \phi).$$

For $\cos \phi = 1/4$,

$$v_{\min}^2 = \frac{5}{2}gr.$$

At equality, bottom squared speed is $5gr$ and greatest tension is $6mg$.

Use the actual height difference and retain the non-zero limiting top speed.

Worked: recover a position from a tension ratio

In a complete loop, $T_b - T_t = 6mg$. If $T_b : T_t = 9 : 1$,

$$8T_t = 6mg, \quad T_t = \frac{3}{4}mg, \quad v_t^2 = \frac{7}{4}gr, \quad u^2 = \frac{23}{4}gr.$$

At a position where $v^2 = 4gr$, energy gives

$$4gr = \frac{23}{4}gr - 2gr + 2gr \cos \theta \quad \Rightarrow \quad \cos \theta = \frac{1}{8}.$$

Use two energy states; equal top and bottom speeds would be inconsistent.

A turning point and a slack point are different

Bottom squared speed $u^2 = gr$

$v = 0$ gives $\cos \theta = 1/2$. Tension is $mg/2 > 0$, so the particle turns back at 60° .

Bottom squared speed $u^2 = 3gr$

$T = 0$ gives $\cos \theta = -1/3$, about 109.5° . Here $v^2 = gr/3 > 0$: the moving particle leaves the circle when the string slackens.

Test which event occurs first. Stop using the circular-string equation after slackening.

An inner shell can push but cannot pull

With θ from downward vertical, the inward normal contact force satisfies

$$N - mg \cos \theta = \frac{mv^2}{r}, \quad N \geq 0.$$

Energy and the complete-loop condition are the same as for an inextensible string, replacing T by N .

A negative reaction would require the surface to pull. Test the zero-reaction boundary: if continuing needs $N < 0$, contact is lost. The limiting complete loop can have $N = 0$ at the top without separating.

Worked: use a reaction ratio to locate a rim

A particle starts downwards from the side of an inner shell with $u^2 = 2gr$. A rim is kr above centre height.

$$N_A = 2mg, \quad v_B^2 = (2 - 2k)gr, \quad \frac{N_B}{m} = \frac{v_B^2}{r} - gk = (2 - 3k)g.$$

If $N_B = N_A/4$, then $2 - 3k = 1/2$, giving $k = 1/2$.

To retain contact as far as that rim needs $u^2 \geq 3gr/2$; the energy-only test $u^2 \geq gr$ is too weak.

Worked: an outer surface reverses the normal sign

At the top of a smooth sphere, let squared speed be $gr/2$. With ϕ from upward vertical,

$$v^2 = \frac{1}{2}gr + 2gr(1 - \cos \phi).$$

The outside normal points outwards:

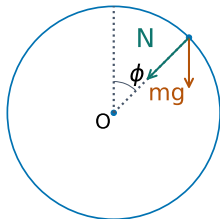
$$mg \cos \phi - N = \frac{mv^2}{r} \quad \Rightarrow \quad N = mg \left(3 \cos \phi - \frac{5}{2} \right).$$

Contact ends at $\cos \phi = 5/6$, with $v^2 = 5gr/6 > 0$.

Set the reaction to zero, not the speed.

Inner and outer surfaces have opposite normal directions

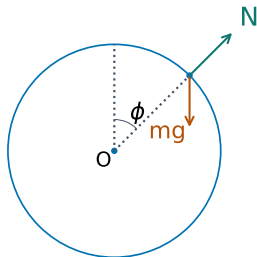
Inside



$$N + mg \cos \phi = mv^2/r$$

ϕ measured from upward vertical

Outside



$$mg \cos \phi - N = mv^2/r$$

ϕ measured from upward vertical

Here both diagrams measure ϕ from upward vertical; write the inward force sum from the arrows.

Release gives a tangent velocity, then a projectile path

At the slack point for $u^2 = 3gr$, $\cos \theta = -1/3$ and $\sin \theta = 2\sqrt{2}/3$.

$$(x, y) = \left(\frac{2\sqrt{2}r}{3}, \frac{r}{3} \right), \quad v^2 = \frac{gr}{3}.$$

On the upward journey along the right side,

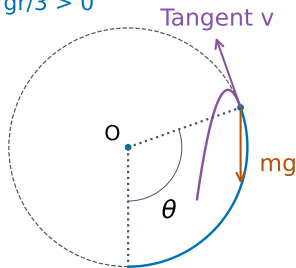
$$(v_x, v_y) = \left(-\frac{v}{3}, \frac{2\sqrt{2}v}{3} \right).$$

Position and velocity stay continuous at release. Gravity curves the later flight; the straight tangent gives only its initial direction.

A slack string releases a moving particle

Slack: $\cos \theta = -1/3$

$v^2 = gr/3 > 0$



Blue: circular path; purple: initial free flight.

The purple curve shows the initial free flight, before any later constraint or collision.

The hammer starts its flight along a tangent



The wire supplies an inward force during the turn. At release, velocity is tangent and gravity then bends the flight.

Worked: fly across the opening of a shell

The rim is at height $r/2$ above the centre. The horizontal gap is $\sqrt{3}r$ and release velocity is $(-v/2, \sqrt{3}v/2)$.

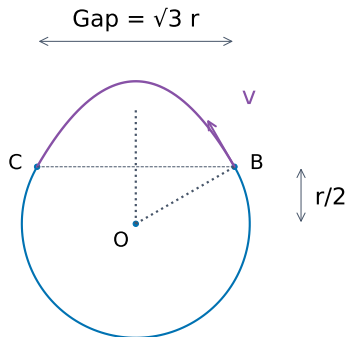
$$t = \frac{\sqrt{3}v}{g}, \quad \sqrt{3}r = \frac{v}{2}t \quad \Rightarrow \quad v^2 = 2gr.$$

From side level to the rim, energy gives $u^2 = v^2 + gr = 3gr$.

$$N_B = m \left(\frac{v^2}{r} - \frac{g}{2} \right) = \frac{3}{2}mg > 0.$$

The shell ends while contact is still possible; its open rim creates the release.

The rim geometry fixes the launch direction



A positive reaction may remain at an open rim.

Use the non-zero same-height return time, then reconnect the release speed to circular-motion energy.

Choose the first event, then the next model

- 1 Identify the orbit radius, actual forces and angle reference.
- 2 For a steady horizontal circle, resolve vertically and inwards.
- 3 For a lossless vertical circle, combine energy and inward dynamics.
- 4 Compare a turning point, slack string, loss of contact and the end of the surface.

After release use projectile motion. Across an impact use a separate collision argument; do not carry the lossless circular energy equation through it automatically.

Retrieval: radius, force feasibility and release

- 1 A string of length 0.8 m carries a vertical loop. With $g = 10\text{ m/s}^2$, find the least bottom speed and limiting top speed.
- 2 A hemispherical bowl has radius 2 m . Its normal makes 60° with vertical. Find the orbit radius and angular speed.
- 3 Does a released particle travel forever along the straight tangent if gravity still acts?

State the geometry and constraint before substituting into a formula.

Retrieval answers: combine geometry with the force model

- 1 $u_{\min} = \sqrt{5gr} = 2\sqrt{10} \text{ m/s}$ and $v_{\text{top}} = \sqrt{gr} = 2\sqrt{2} \text{ m/s}$. The limiting top tension is zero.
- 2 $r = 2 \sin 60^\circ = \sqrt{3} \text{ m}$ and $\omega^2 = g/(2 \cos 60^\circ) = 10$, so $\omega = \sqrt{10} \text{ rad/s}$.
- 3 No. The velocity is tangent at release; gravity then produces a curved projectile path until another constraint acts.

The inward radial acceleration is v^2/r . At changing speed, the full acceleration also has a tangential component.

3

Elasticity

Why a jump needs a model

Start with the model

A bungee rope stores energy as it stretches. Its pull changes during the jump, so constant-acceleration formulae are usually unsuitable.

Hooke's law 胡克定律 models force proportional to extension within a stated range. Real ropes need not follow it over a whole jump. Here use ideal light **elastic** 弹性 strings and springs with no energy loss unless stated.

Measure extension from natural length

Start with the model

The **natural length** 自然长度 L is the unloaded length. If the actual length is ℓ , the **extension** 伸长量 is $x = \ell - L$, not ℓ . A taut elastic string has tension $T = kx$.

With **modulus of elasticity** 弹性模量 λ , write $k = \lambda/L$, so $T = \lambda x/L$. For a spring the signed restoring force opposes displacement from its natural length.

Modulus and stiffness have different units

Start with the model

Stiffness 劲度系数 k is measured in $\frac{\text{N}}{\text{m}}$; this string model's λ is measured in newtons. It includes material and cross-section effects; it is not Young modulus, whose unit is pascals.

Cut a uniform string into two equal pieces: each has the same λ but half the natural length, so each piece has twice the stiffness. Do not keep k fixed when changing L in this way.

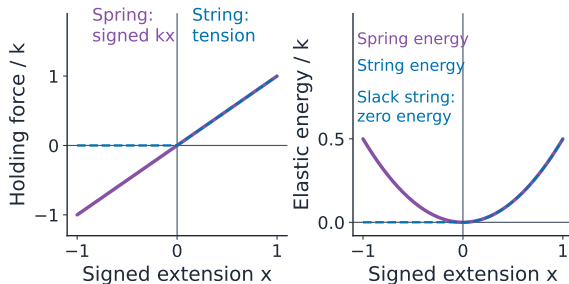
A string and a spring have different branches

Start with the model

A string pulls only when **taut** 绷紧. For $\ell \leq L$ it is **slack** 松弛: $T = 0$ and its stored elastic energy is zero. A negative value of $\ell - L$ is not a negative string tension.

An ideal spring can pull in extension and push in **compression** 压缩. Its signed x may be negative, while the force magnitude is $k|x|$. Continue to use the spring law only within the stated model range.

A string has a slack branch; a spring can store compression energy



Both obey $T = \frac{\lambda x}{l}$, but a string with $x < 0$ is slack ($T = 0$); a compressed spring pushes back and still stores $\frac{\lambda x^2}{2l}$.

Store the area under the graph

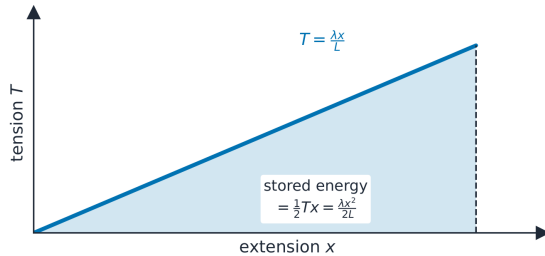
Start with the model

Stretching slowly from natural length does work equal to the triangular area under the force–extension line. The stored **elastic potential energy** 弹性势能 is

$$E_e = \frac{1}{2} kx^2 = \frac{\lambda x^2}{2L}.$$

For a spring this also applies in compression. For a string use it only when $x \geq 0$; the slack branch has zero energy. Doubling extension doubles tension but multiplies energy by four. A proof of the energy formula is not required.

Force rises linearly with extension; the triangular area is stored energy



$$T = \frac{\lambda}{L}x \cdot E = \frac{1}{2}Tx = \frac{\lambda x^2}{2L} \quad (\text{area under graph})$$

Elastic potential energy $= \frac{1}{2} \cdot \frac{\lambda x}{l} \cdot x = \frac{\lambda x^2}{2l}$: the area under the force–extension line.

Use force and energy on the same string

Start with the model

A string has $L = 2$ m, $\lambda = 50$ N and extension 0.5 m. Its actual length is 2.5 m. First use $T = \lambda x/L$, giving $T = 50(0.5)/2 = 12.5$ N.

For stored energy use $E_e = \lambda x^2/(2L)$, giving $E_e = 50(0.5)^2/4 = 3.125$ J. Force and energy have different units; do not substitute actual length for extension.

Infer a natural length from measurements

Start with the model

A string has tension 6 N at length 1.3 m and 10 N at length 1.5 m. Since $T = k(\ell - L)$, the gradient is $k = (10 - 6)/(1.5 - 1.3) = 20 \frac{\text{N}}{\text{m}}$.

Then $L = \ell - T/k = 1.3 - 6/20 = 1 \text{ m}$ and $\lambda = kL = 20 \text{ N}$. Check the other measurement: $20(1.5 - 1) = 10 \text{ N}$. The line's horizontal intercept gives natural length, not zero length.

Find a hanging equilibrium

Start with the model

A mass m hangs at rest from one vertical string attached above it. **Equilibrium** 平衡 requires $T = mg$, so its equilibrium extension is $e = mg/k = mgL/\lambda$.

A 0.6 kg mass on $k = 30 \frac{\text{N}}{\text{m}}$ has $e = 0.2 \text{ m}$ when $g = 10 \frac{\text{m}}{\text{s}^2}$. If the mass is moving through that position, acceleration is momentarily zero; speed need not be zero.

Release at natural length is not slow loading

Start with the model

Attach a mass to the lower end at natural length and release it from rest, with no resistance. After a downward extension x , energy gives $mgx = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

At maximum stretch $v = 0$, so the non-zero root is $x = 2mg/k = 2e$. Maximum speed occurs earlier at $x = e$, where $mg - kx = 0$; $v_{\max}^2 = g^2 m/k$. Slow loading instead ends at e and removes energy from the system.

A horizontal spring can pass natural length

Start with the model

A 2 kg particle on a smooth horizontal line is attached to a spring with $k = 50 \frac{\text{N}}{\text{m}}$. Release it from rest at extension 0.4 m. Its initial energy is $\frac{1}{2}k(0.4)^2 = 4 \text{ J}$.

At natural length the energy is all kinetic: $\frac{1}{2}mv^2 = 4$, so $v = 2 \frac{\text{m}}{\text{s}}$. The particle continues into compression and first stops at $x = -0.4 \text{ m}$, provided no other physical constraint intervenes.

A horizontal string changes model at natural length

Start with the model

Replace that spring by an elastic string with the same k and initial extension. It has the same speed at natural length, but beyond that point it has no tension and no elastic energy.

On the smooth horizontal surface the particle then moves at constant velocity until another constraint acts. Do not predict spring-like compression and reversal. Specify geometry before following a particle past its anchor.

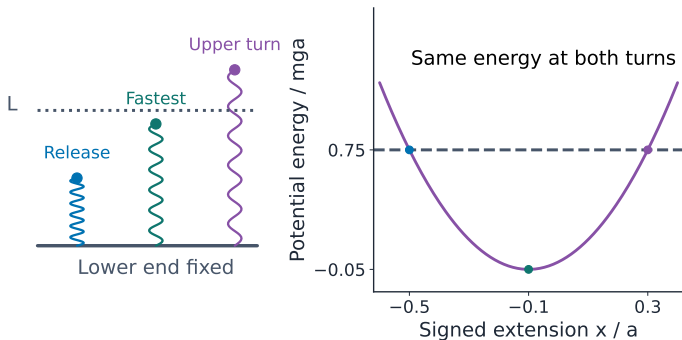
Release a compressed vertical spring

Start with the model

A spring has lower end fixed, natural length a , and $\lambda = 10mg$. A mass m is attached to its top and released from rest with compression $a/2$. Let x be signed extension upwards. Relative to natural length, total potential energy is $U = 5mgx^2/a + mgx$.

Initially $x = -a/2$ and $U = 3mga/4$. At the next rest point, put $z = x/a$: $5z^2 + z = 3/4$. The roots are $-1/2$ and $3/10$. The first is the initial state; the maximum extension is $3a/10$.

The two turning points share an energy; maximum speed lies between them



At both rest points $v = 0$, so equate the total energy there: the roots are the two turning points. Speed is greatest where the resultant force is zero.

Maximum speed is not a turning point

Start with the model

In that spring example, $-kx - mg = 0$ at $x = -a/10$. Potential energy there is $-mga/20$, so the maximum kinetic energy is $3mga/4 + mga/20 = 4mga/5$.

Hence $v_{\max}^2 = 8ag/5$. At maximum extension $3a/10$, the mass is instantaneously at rest but the acceleration is $-4g$, downwards. A **turning point** 转折点 has $v = 0$; it need not have zero resultant force.

Record both end states before writing energy

Start with the model

Choose one height reference. Write kinetic energy, gravitational potential energy mgh , and every active elastic energy at the initial and final positions. With no resistance their total is conserved; with friction include its negative work.

Use actual lengths to find each extension independently. A string at or below natural length contributes zero energy, not $\lambda(\ell - L)^2/(2L)$. If a path crosses a slack boundary, use the right force law in each interval.

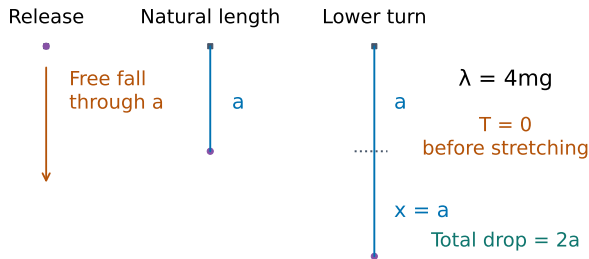
Free fall can come before stretching

Start with the model

A mass m is released from rest at its anchor, on a vertical string with natural length a and $\lambda = 4mg$. It first falls freely through a ; the string then begins stretching. At later extension x , energy gives $mg(a+x) = \frac{1}{2}mv^2 + 2mgx^2/a$.

At the lowest point let $z = x/a$. Then $2z^2 - z - 1 = 0$, so $z = 1$; the negative root is outside the stretched branch. The total drop is $2a$. Maximum speed is at $x = a/4$, giving $v_{\max}^2 = 9ag/4$.

Free fall through the natural length comes before the string stretches



A slack string stores no compressive energy.

Work in two stages: free fall under gravity through the natural length l , then energy conservation including $\frac{\lambda x^2}{2l}$ once the string is taut.

Removing a mass changes the equilibrium

Start with the model

A vertical string has natural length a and $\lambda = 4mg$. Masses m and $3m$ first hang at equilibrium, so the extension is a . Remove $3m$ without an impulse. The remaining mass starts at rest at length $2a$.

On its taut branch put $z = x/a$ and take height zero at natural length. Energy becomes $\frac{1}{2}mv^2 + mga(2z^2 - z) = mga$. To find the first position where $v^2 = ag$, solve $2z^2 - z = 1/2$. The valid root gives $\ell = (5 + \sqrt{5})a/4$; the other root has negative extension.

A rejected root does not describe the slack flight

Start with the model

At natural length in that example, $v^2 = 2ag$ upwards. After that the string is slack; gravity alone acts until the next event. The taut-string quadratic cannot describe this phase.

After rising a further distance s , use $v^2 = 2ag - 2gs$. The later position with $v^2 = ag$ is $s = a/2$, so the length is $a/2$. The negative-extension root from the old formula gives a different value because it invents compressive string energy.

Resolve along an inclined plane

Start with the model

A mass m rests on a smooth plane inclined at β to horizontal, attached to an anchor upslope by a string along the plane. Normal reaction is $mg \cos \beta$; downslope weight is $mg \sin \beta$.

Equilibrium requires $kx = mg \sin \beta$. With $m = 1 \text{ kg}$, $g = 10 \frac{\text{m}}{\text{s}^2}$, $\sin \beta = 3/5$ and $k = 20 \frac{\text{N}}{\text{m}}$, $x = 0.3 \text{ m}$. Friction, if present, must be included with its actual direction.

Follow a string past its anchor

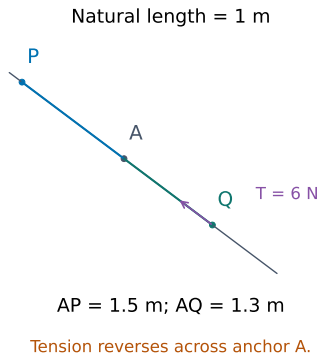
Start with the model

Use that smooth plane and particle with natural length 1 m. Release from rest 1.5 m above the anchor. Initially the string pulls downslope; it then goes slack around the anchor and becomes taut again below it. Maximum speed is below the anchor at extension 0.3 m, where tension balances the 6 N downslope weight.

Total travel to that point is $1.5 + 1 + 0.3 = 2.8$ m. Work–energy gives

$\frac{1}{2}mv^2 = \frac{1}{2}k(0.5)^2 + mg\sin\beta(2.8) - \frac{1}{2}k(0.3)^2 = 18.4$ J. Thus $v_{\max}^2 = 36.8 \frac{\text{m}^2}{\text{s}^2}$. Before this point the downslope resultant stays positive.

Tension reverses direction across an inclined-plane anchor



Above the anchor a taut string pulls down the slope; within its natural length it is slack; below, it pulls up the slope. Treat each stage in turn.

Friction removes mechanical energy

Start with the model

On the same incline let $\mu = 1/4$, and release the particle from natural length below the anchor. For the first downslope motion, normal reaction is 8 N and friction is 2 N upslope. Net driving force before elasticity is $6 - 2 = 4$ N.

Work-energy gives $\frac{1}{2}mv^2 = 4x - 10x^2$. The first maximum extension is 0.4 m, and maximum speed occurs at $x = 0.2$ m with $v_{\max}^2 = 0.8 \frac{\text{m}^2}{\text{s}^2}$. Recheck friction's direction on any return journey; this equation describes the first descent.

Two opposing strings need two extensions

Start with the model

Fix anchors 6 m apart on a smooth horizontal line. Both strings have natural length 2 m; the left has $\lambda = 20$ N and the right 40 N. At coordinate q from the left, both are taut only when $2 < q < 4$.

Balance $10(q - 2) = 20(4 - q)$ to find equilibrium $q = 10/3$ m. Total elastic energy is $U = 5(q - 2)^2 + 10(4 - q)^2$. A 2 kg mass released from $q = 3$ m has $U = 15$ J; at equilibrium $U = 40/3$ J, giving $v_{\max}^2 = 5/3 \frac{\text{m}^2}{\text{s}^2}$. The other turn is $q = 11/3$ m, still in the same branch.

Series strings carry a common tension

Start with the model

For two strings in **series** 串联 with a light unloaded joining point, tensions are equal and extensions add. Use $x_i = T/k_i$ for each. The equivalent stiffness satisfies $1/k_{\text{eq}} = 1/k_1 + 1/k_2$.

Natural lengths 1, 2 m and moduli 20, 30 N give $k_1 = 20$, $k_2 = 15 \frac{\text{N}}{\text{m}}$. Under 6 N tension their extensions are 0.3, 0.4 m. Total extension is 0.7 m; energy is $\frac{1}{2} T(x_1 + x_2) = 2.1 \text{ J}$.

Parallel springs share an extension

Start with the model

Two springs in **parallel** 并联 between the same moving and fixed supports have the same extension if their natural lengths and geometry match. Their forces add: $T = (k_1 + k_2)x$, so $k_{\text{eq}} = k_1 + k_2$.

For $k_1 = 20$ and $k_2 = 30 \frac{\text{N}}{\text{m}}$ at $x = 0.2$ m, the forces are 4 and 6 N; total force is 10 N and total energy 1 J. Equal extensions do not require equal tensions.

Series elements share tension; matching parallel elements share extension

Series

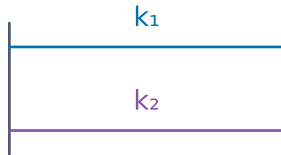


Same tension T

Extensions add

Light, unloaded join

Parallel



Same extension x

Forces add

In series one tension runs through both and the extensions add; in parallel the extensions are equal and the tensions add, though they need not be equal.

A lower string pulls downwards

Start with the model

Two vertical anchors A above B are $14a$ apart. A particle of mass κm is joined to A by a string of natural length $2a$, and to B by one of natural length $4a$. Both moduli are $4mg$. Let $BP = ya$; then upper and lower extensions are $(12 - y)a$ and $(y - 4)a$.

Both are taut for $4 < y < 12$. Their tensions are $2mg(12 - y)$ upwards and $mg(y - 4)$ downwards. Equilibrium gives $2(12 - y) - (y - 4) - \kappa = 0$, so $y = (28 - \kappa)/3$. Do not add both tensions in the same direction.

Two rest positions can determine the mass

Start with the model

Release the particle from rest at $y = 12$, and suppose its first lower turning point is $y = 4$. Take gravitational potential zero at B . Total potential energy divided by mga is $V = (12 - y)^2 + \frac{1}{2}(y - 4)^2 + \kappa y$.

The endpoints give $32 + 12\kappa = 64 + 4\kappa$, hence $\kappa = 4$. Between them both extensions are nonnegative. With $\kappa = 4$, $V - 80 = \frac{3}{2}(y - 4)(y - 12) \leq 0$, so the intermediate kinetic energy is feasible.

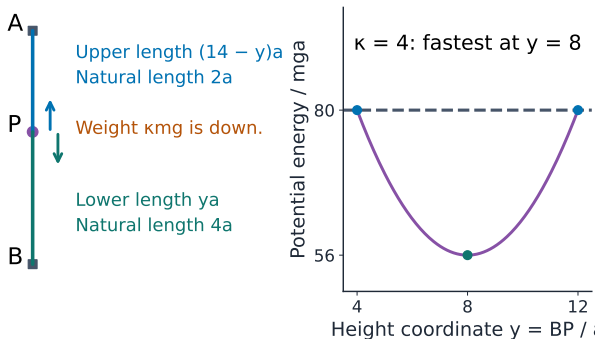
Use both strings to find maximum speed

Start with the model

With $\kappa = 4$, equilibrium in that example is $y = 8$. At release the total energy is $80mga$; at $y = 8$ the potential energy is $(16 + 8 + 32)mga = 56mga$.

Therefore $\frac{1}{2}(4m)v_{\max}^2 = 24mga$, giving $v_{\max}^2 = 12ag$. Use the actual moving mass $4m$ in kinetic energy; the symbol m in the moduli is only the reference mass.

Two vertical strings pull in opposite directions and store separate energies



Give each string its own modulus, natural length and extension, add their elastic energies separately, and use the actual moving mass in $\frac{1}{2}mv^2$.

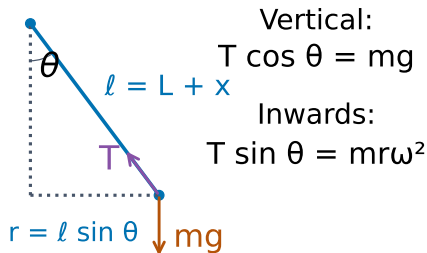
An elastic conical pendulum stretches

Start with the model

A mass 0.6 kg rotates steadily with a string of natural length 1.5 m and modulus 30 N. Let its angle to vertical have $\cos \theta = 4/5$, $\sin \theta = 3/5$, and take $g = 10 \frac{\text{m}}{\text{s}^2}$. Vertical balance gives $T \cos \theta = mg$, hence $T = 7.5 \text{ N}$.

Hooke's law gives $x = TL/\lambda = 0.375 \text{ m}$, so actual length $\ell = 1.875 \text{ m}$ and orbit radius $r = \ell \sin \theta = 1.125 \text{ m}$. The radial equation $T \sin \theta = mr\omega^2$ gives $\omega^2 = 20/3 \frac{\text{rad}^2}{\text{s}^2}$. Use stretched length in the geometry.

The stretched string length determines the conical orbit geometry



The radius is $r = (\ell + x) \sin \theta$. Resolve: $T \cos \theta = mg$ vertically and $T \sin \theta = mr\omega^2$ horizontally, with $T = \frac{\lambda x}{l}$.

A positive length need not give a possible orbit

Start with the model

For a non-zero elastic conical orbit, $T = m\ell\omega^2$ and $T = \lambda(\ell - L)/L$. Therefore $\ell = \lambda L/(\lambda - mL\omega^2)$. Its denominator must be positive. Vertical balance also needs $\ell \cos \theta = g/\omega^2$, with $0 < \cos \theta < 1$.

Take $L = 1$ m, $\lambda = 20$ N, $m = 1$ kg and $g = 10 \frac{\text{m}}{\text{s}^2}$. If $\omega^2 = 5$, $\ell = 4/3$ m but the required vertical height is 2 m: impossible. If $\omega^2 = 10$, $\ell = 2$ m and $r = \sqrt{3}$ m: possible. At $\omega^2 = 20$ no finite length solves the equations.

Select a physically valid event

Start with the model

Mark the initial state, natural-length boundaries, equilibrium positions and candidate turning points. State the active model in each interval: a taut string pulls, a slack string has no force, and a spring may compress.

Find roots, then check lengths, nonnegative string tensions, nonnegative kinetic energy and the order of events. Maximum speed requires zero tangential resultant within a smooth branch, or comparison with a boundary; instantaneous rest requires $v = 0$.

Connect force, energy and geometry

Start with the model

First find $x = \ell - L$ for every element. Hooke's law gives force; $\lambda x^2/(2L)$ gives energy on its valid branch. Resolve directions before adding forces, and keep one height reference throughout energy calculations.

Use original data to practise both one-element and connected systems. In circular motion, stretching determines actual length before orbit geometry. Check each final answer against its branch and units.

Independent check 1

Work independently before revealing the model

A string has tension 8 N at actual length 1.4 m and 14 N at 1.7 m. Find its natural length and modulus, then predict the tension at 1.5 m.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 1

Two readings of the line give the stiffness, then the natural length:

$$k = \frac{\Delta T}{\Delta \ell} = \frac{14 - 8}{1.7 - 1.4} = 20 \text{ N m}^{-1}$$

$$L = 1.4 - \frac{T}{k} = 1.4 - \frac{8}{20} = 1 \text{ m}, \quad \lambda = kL = \mathbf{20 \text{ N}}$$

$$T(1.5) = k(\ell - L) = 20(1.5 - 1) = \mathbf{10 \text{ N}}$$

Uses the force-length gradient. Finds the zero-force intercept and checks a prediction.

Independent check 2

Work independently before revealing the model

A vertical spring has its lower end fixed, natural length a and modulus $8mg$. A mass m is attached to its upper end and released at rest with compression $a/2$. Find its other turning extension.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 2

Energy, with upward extension $z = x/a$ (divide through by mga):

$$\frac{E}{mga} = \underbrace{4z^2}_{\text{elastic}} + \underbrace{z}_{\text{gravity}}, \quad \text{start } z = -\frac{1}{2} : \quad \frac{E}{mga} = \frac{1}{2}$$

$$4z^2 + z = \frac{1}{2} \Rightarrow (4z - 1)(2z + 1) = 0$$

$$z = -\frac{1}{2} \text{ (start, reject) } \quad \text{or} \quad z = \frac{1}{4} \Rightarrow x = \mathbf{a/4}$$

Includes gravity with the chosen upward coordinate. Rejects the initial root for the requested later event.

Independent check 3

Work independently before revealing the model

A mass m is at rest at extension a on a vertical string of natural length a and modulus $3mg$, just after another mass has been removed without an impulse. It then moves under gravity and the string force. Find its squared speed at natural length and the later actual length where squared speed is $ag/2$, before its first up-per turn.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 3

At natural length all the elastic energy $\frac{1}{2}mga$ is kinetic; then the string is slack:

$$v^2 = ag \quad (\text{upwards})$$

$$\text{slack: } v^2 = ag - 2gs = \frac{1}{2}ag \Rightarrow s = \frac{a}{4} \Rightarrow \text{length} = \mathbf{3a/4}$$

$$\text{upper turn: } v^2 = 0 \Rightarrow s = \frac{a}{2}, \text{ length } \frac{a}{2} < \frac{3a}{4} \quad \checkmark$$

Uses gravity-only motion while slack. Checks the target lies before the upper turning point.

Independent check 4

Work independently before revealing the model

Anchors 5 m apart are connected to a 1 kg particle on a smooth horizontal line by strings of natural length 2 m each. The left and right stiffnesses are 10 and 30 N/m. Find equilibrium q from the left anchor and maximum squared speed when released from rest at $q = 2.5$ m. Check the branch.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 4

Both strings taut for $2 < q < 3$; speed is greatest at equilibrium:

$$10(q - 2) = 30(3 - q) \Rightarrow q = \frac{11}{4} \text{ m}$$

$$E_{\min} = \frac{1}{2}(10)\left(\frac{3}{4}\right)^2 + \frac{1}{2}(30)\left(\frac{1}{4}\right)^2 = \frac{15}{4} \text{ J}$$

$$\frac{1}{2}mv_{\max}^2 = 5 - \frac{15}{4} \Rightarrow v_{\max}^2 = \mathbf{5/2} \text{ m}^2\text{s}^{-2}$$

Other turn: $q = 3 \text{ m}$, where the right string just reaches natural length.

Uses separate extensions. Checks the endpoint branch as well as equilibrium.

Independent check 5

Work independently before revealing the model

Vertical anchors A above B are $12a$ apart. A particle of mass κm is joined to A by a string of natural length $2a$ and stiffness $3mg/a$, and to B by a string of natural length $3a$ and stiffness mg/a . It is released from rest at $BP = 10a$ and first turns below at $BP = 3a$. Find κ and check that intermediate kinetic energy is feasible.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 5

Potential energy over mga (both strings and gravity):

$$V(y) = \frac{3}{2}(10 - y)^2 + \frac{1}{2}(y - 3)^2 + \kappa y$$

$$V(3) = V(10) : \frac{49}{2} + 10\kappa = \frac{147}{2} + 3\kappa \Rightarrow \kappa = \mathbf{7}$$

$$\text{check: } V - \frac{189}{2} = 2(y - 3)(y - 10) \leq 0 \quad \text{for } 3 \leq y \leq 10$$

So kinetic energy is never negative between the rest points.

Includes both elastic energies and gravity. Checks the path between rest endpoints.

Independent check 6

Work independently before revealing the model

A 0.5 kg mass rotates steadily with a string of natural length 1 m and modulus 20 N, making an angle to vertical with $\cos\theta = 4/5$ and $\sin\theta = 3/5$. Take $g = 10 \text{ m/s}^2$. Find tension, actual length, orbit radius and squared angular speed.

Draw the geometry, choose the force or energy model, and check its valid branch.

Model and check 6

Vertical balance, Hooke's law, then the horizontal circle:

$$\begin{aligned}T &= \frac{mg}{\cos \theta} = \frac{25}{4} \text{ N}, & x &= \frac{TL}{\lambda} = \frac{5}{16} \text{ m} \\ \ell &= L + x = \frac{21}{16} \text{ m}, & r &= \ell \sin \theta = \frac{63}{80} \text{ m} \\ T &= m\ell\omega^2 \Rightarrow \omega^2 = \mathbf{200/21} \text{ rad}^2\text{s}^{-2}\end{aligned}$$

Finds tension from vertical balance. Uses stretched length and then the horizontal orbit radius.

4

Variable forces

Why a fast object slows differently

Start with the force model

A bicycle usually feels more air resistance when it moves faster. Its **drag** 阻力 may depend on **speed** 速率. A different force might depend on **position** 位置 or time; these are different models.

A **variable force** 变力 can produce changing acceleration. Start with Newton's law and set up a **differential equation** 微分方程 for the motion; constant-acceleration formulae need not apply.

Choose a positive direction first

Start with the force model

Take displacement x positive to the right, and write **velocity** 速度 $v = dx/dt$. It can be negative. Speed is $|v|$, so a negative velocity does not mean a negative speed.

Add signed forces on the particle: $m dv/dt = \sum F$. For motion to the right, resistance is negative. After reversal, ordinary resistance reverses too; a prescribed force towards a fixed point follows its own direction rule.

Change the variable using the chain rule

Start with the force model

If velocity is written as a function of position along a branch of the motion, the **chain rule** 链式法则 gives

$$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}.$$

This is useful for finding v at a position without first finding time. At a **turning point** 转折点 $v(x)$ may have a vertical tangent; do not assume $a = 0$ merely because $v = 0$. Use the original force law there.

Choose the variables for the requested answer

Start with the force model

For a time-dependent force, $m dv/dt = F(t)$ often gives $v(t)$ directly. For a position-dependent force, $mv dv/dx = F(x)$ often gives $v(x)$ directly.

If force depends on velocity, either form may help: use dv/dt for a time question and $v dv/dx$ for a distance question. These are choices of variables for the same Newton equation, not different laws.

Choose acceleration variables to match the force model and requested result

Start with signed Newton equation: $ma = \Sigma F$

Force uses time



$$a = dv/dt$$

Find $v(t)$, then $x(t)$

Force uses position



$$a = v dv/dx$$

Find $v(x)$, choose its sign

Force uses velocity



Either form

Choose time or distance

Separate → integrate → use the initial state
Check force signs and the valid branch.

Force in t : use $a = \frac{dv}{dt}$. Force in x : use $a = v \frac{dv}{dx}$. Force in v : $v \frac{dv}{dx}$ if a distance is

The original acceleration forms connect time and position

which form of acceleration?

force depends on **time** t

$$a = \frac{dv}{dt}$$

gives v as a function of t

force depends on **position** x

$$a = v \frac{dv}{dx}$$

gives v as a function of x

both come from $a = \frac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

The chain rule connects both forms of acceleration.

Use each stated condition

Start with the force model

A **separable differential equation** 可分离微分方程 can be rearranged so each side contains one variable and its differential. **Separate** 分离 the variables, **integrate** 积分, include a constant, then use an **initial condition** 初始条件 to determine it.

A condition at a specified position is often called a **boundary condition** 边界条件. Here use only **separable** 可分离 equations and Pure Mathematics 3 calculus. Check any value divided out: division by a factor that can be zero may remove a constant solution.

A force depending on time can reverse the velocity

Start with the force model

A 2 kg particle has resultant force $6t$ N to the right. At $t = 0$, $v = -1 \frac{\text{m}}{\text{s}}$ and $x = 0$.

Newton's law gives $2 dv/dt = 6t$, so $v = \frac{3}{2}t^2 + C$; the initial velocity gives $C = -1$.

Now integrate $dx/dt = -1 + \frac{3}{2}t^2$. The initial position gives $x = -t + \frac{1}{2}t^3$. There are two integrations and two initial conditions. The particle initially moves left even though acceleration is to the right for $t > 0$.

Displacement and distance differ across a turn

Start with the force model

In that example, $v = 0$ first at $t_* = \sqrt{2/3}$ s. Its position is $x_* = -2t_*/3$ m. At $t = 2$ s, displacement is $x = 2$ m.

Total distance is the leftward distance $-x_*$ plus the rightward distance $2 - x_*$. Thus distance is $2 + 4\sqrt{2/3}/3$ m, greater than the displacement. Integrating signed velocity gives displacement; split at each reversal to find distance.

A position-dependent force links work to speed

Start with the force model

With $F = mv dv/dx$, separation gives $mv dv = F(x) dx$. Integrating between two known states gives

$$\frac{1}{2}m(v^2 - u^2) = \int_{x_0}^x F(s) ds.$$

The signed area under a force–position graph is **work** 功. It equals the change in kinetic energy for the resultant force. This relation determines squared speed; use the direction of motion to choose the sign of velocity.

Work through a polynomial force

Start with the force model

A particle of mass 8 kg starts at $x = 0$ with $v = 1 \frac{\text{m}}{\text{s}}$. For its outward motion the force is $(x^3 + 4x)$ N. Newton's law gives $8v dv/dx = x^3 + 4x$.

Separate and integrate: $4v^2 = x^4/4 + 2x^2 + C$. Initially $C = 4$, so $v^2 = x^4/16 + x^2/2 + 1 = (x^2/4 + 1)^2$. The outward velocity is $v = x^2/4 + 1$; the negative square root would describe a different direction.

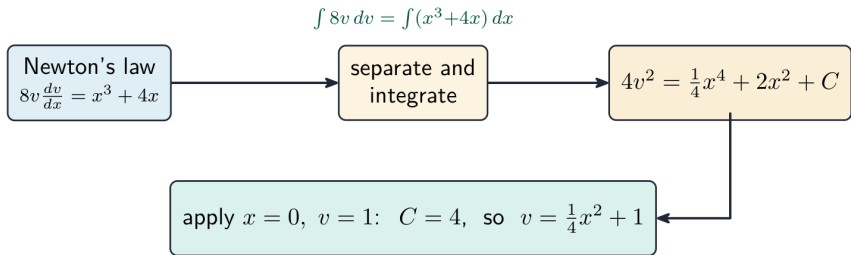
A second separation can give position in time

Start with the force model

For that outward branch, $dx/dt = 1 + x^2/4$. Separate as $dt = 4 dx/(x^2 + 4)$ and integrate: $t = 2 \tan^{-1}(x/2) + C$. The initial position gives $C = 0$.

Hence $x = 2 \tan(t/2)$ on the branch $0 \leq t < \pi$. The mathematical model predicts unbounded position as $t \rightarrow \pi$ from below. Do not continue the periodic tangent formula across that boundary or assume a real system can sustain the force model indefinitely.

The original polynomial-force worked sequence



a position-dependent force becomes a **differential equation** in v and x

Use the initial state to find the integration constant, then select the velocity sign.

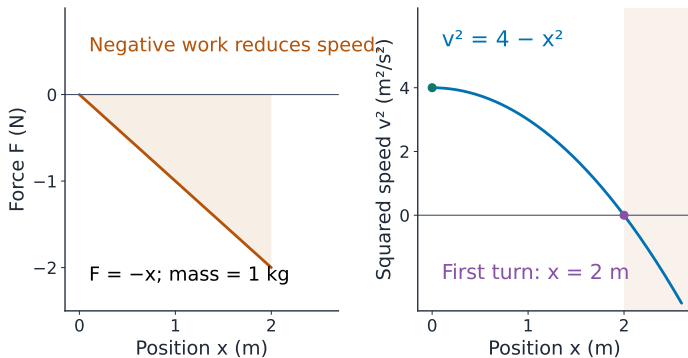
A turning root limits a position formula

Start with the force model

A 1 kg particle starts at $x = 0$ with velocity $2 \frac{\text{m}}{\text{s}}$, under force $-x$ N. Newton's law gives $v dv/dx = -x$, hence $v^2 = 4 - x^2$ from the initial state.

It first stops at $x = 2$ m, where acceleration is $-2 \frac{\text{m}}{\text{s}^2}$, so it turns back. Use $v = +\sqrt{4 - x^2}$ before that turn and $v = -\sqrt{4 - x^2}$ immediately afterwards. The region $|x| > 2$ m would require negative squared speed and is inaccessible at this energy.

Negative work reduces squared speed to zero at the first turning point



With a variable force, $\int F dx = \Delta \left(\frac{1}{2} m v^2 \right)$; set $v = 0$ to locate the first turning point.

Linear resistance gives an exponential speed

Start with the force model

A 2 kg particle moves horizontally with initial velocity $3 \frac{\text{m}}{\text{s}}$, opposed by resistance of magnitude $4v$ N while $v > 0$. Newton's law gives $2 dv/dt = -4v$, so $dv/v = -2 dt$.

Integration gives $\ln v = -2t + C$. The initial value gives $v = 3e^{-2t}$. It remains positive for every finite $t \geq 0$ and tends to zero as time grows. An exponential is not zero merely because a calculator rounds it to zero.

A limiting distance is not a finite stopping point

Start with the force model

Integrating $v = 3e^{-2t}$ with $x(0) = 0$ gives $x = \frac{3}{2}(1 - e^{-2t})$. Thus the particle travels towards a limiting displacement of 1.5 m.

It never reaches that limit in finite time under this ideal model. Equivalently $v = 3 - 2x$ on the moving branch, but putting $v = 0$ in that relation gives a limit, not a finite arrival time. Check the time law before describing a stop.

Quadratic resistance has a different distance limit

Start with the force model

Suppose horizontal motion satisfies $dv/dt = -v^2/2$ with $v(0) = 4$. Separation gives $-1/v = -t/2 + C$, so $v = 4/(1 + 2t)$ for $t \geq 0$.

With $x(0) = 0$, integration gives $x = 2 \ln(1 + 2t)$. Speed tends to zero, but displacement has no finite upper limit. A decreasing speed does not by itself imply a finite total distance.

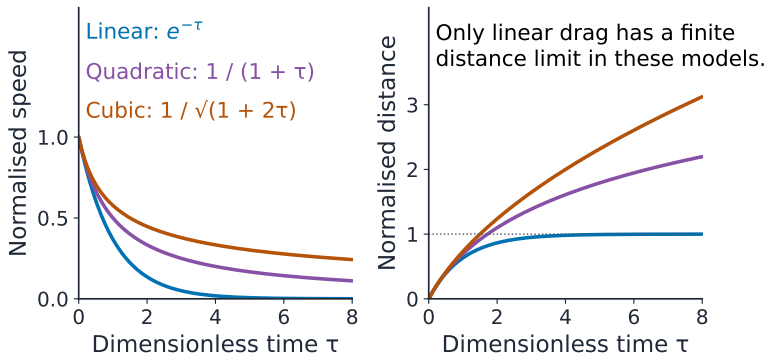
A cubic resistance gives another power law

Start with the force model

Suppose $dv/dt = -3v^3$ with $v(0) = 2$. Separate as $v^{-3} dv = -3 dt$. Integration and the initial value give $1/v^2 = 6t + 1/4$.

Choose the positive branch: $v = 2/\sqrt{1 + 24t}$. With $x(0) = 0$, integration gives $x = (\sqrt{1 + 24t} - 1)/6$. Both this model and the quadratic model slow forever while covering an unbounded distance; their rates of slowing differ.

Different resistance laws can give different limiting distances



Resistance kv : the particle travels only a finite distance $\frac{mu}{k}$. Resistance kv^2 : $v = ue^{-kx/m}$, so the distance grows without limit.

Falling with linear drag approaches a balance

Start with the force model

Take downward as positive. A mass m falling under gravity and linear resistance kv obeys $m dv/dt = mg - kv$. Its **terminal velocity** 终端速度 is $V = mg/k$, where the resultant is zero.

For $m = 2 \text{ kg}$, $k = 4 \frac{\text{Ns}}{\text{m}}$ and $g = 10 \frac{\text{m}}{\text{s}^2}$, $V = 5 \frac{\text{m}}{\text{s}}$. Released from rest, separation gives $v = 5(1 - e^{-2t})$ and downward distance $y = 5t - \frac{5}{2}(1 - e^{-2t})$.

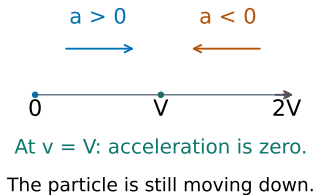
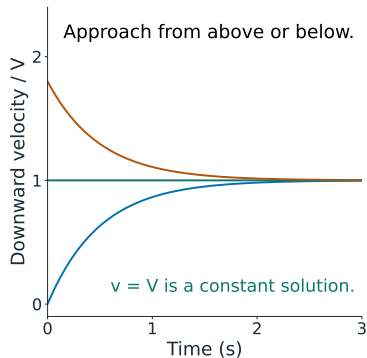
Check a constant solution before dividing

Start with the force model

In $dv/dt = (k/m)(V - v)$, dividing by $V - v$ excludes $v = V$. Test it in the original equation: it is a valid constant-velocity solution if that is the initial velocity.

An initial downward velocity below V increases towards V ; one above V decreases towards it. At terminal velocity the particle is moving with zero acceleration. It is not at rest, and the model need not reach the limit in finite time.

Terminal velocity is approached from above or below and is also a constant solution



Terminal speed V is where the resultant force is zero: a particle starting faster slows towards V , one starting slower speeds up towards it.

Separate the falling quadratic-drag equation

Start with the force model

Take downward speed $v \geq 0$ with resistance per unit mass gv^2/V^2 . Then $dv/dt = g(1 - v^2/V^2)$. From rest, $0 \leq v < V$, so separation gives $dv/(V^2 - v^2) = (g/V^2) dt$.

$$\frac{1}{V^2 - v^2} = \frac{1}{2V} \left(\frac{1}{V + v} + \frac{1}{V - v} \right).$$

Partial fractions 部分分式 give $\ln[(V + v)/(V - v)] = 2gt/V$ after using $v(0) = 0$.

Rearranging gives $v = V(e^{2gt/V} - 1)/(e^{2gt/V} + 1)$. The formula approaches V from below; keep logarithm arguments and the initial branch consistent.

Falling speed can be found from distance instead

Start with the force model

For the same quadratic-drag fall from rest, choose $v dv/dy = g(1 - v^2/V^2)$ with downward distance y . Separation and the initial state give

$$v^2 = V^2(1 - e^{-2gy/V^2}).$$

With $g = 10 \frac{\text{m}}{\text{s}^2}$ and $V = 10 \frac{\text{m}}{\text{s}}$, this is $v^2 = 100(1 - e^{-y/5})$. Gravity's lost potential energy does not all become kinetic energy: resistance does negative work.

Upward motion puts gravity and drag together

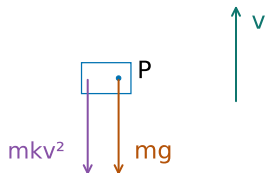
Start with the force model

Launch a particle vertically upwards with speed u . Take upwards as positive on the ascent, so $v > 0$. Gravity and resistance both act downwards. For quadratic resistance per unit mass kv^2 , the equation is $dv/dt = -g - kv^2$.

Define $V^2 = g/k$ to simplify the algebra. Here V is the terminal speed for a later downward fall, not a possible steady upward speed. Using $g - kv^2$ during ascent would reverse the resistance incorrectly.

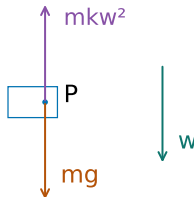
Gravity and drag act together on ascent; drag reverses on descent

Ascent



$$dv/dt = -g - kv^2$$

Descent



$$dw/dt = g - kw^2$$

Upwards $\ddot{x} = -g - kv^2$; downwards $\ddot{x} = g - kv^2$. Solve the two stages separately:
 $V = \sqrt{g/k}$ is a terminal speed only on the way down.

Find maximum height from the position equation

Start with the force model

On ascent use $v dv/dx = -g - kv^2$. Separate and use the launch state in the integration:

$$\int_u^v \frac{w}{g + kw^2} dw = - \int_0^x ds.$$

$$\ln \left(\frac{g + kv^2}{g + ku^2} \right) = -2kx.$$

Rearranging gives $v^2 = (u^2 + V^2)e^{-2kx} - V^2$ on ascent until its first zero. For $g = 10$, $k = 1/10$ and $u = 10$ in consistent SI units, $V = 10$ and $v^2 = 200e^{-x/5} - 100$. Maximum height is $H = 5 \ln 2$ m. It is less than the no-drag height $u^2/(2g) = 5$ m.

Find ascent time by an inverse tangent

Start with the force model

Separate $dv/dt = -k(V^2 + v^2)$. Integration gives $\tan^{-1}(v/V) = \tan^{-1}(u/V) - gt/V$. Use radians for the inverse tangent in this calculus calculation.

In the example $u = V = 10$ and $g = 10$, $v = 10 \tan(\pi/4 - t)$. The first zero gives ascent time $t = \pi/4$ s. Stop using this ascent equation at the top; resistance reverses during the descent.

Join the descent to the top state

Start with the force model

At the top, velocity is continuous and equals zero. Restart a downward coordinate y from the top, with downward speed $w \geq 0$. The new equation is $dw/dt = g - kw^2$, because resistance now acts upwards.

The falling-distance result gives $w^2 = V^2(1 - e^{-2ky})$. The ascent and descent each have a consistent sign convention; record the new initial state rather than carrying a negative ascent speed into an unchanged force equation.

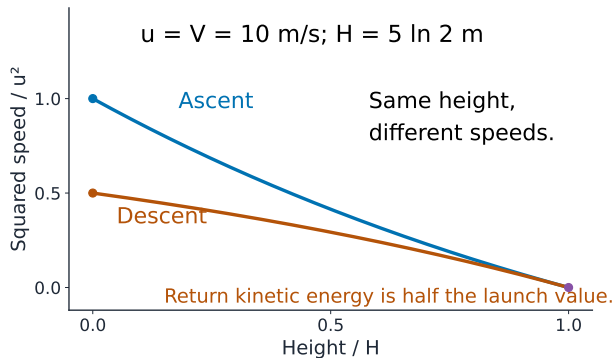
Compare the return speed with the launch speed

Start with the force model

In the example $u = V = 10$, $k = 1/10$ and $H = 5 \ln 2$, the particle returns to launch height after falling distance H . Its downward squared speed is $w^2 = 100(1 - e^{-H/5}) = 50$.

The return speed is $5\sqrt{2} \frac{\text{m}}{\text{s}}$, less than the launch speed $10 \frac{\text{m}}{\text{s}}$. Gravity has the same potential energy at both endpoints, but resistance has removed mechanical energy during both parts.

Energy lost to resistance makes the return speed smaller



Work done against resistance on the way up and on the way down is lost, so the particle returns to its start slower than it was projected.

A force towards a point is not always drag

Start with the force model

Suppose acceleration is $-(v^2 + 9)/3$ while a particle is to the right of O . It is prescribed to point towards O , even after the particle reverses and moves left. It then speeds up leftwards rather than resisting that motion.

With $x(0) = 0$ and $v(0) = 3$, separation gives $\tan^{-1}(v/3) = \pi/4 - t$, so $v = 3 \tan(\pi/4 - t)$. The particle first stops at $t = \pi/4$. A negative velocity after this time correctly describes the return towards O .

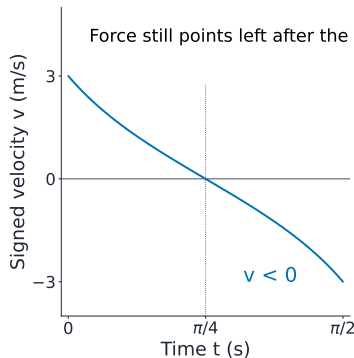
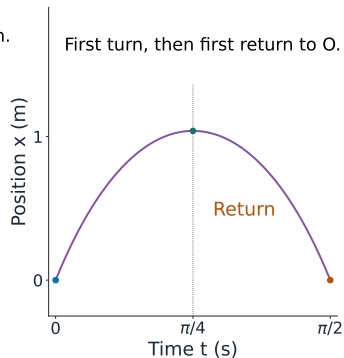
Integrate the return branch with its domain

Start with the force model

Integrating that velocity and using $x(0) = 0$ gives $x = 3 \ln[\cos(t - \pi/4)] + \frac{3}{2} \ln 2$ before the next return to O . At the turn the maximum displacement is $\frac{3}{2} \ln 2$.

The next zero of x is $t = \pi/2$, found from the first return branch, not a later periodic tangent value. At O the direction rule changes; a formula derived for $x > 0$ is not automatically valid beyond that event.

A prescribed force towards O keeps pointing left during the return



“Towards O ” depends on position, not on the direction of motion: for $x > 0$ the force is $-f(x)$ on the way out and on the way back.

Infer a force from a velocity relation

Start with the force model

If a 2 kg particle has velocity $v = 1 + x^2$ on its outward branch, then $a = v dv/dx = (1 + x^2)(2x)$. Its resultant force is $F = ma = 4x(1 + x^2)$ in consistent SI units.

To infer unknown parameters, substitute measured states into the integrated model and solve for the constants. Then check another state when available. A force coefficient has units chosen so the whole expression is a force, not necessarily newtons alone.

Check a proposed solution in the original equation

Start with the force model

Differentiate the proposed $v(t)$ or $v(x)$, substitute into Newton's law, and check the initial value. For $v = 4/(1 + 2t)$, differentiation gives $dv/dt = -8/(1 + 2t)^2 = -v^2/2$, as required.

Check signs, units and domains as well as algebra. A positive drag magnitude must oppose velocity; a squared-speed expression must stay nonnegative; logarithms need valid arguments; a new branch needs its own physical assumptions.

Decide which event ends the current model

Start with the force model

Mark any turning point, crossing of a force-direction boundary, change of resistance law, or endpoint of the allowed motion. Decide whether a zero-speed state occurs at a finite time or only as a limit.

A terminal solution can disappear when dividing by a zero factor, and a square root can hide a return branch. Restore such possibilities by checking the original equation and the order of events.

Keep the force law tied to the motion

Start with the force model

Choose a signed direction, add the real forces, then select $a = dv/dt$ or $a = v dv/dx$ to suit the variables. Separate, integrate and apply the initial conditions at every stage.

Distinguish force, acceleration, velocity and speed. Check each answer against the original equation, its branch and its limits; after a physical event, reconsider the model before continuing.

Variable-force independent check 1

Work independently before revealing the model

A 2 kg particle has resultant force $4t$ N to the right. At $t = 0$ it has $x = 0$ and $v = -2$ m/s. Find $v(t)$, $x(t)$, and its first turning time.

State force directions, integration constants and the valid branch.

Variable-force model and check 1

Newton's second law, then integrate twice, fixing a constant each time:

$$2\frac{dv}{dt} = 4t \Rightarrow v = t^2 + C, \quad v(0) = -2 \Rightarrow C = -2$$

$$x = \frac{1}{3}t^3 - 2t + D, \quad x(0) = 0 \Rightarrow D = 0$$

$$v = 0 \Rightarrow t = \sqrt{2} \text{ s} \quad (\text{first turn: from } v, \text{ not } x)$$

Uses both initial conditions after separate integrations. Finds the first turning time from velocity, not displacement.

Variable-force independent check 2

Work independently before revealing the model

A 2 kg particle starts at $x = 0$ with velocity 2 m/s to the right. The resultant force is $F = 8x + 4x^3$ N for $x \geq 0$. Find $v(x)$ on its outward branch.

State force directions, integration constants and the valid branch.

Variable-force model and check 2

Force depends on x , so use $v \frac{dv}{dx}$:

$$2v \frac{dv}{dx} = 8x + 4x^3 \Rightarrow v^2 = 4x^2 + x^4 + C, \quad C = 4$$

$$v^2 = (x^2 + 2)^2 \Rightarrow v = x^2 + 2 \quad (\text{outward branch})$$

Uses $v \, dv/dx$ and includes the mass. Applies the initial value and selects the velocity sign.

Variable-force independent check 3

Work independently before revealing the model

A particle moving right obeys $dv/dt = -v^2$, starts with $v = 3$ m/s at $x = 0$, and remains on the positive-velocity branch. Find $v(t)$, $x(t)$, and its limiting total distance.

State force directions, integration constants and the valid branch.

Variable-force model and check 3

Separate, then integrate the velocity:

$$\frac{1}{v} = t + \frac{1}{3} \Rightarrow v = \frac{3}{1 + 3t}$$
$$x = \int_0^t \frac{3}{1 + 3u} du = \ln(1 + 3t)$$

$v \rightarrow 0$ but $x \rightarrow \infty$: there is **no** finite limiting distance.

Uses the positive branch and both initial conditions. Does not infer finite distance merely from speed tending to zero.

Variable-force independent check 4

Work independently before revealing the model

A particle is launched upwards at 5 m/s with acceleration $dv/dt = -10 - 2v^2/5$. Let x be upward displacement from launch. Find $v^2(x)$ and maximum height.

State force directions, integration constants and the valid branch.

Variable-force model and check 4

Upward launch, gravity and quadratic drag both oppose motion:

$$v \frac{dv}{dx} = -10 - \frac{2}{5}v^2 \Rightarrow v^2 = 50e^{-4x/5} - 25$$
$$v^2 = 0 \Rightarrow e^{-4H/5} = \frac{1}{2} \Rightarrow H = \frac{5}{4} \ln 2 \text{ m}$$

Uses the upward branch and launch condition. Finds the first zero of squared speed.

Variable-force independent check 5

Work independently before revealing the model

A particle starts at $x = 0$ with velocity 2 m/s . While $x > 0$ its acceleration is $-(v^2 + 4) \text{ m/s}^2$ and its velocity is $2\tan(\pi/4 - 2t)$. Find $x(t)$, maximum displacement and the first return time to O .

State force directions, integration constants and the valid branch.

Variable-force model and check 5

Integrate the velocity and use $x(0) = 0$:

$$x = \ln\left[\cos\left(2t - \frac{\pi}{4}\right)\right] + \frac{1}{2} \ln 2$$
$$x_{\max} = \frac{1}{2} \ln 2 \text{ m at } t = \frac{\pi}{8} \text{ s,} \quad x = 0 \text{ again at } t = \frac{\pi}{4} \text{ s}$$

Beyond that return the direction rule must be looked at again.

Uses the integration constant and correct logarithm sign. Selects the first physical return rather than a periodic continuation.

Variable-force independent check 6

Work independently before revealing the model

A falling particle obeys $dv/dt = 2(5 - v)$, with downward velocity positive. It starts at $v = 8$ m/s. Find $v(t)$ and initial acceleration, and describe the limit. Also test the initial value $v = 5$.

State force directions, integration constants and the valid branch.

Variable-force model and check 6

Solve, then read the signs separately:

$$v = 5 + 3e^{-2t} \quad (v_0 = 8), \quad a(0) = 2(5 - 8) = -6 \text{ m s}^{-2} \text{ (upwards)}$$

- Downward speed **decreases towards** 5 m s^{-1} from above.
- If $v_0 = 5$, then $v \equiv 5$: dividing by $5 - v$ must not lose this solution.

Distinguishes velocity direction from acceleration direction. Keeps the constant terminal solution.

5

Momentum

A bounce needs a system and a direction

Choose the system and contact direction

A **collision** 碰撞 brings two low-friction carts into contact. Their velocities change, but their combined **momentum** 动量 stays constant if the external impulse during contact is negligible.

A ball hitting a fixed wall is different: the wall gives the ball an impulse. Define your system before writing a conservation equation.

Decide which momentum is conserved

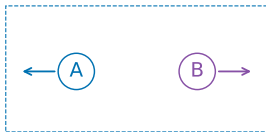
Choose the system and contact direction

Linear momentum 线动量 is a vector: $\mathbf{p} = m\mathbf{v}$. The law is **conservation of linear momentum** 动量守恒, often shortened to **conservation of momentum**. It applies to a system with zero resultant external impulse, or to a component with zero external impulse in that direction.

For two smooth spheres on a smooth horizontal plane, their mutual impulses cancel. For the sphere alone at a fixed wall, normal momentum changes; the wall belongs outside that one-sphere system.

Conservation requires a stated system and a justified momentum component

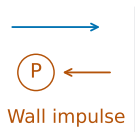
Two-sphere system



Internal impulses cancel.

External horizontal impulse ≈ 0

Sphere alone at a wall



Normal momentum changes.

Tangential component unchanged.

Name the system and the direction: momentum is conserved only where there is no external impulse, e.g. along the line of centres for two free spheres.

Draw the line of impact

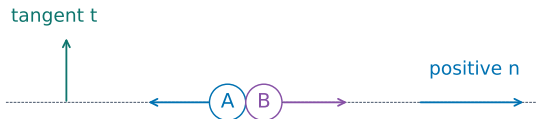
Choose the system and contact direction

For two smooth spheres, the **line of impact** 碰撞线 is their **line of centres** 球心连线 at contact. The collision impulse acts along it.

Call that direction normal, n , and a perpendicular direction tangent, t . A **direct impact** 正碰撞 has motion along n ; an **oblique impact** 斜碰撞 also has tangential motion.

Smooth contact changes normal components; each tangential component is unchanged

Smooth contact: impulse acts along the line of centres



Normal: momentum + restitution

Tangent: each component is unchanged

Restitution uses relative normal velocities.

Smooth spheres exchange impulse only along the line of centres: apply momentum and e there, and keep each perpendicular component unchanged.

Write restitution with signed components

Choose the system and contact direction

Choose positive n from A towards B , with $u_{An} > u_{Bn}$ before contact. **Newton's experimental law** 牛顿实验定律 is

$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}).$$

The **coefficient of restitution** 恢复系数 e is normal separation speed divided by normal approach speed. In this syllabus model, $0 \leq e \leq 1$; use relative components, not the two objects' separate speed magnitudes.

Interpret the two limiting cases

Choose the system and contact direction

A **perfectly elastic** 完全弹性 impact has $e = 1$: **kinetic energy** 动能 is conserved for the ideal smooth collision. An **inelastic** 非弹性 impact with $e = 0$, also called **perfectly inelastic** 完全非弹性, gives equal final normal velocities.

In a direct collision, $e = 0$ gives a common velocity. In an oblique collision, tangential velocities can differ, so equal normal velocities do not mean that the spheres become glued together.

A particle glancing along a smooth wall with $e = 0$ may keep moving along it. Apply the law to the normal component first.

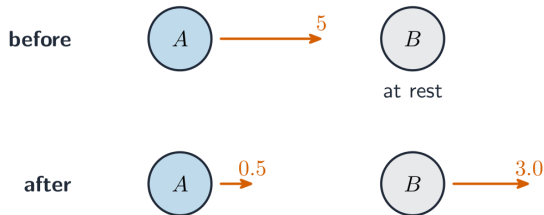
Solve the original direct-collision example

Choose the system and contact direction

A sphere A of mass 2 kg moves at $5\frac{\text{m}}{\text{s}}$ into stationary B of mass 3 kg on a smooth horizontal plane. Take rightwards positive and $e = 0.5$.

Momentum gives $2v_A + 3v_B = 10$; restitution gives $v_B - v_A = 2.5$. Thus $v_A = 0.5\frac{\text{m}}{\text{s}}$ and $v_B = 3\frac{\text{m}}{\text{s}}$, both rightwards.

The original direct impact uses the same signed normal equations



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

Define the direction and use the stated masses and coefficient.

A head-on collision needs negative velocities

Choose the system and contact direction

Let A have mass 2 kg and velocity $+3 \frac{\text{m}}{\text{s}}$, while B has mass 1 kg and velocity $-3 \frac{\text{m}}{\text{s}}$.

They approach at $6 \frac{\text{m}}{\text{s}}$, not zero.

With $e = 0.5$, $2v_A + v_B = 3$ and $v_B - v_A = 3$. Hence $v_A = 0$ and $v_B = 3 \frac{\text{m}}{\text{s}}$; A is instantaneously at rest after impact.

Check whether contact can occur

Choose the system and contact direction

If A is left of B and both move right, a first collision requires $u_A > u_B$. A negative answer for v_A means rebound, not a negative speed.

After contact, check $v_B - v_A \geq 0$. With $e > 0$ the bodies separate; with $e = 0$ their normal separation speed is zero at that instant.

Derive a reusable direct-impact result

Choose the system and contact direction

Solving normal momentum and restitution gives

$$v_A = \frac{(m_A - em_B)u_A + (1 + e)m_B u_B}{m_A + m_B}.$$

Then $v_B = v_A + e(u_A - u_B)$. If B starts at rest, $v_B = \frac{(1 + e)m_A u_A}{m_A + m_B}$; use the equations to check any memorised formula.

Infer a mass ratio from rebound

Choose the system and contact direction

A sphere of mass m hits a stationary sphere of mass km in a perfectly elastic direct impact. If its final velocity is $-u/3$, restitution gives $v_B = 2u/3$.

Momentum gives $mu = -mu/3 + km(2u/3)$, so $k = 2$. The sign of the rebound matters in both equations.

A momentum ratio is not a speed ratio

Choose the system and contact direction

A sphere of mass $3m$ moving at u hits a stationary sphere of mass m . Afterwards $p_A = 2p_B$, both rightwards. Since total momentum is $3mu$, $p_A = 2mu$ and $p_B = mu$.

Thus $v_A = 2u/3$, $v_B = u$ and $e = 1/3$. Dividing momenta without accounting for the masses would give the wrong velocity ratio.

Keep energy loss separate from momentum

Choose the system and contact direction

In the original 2 kg and 3 kg example, initial kinetic energy is 25 J and final kinetic energy is 13.75 J. The loss is 11.25 J, although momentum is conserved.

For an isolated smooth two-sphere collision, the loss is $\frac{1}{2}\mu(1 - e^2)(u_{An} - u_{Bn})^2$, where the **reduced mass** 约化质量 is $\mu = m_A m_B / (m_A + m_B)$. This checks a calculation; it does not replace the signed impact equations.

Resolve an oblique sphere impact

Choose the system and contact direction

Smooth spheres exert no tangential impulse on each other. Therefore $v_{At} = u_{At}$ and $v_{Bt} = u_{Bt}$ separately; normal momentum and restitution determine v_{An} , v_{Bn} .

Draw a component table before solving. Recombine each sphere's own components using $v = \sqrt{v_n^2 + v_t^2}$; the final normal component alone is usually not its speed.

Work through an oblique example

Choose the system and contact direction

Sphere A has mass 2 kg and initial components $(3, 4) \frac{\text{m}}{\text{s}}$ in (n, t) . Stationary B has mass 1 kg . For $e = 0.5$, $2v_{An} + v_{Bn} = 6$ and $v_{Bn} - v_{An} = 1.5$.

Hence $\mathbf{v}_A = (1.5, 4)$ and $\mathbf{v}_B = (3, 0)$ in the same units. Their speeds are $\sqrt{73}/2$ and $3 \frac{\text{m}}{\text{s}}$; A keeps its tangential component.

Recombine each sphere's own components after solving the normal equations

Worked oblique impact: A has mass 2 kg; B has mass 1 kg

Before (m/s)

After, $e = 0.5$ (m/s)

A: $n = 3$ A: $t = 4$

A: $n = 1.5$ A: $t = 4$

B: $n = 0$ B: $t = 0$

B: $n = 3$ B: $t = 0$

Only the normal components change.

Final speed of A = $\sqrt{(1.5^2 + 4^2)} = \sqrt{73} / 2$ m/s

After solving along the line of centres, combine each sphere's new normal component with its own unchanged tangential one to get its speed and direction.

Zero restitution does not remove tangential motion

Choose the system and contact direction

Two equal-mass smooth spheres have initial components $\mathbf{u}_A = (4, 3)$ and $\mathbf{u}_B = (0, -1)$, with positive normal from A to B . If $e = 0$, both final normal velocities are 2.

Their final vectors are $(2, 3)$ and $(2, -1)$ in the stated SI units. They have equal normal velocities but different tangential velocities; the particles do not move as a single rigid object.

Recover direction and deflection

Choose the system and contact direction

For final components (v_n, v_t) with $v_n > 0$, the angle to positive normal satisfies $\tan \phi = v_t/v_n$. If $v_n < 0$, draw the quadrant; a calculator's acute angle alone can hide a rebound.

In the previous $A = (1.5, 4)$ example, $\tan \phi = 8/3$. The **deflection** 偏转 is the change from its incoming direction, whose angle satisfies $\tan \theta = 4/3$, not simply ϕ itself.

Infer restitution from final kinetic energies

Choose the system and contact direction

Equal-mass spheres collide with $\mathbf{u}_A = (4, 3)$ and $\mathbf{u}_B = (0, 0)$. The final normal velocities are $2(1 - e)$ and $2(1 + e)$, while their tangential velocities remain 3 and 0.

Equal final kinetic energies require $4(1 - e)^2 + 9 = 4(1 + e)^2$, so $e = 9/16$. Both final speeds are $25/8 \frac{\text{m}}{\text{s}}$; keep both components in the energy equation.

A fixed wall changes the sphere's momentum

Choose the system and contact direction

At a smooth **fixed surface** 固定表面, take positive normal towards the wall. The surface has zero velocity, so the sphere's final component is $v_n = -eu_n$ and its tangential component is $v_t = u_t$.

Do not conserve the sphere's normal momentum. The wall's impulse supplies the change: $\Delta p_n = -m(1 + e)u_n$. The wall–Earth system can take momentum while its speed is negligible.

Recombine components after a wall impact

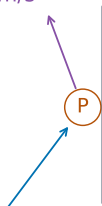
Choose the system and contact direction

A 2 kg sphere approaches a smooth fixed wall with components $(6, 8) \frac{\text{m}}{\text{s}}$ in normal/tangent directions. For $e = 0.5$, the outgoing components are $(-3, 8)$.

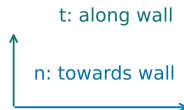
Speed changes from 10 to $\sqrt{73} \frac{\text{m}}{\text{s}}$. Kinetic energy loss is $100 - 73 = 27 \text{ J}$; the overall speed ratio is not e .

A smooth fixed wall reverses and scales the normal component only

Outgoing: $(-3, 8)$ m/s



Incoming: $(6, 8)$ m/s



$e = 0.5$

Signed normal reverses and scales; tangent stays the same.

With α measured from the wall, $u \cos \alpha$ along the wall is unchanged and the normal component becomes $eu \sin \alpha$, so $\tan \beta = e \tan \alpha$.

State which line an angle uses

Choose the system and contact direction

If incoming and outgoing acute angles α, β are measured from the wall, $u_n = u \sin \alpha$ and $u_t = u \cos \alpha$. Thus $\tan \beta = e \tan \alpha$.

If θ, ϕ are measured from the normal, $\tan \phi = \tan \theta / e$ for $e > 0$. At $e = 0$, use components directly; a nonzero tangent component leaves motion parallel to the wall.

Angles from the wall and angles from the normal use complementary components

Name the reference line before resolving

Angles from wall

$$\tan \beta = e \tan \alpha$$

$$\text{Normal} = u \sin \alpha$$

$$\text{Tangent} = u \cos \alpha$$

Angles from normal

$$\tan \phi = \tan \theta / e$$

$$\text{Normal} = u \cos \theta$$

$$\text{Tangent} = u \sin \theta$$

For $e = 0$, use components; do not divide by e .

From the wall, the normal component is $u \sin \alpha$; from the normal, it is $u \cos \alpha$.
Check which angle the question gives before resolving.

Infer a wall coefficient from angles

Choose the system and contact direction

A sphere strikes a smooth fixed wall at 45° to the wall and leaves at angle β , with $\tan \beta = 1/3$. Then $e = \tan \beta / \tan 45^\circ = 1/3$.

Its final speed is $u\sqrt{\cos^2 45^\circ + e^2 \sin^2 45^\circ} = u\sqrt{5}/3$. A change in direction is why the speed ratio differs from $1/3$.

Check wall-impact energy and limits

Choose the system and contact direction

At a smooth fixed wall, lost kinetic energy is $\frac{1}{2}m(1 - e^2)u_n^2$. Only the normal contribution changes; at $e = 1$, speed is unchanged and the normal component reverses.

At $e = 0$, normal speed becomes zero. If $u_t \neq 0$, the sphere continues along the wall; if both final components are zero, it is at rest immediately after contact.

A time ratio can reveal restitution

Choose the system and contact direction

A particle travels between two parallel smooth fixed walls separated by d . If its first flight takes T , its normal speed is d/T . After a bounce with coefficient $e > 0$, the next flight takes T/e .

Thus if the second time is kT , $e = 1/k$ and $k \geq 1$. The case $e = 0$ gives no finite second crossing; do not cancel a zero normal speed as though a return occurs.

Treat successive impacts as separate events

Choose the system and contact direction

Equal-mass particles collide directly: A initially has velocity $6 \frac{\text{m}}{\text{s}}$, B is at rest and $e_{AB} = 0.5$. They leave with velocities 1.5 and $4.5 \frac{\text{m}}{\text{s}}$.

If B then hits a smooth wall with $e_{BW} = 1/3$, its velocity becomes $-1.5 \frac{\text{m}}{\text{s}}$. Their next mutual impact uses incoming velocities $+1.5, -1.5$ and e_{AB} , giving $v_A = -0.75$, $v_B = +0.75 \frac{\text{m}}{\text{s}}$.

Check the next event using distances

Choose the system and contact direction

In that example, put the first mutual collision at $x = 0$ and the wall at $x = 9$ m. Particle B reaches the wall at $t = 2$ s; then A is at $x = 3$ m.

The gap is 6 m and their closing speed after the wall bounce is $3 \frac{\text{m}}{\text{s}}$. They meet 2 s later at $x = 6$ m, before A can reach the wall. Different pairs can have different restitution coefficients.

Use event positions and closing speeds to determine the next collision

Positions identify which collision happens next

First impact: $t = 0$ s



Wall bounce: $t = 2$ s



Next meeting: $t = 4$ s



$x = 0$

wall: 9 m

A: blue B: purple; position markers use metres.

After each impact, find positions and velocities: another collision happens only if the gap is closing, and its time is $\text{gap} \div \text{closing speed}$.

Use a fresh projectile segment after a barrier

Choose the system and contact direction

A particle has incoming velocity $(4, -4) \frac{\text{m}}{\text{s}}$ at a smooth barrier inclined at 45° upwards to the right. Its velocity is purely normal to the barrier; $e = 0.5$ gives outgoing velocity $(-2, 2) \frac{\text{m}}{\text{s}}$.

If impact is 4 m above horizontal ground and $g = 10 \frac{\text{m}}{\text{s}^2}$, solve $0 = 4 + 2t - 5t^2$. The positive flight time is $(1 + \sqrt{21})/5$ s and horizontal distance is $2(1 + \sqrt{21})/5$ m to the left.

Link a collision to a vertical circle

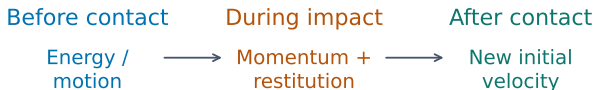
Choose the system and contact direction

At a circle's bottom, a particle of mass m and incoming speed $3\sqrt{5ag}$ hits a stationary particle of mass $2m$ in a perfectly elastic direct impact. Momentum and restitution give signed velocities $-\sqrt{5ag}$ and $2\sqrt{5ag}$.

The first particle rebounds with squared speed $5ag$. Its later energy equation gives top squared speed ag , the limiting $T = 0$ condition for a taut string. Use energy on the smooth arcs and collision laws at contact; zero tension is a boundary test, not by itself proof of subsequent separation.

Use collision laws at contact and a fresh motion model on the following segment

Use a different model for each physical segment



A smooth arc or projectile can conserve mechanical energy.

An inelastic impact loses kinetic energy at the velocity jump.

Check constraints and the next event on the new segment.

Momentum and restitution apply only at the instant of impact; between impacts, model the motion afresh with energy, gravity or resistance.

Reject a physically impossible model

Choose the system and contact direction

An algebraic candidate must satisfy $0 \leq e \leq 1$, the stated incoming approach and outgoing directions. Reject inadmissible roots with a reason; do not turn a negative signed velocity into a positive one midway through the equations.

Check momentum for the correct system, the restitution relation, unchanged smooth tangential components and nonnegative kinetic-energy loss. A fixed wall can violate one-sphere momentum conservation without violating the model.

Organise an independent impact solution

Choose the system and contact direction

Draw the contact normal, define signed components and label the system. Write the normal momentum equation only where justified, then restitution and unchanged tangential components.

Recombine velocities, state directions and check the next physical event. For a question combining topics, carry the outgoing velocity into a new motion segment with its own initial state.

Retrieve the model before calculating

Choose the system and contact direction

Direct and oblique impacts use the same normal equations; tangential components survive smooth contact unchanged. For a fixed surface, $v_n = -eu_n$, not conservation of the sphere's normal momentum.

Always define the system, the angle convention and the event order. Distinguish normal separation from total speed, equal normal velocity from sticking, and energy between impacts from energy across them.

Momentum independent check 1

Work independently before revealing the model

A of mass 1 kg moves at +4 m/s towards B of mass 3 kg moving at -2 m/s on a smooth horizontal line. For $e = 1/2$, find their final velocities and directions.

Define the system, normal and tangent; check the physical event.

Momentum model and check 1

Conservation of momentum and Newton's law of restitution:

$$v_A + 3v_B = -2 \quad (\text{momentum})$$

$$v_B - v_A = e \times 6 = 3 \quad (\text{restitution})$$

$$v_A = -\frac{11}{4}, \quad v_B = \frac{1}{4} \text{ m s}^{-1}$$

A rebounds left; B reverses to the right.

Includes the negative incoming velocity and interprets both final signs.

Momentum independent check 2

Work independently before revealing the model

On a smooth horizontal plane, A of mass 3 kg has incoming normal/tangent components (4,3) m/s. B of mass 1 kg is at rest. For $e = 1/3$, find the final velocity vectors and speeds.

Define the system, normal and tangent; check the physical event.

Momentum model and check 2

Along the line of centres only; tangential parts are unchanged:

$$3v_{An} + v_{Bn} = 12, \quad v_{Bn} - v_{An} = \frac{4}{3}$$

$$A: \left(\frac{8}{3}, 3\right), \quad |v_A| = \frac{\sqrt{145}}{3} \text{ m s}^{-1}$$

$$B: (4, 0), \quad |v_B| = \mathbf{4} \text{ m s}^{-1}$$

Solves normal equations and retains each tangential component.

Momentum independent check 3

Work independently before revealing the model

A sphere strikes a smooth fixed wall at 45 degrees to the wall. Its final speed is $\sqrt{10}/4$ times its incoming speed. Find e and the tangent of the outgoing angle to the wall.

Define the system, normal and tangent; check the physical event.

Momentum model and check 3

Tangential part kept, normal part scaled by e :

$$\begin{aligned}\frac{v'^2}{v^2} &= \frac{10}{16} = \cos^2 45^\circ + e^2 \sin^2 45^\circ = \frac{1 + e^2}{2} \\ e^2 &= \frac{1}{4} \Rightarrow e = \frac{1}{2} \quad (e \geq 0), \quad \tan(\text{outgoing}) = \frac{1}{2}\end{aligned}$$

Uses the speed ratio squared and selects the admissible nonnegative e .

Momentum independent check 4

Work independently before revealing the model

Equal-mass particles collide directly on a smooth line. A initially moves right at 8 m/s, B is at rest and $e_{AB} = 1/2$. B then hits a wall to its right with $e_{BW} = 1/2$. Find their velocities after their next mutual impact.

Define the system, normal and tangent; check the physical event.

Momentum model and check 4

Track every impact with fresh incoming velocities:

$$\text{1st impact: } v_A = 2, v_B = 6; \quad \text{wall: } v_B = -3$$

$$\text{2nd impact: } v_A + v_B = -1, \quad v_B - v_A = \frac{1}{2} [2 - (-3)] = \frac{5}{2}$$

$$v_A = -\frac{7}{4}, \quad v_B = \frac{3}{4} \text{ m s}^{-1}$$

Uses fresh incoming velocities and the coefficient for the correct pair.

Momentum independent check 5

Work independently before revealing the model

A particle has incoming velocity $(6, -6)$ m/s when it strikes a smooth fixed barrier inclined at 45 degrees upwards to the right. Restitution is $1/3$. Impact is 6 m above the ground; take $g = 10 \text{ m/s}^2$. Find its outgoing velocity and horizontal distance before landing.

Define the system, normal and tangent; check the physical event.

Momentum model and check 5

Velocity normal to the barrier reverses; then a new projectile starts:

$$\begin{aligned} \mathbf{v}' &= (-2, 2) \text{ m s}^{-1}, & y &= 6 + 2t - 5t^2 \\ y = 0 &\Rightarrow t = \frac{1+\sqrt{31}}{5} \text{ s}, & \text{distance} &= \frac{2(1+\sqrt{31})}{5} \text{ m (left)} \end{aligned}$$

Resolves at contact and starts a new projectile segment with the outgoing velocity.

Momentum independent check 6

Work independently before revealing the model

At the bottom of a vertical circle of radius a , a particle of mass m moving at $2\sqrt{5ag}$ collides perfectly elastically with a stationary particle of mass $3m$. Find its signed final velocity and test the limiting top tension on its rebound arc.

Define the system, normal and tangent; check the physical event.

Momentum model and check 6

With the incoming direction positive, restitution and momentum give $v_A = -\sqrt{5ag}$ and $v_B = \sqrt{5ag}$. A rebounds with squared speed $5ag$. Energy to the top removes $4ag$ from squared speed, leaving ag . The top radial equation gives $T + mg = m(ag)/a$, so $T = 0$, the limiting taut-circle boundary. This alone does not prove actual later separation.

Combines the signed collision with a separate circular-motion energy and radial test.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, choose $a = dv/dt$ or $a = v dv/dx$ to suit the variables. Integrate, use the initial state and check the branch.

6

Recap

Topic 3 – key takeaways

1

Projectiles

range $u^2 \sin 2\alpha / g$; parabolic path

2

Circular

$v = r\omega$, $a = v^2/r$ toward the centre

3

Hooke

$$T = \lambda x / L; E = \lambda x^2 / 2L$$

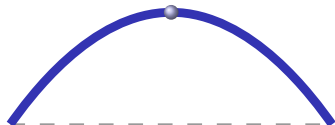
4

Restitution

$e = \text{separation} / \text{approach speed}$

Check yourself

- 1 Direction of the centripetal acceleration?
- 2 Tension formula for an elastic string?
- 3 What is e for a perfectly elastic collision?
- 4 Which form of a suits a force depending on x ?



Answers 1. toward the centre 2. $T = \lambda x/L$ 3. $e = 1$ 4. $a = v dv/dx$