Further Pure Mathematics 2

A-Level Further Mathematics ▪ Topic 2

A-Level Further Mathematics

Six sections: connect the methods

1. Hyperbolic functions: definitions, graphs, identities and inverses.
2. Matrices: systems, eigenvectors, diagonalisation and Cayley–Hamilton.
3. Differentiation: branches, implicit/parametric derivatives and local series.
4. Integration: substitutions, reductions, bounds, length and surface area.
5. Complex numbers: geometry, de Moivre, roots, identities and sums.
6. Differential equations: integrating factors, CF/PI, substitutions and models.

1

2.1 Hyperbolic functions

A hanging cable needs a hyperbolic curve

A flexible, uniform chain hanging under gravity forms a **catenary** 悬链线. A shifted and scaled model is

$$y = a\cosh((x-h)/a) + k, \qquad a > 0.$$

Its lowest point is $(h, a+k)$: the input is zero at $x = h$ and $\cosh 0 = 1$. The model assumes uniform weight per unit length, flexibility and no other loads.

> Start with the model's assumptions. A hanging chain is not generally a parabola.

Define all six functions from exponentials

$$\sinh x = (e^x - e^{-x})/2, \qquad \cosh x = (e^x + e^{-x})/2,$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}, \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

Since $e^x > 0$, cosh never vanishes. Sinh, an odd function, vanishes only at $x = 0$.

> A reciprocal is not an inverse function. Cosech and coth exclude zero.

Worked: exact values at a logarithm

Set $x = \ln 2$. Then $e^x = 2$ and $e^{-x} = 1/2$.

$$\begin{aligned}\sinh x &= (2 - 1/2)/2 = 3/4,\\ \cosh x &= (2 + 1/2)/2 = 5/4,\\ \tanh x &= (3/4)/(5/4) = 3/5.\end{aligned}$$

The reciprocals give $\operatorname{cosech} x = 4/3$, $\operatorname{sech} x = 4/5$ and $\coth x = 5/3$.

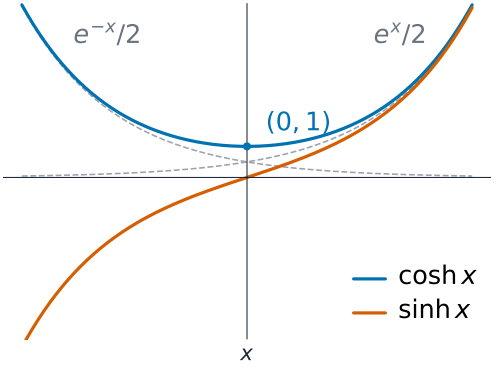
> Check $(5/4)^2 - (3/4)^2 = 1$ before reusing these values.

Record all six domains and ranges

function	domain	range
$\sinh x$	$\mathbb{R}$	$\mathbb{R}$
$\cosh x$	$\mathbb{R}$	$[1, \infty)$
$\tanh x$	$\mathbb{R}$	$(-1, 1)$
$\operatorname{sech} x$	$\mathbb{R}$	$(0, 1]$
$\operatorname{cosech} x$	$\mathbb{R} \setminus \{0\}$	$\mathbb{R} \setminus \{0\}$
$\coth x$	$\mathbb{R} \setminus \{0\}$	$(-\infty, -1) \cup (1, \infty)$

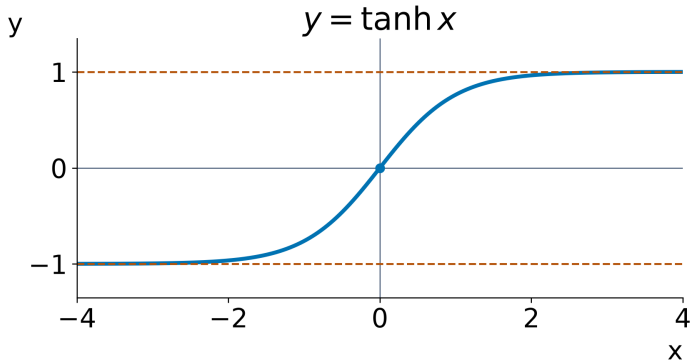
Cosh and sech are even; the other four are odd. Denominator zeros exclude inputs, and limiting values need not be attained.

Basic graphs: symmetry and limiting values



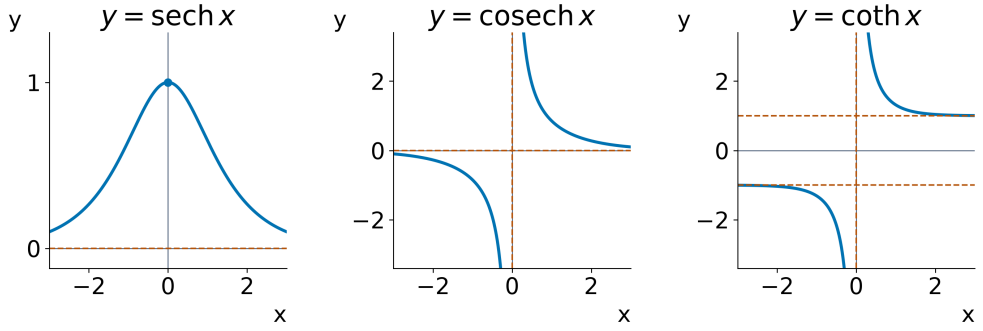
$\cosh x = \frac{1}{2}(e^x + e^{-x})$ is even, with minimum $(0, 1)$; $\sinh x = \frac{1}{2}(e^x - e^{-x})$ is odd and passes through the origin.

Tanh approaches two unattained limits



Domain $\mathbb{R}$, range $(-1, 1)$; horizontal asymptotes $y = \pm 1$.

Reciprocal graphs retain excluded inputs



Sech: $(0, 1]$. Cosech: non-zero real values. Coth: $|y| > 1$. Each cosech/coth branch decreases.

Prove the identity from the definitions

If definitions are requested, expand the exponential expressions:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} \\ &= \frac{4e^x e^{-x}}{4} = 1.\end{aligned}$$

Dividing by $\cosh^2 x$ gives $1 - \tanh^2 x = \operatorname{sech}^2 x$. Dividing by $\sinh^2 x$ instead gives $\coth^2 x - \operatorname{cosech}^2 x = 1$ for $x \neq 0$.

> A quoted identity is not a proof starting from the definitions.

Derive the addition formula by collecting exponentials

Expand the products in $\cosh u \cosh v + \sinh u \sinh v$:

$$\frac{(e^u + e^{-u})(e^v + e^{-v}) + (e^u - e^{-u})(e^v - e^{-v})}{4} = \frac{e^{u+v} + e^{-u-v}}{2}.$$

Thus

$$\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v,$$
$$\sinh(u+v) = \sinh u \cosh v + \cosh u \sinh v.$$

For subtraction replace v by $-v$: cosh is even and sinh odd.

Double angles turn an equation into algebra

Put $u = v = x$ in the addition formulae:

$$\sinh 2x = 2\sinh x\cosh x, \qquad \cosh 2x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1.$$

Solve $\cosh 2x = 4\sinh^2 x$:

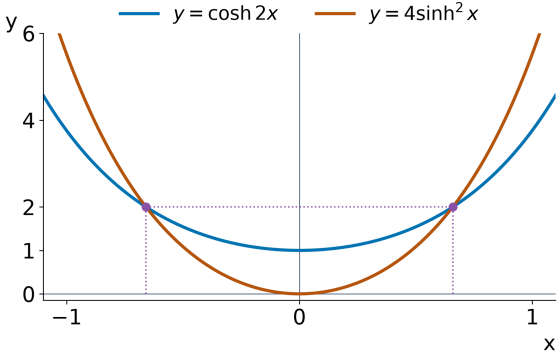
$$1 + 2\sinh^2 x = 4\sinh^2 x \implies \sinh x = \pm 1/\sqrt{2}.$$

Both signs are allowed. Hence

$$x = \pm \ln\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right), \qquad y = 2.$$

> Squaring can hide a sign; an even graph predicts paired intersections.

Two intersections are different from two minima



The two even curves meet at $y = 2$, once on each side of zero.

An inverse exchanges a chosen domain and range

Cosh, an even function, is not one-to-one on $\mathbb{R}$. Restrict it to $x \geq 0$ to define the principal inverse.

$$y = \cosh^{-1} x : \quad x \geq 1, \quad y \geq 0.$$

Reflection in $y = x$ exchanges the domain and range of the restricted curve.

$$\cosh^{-1}(\cosh(-2)) = 2, \qquad \cosh^{-1}(\cosh x) = |x|.$$

> An inverse value uses its chosen branch; it need not recover the original unrestricted input.

Derive the logarithmic form of inverse sinh

Let $y = \sinh^{-1} x$ and $t = e^y > 0$.

$$2x = t - t^{-1}, \qquad t^2 - 2xt - 1 = 0,$$

$$t = x + \sqrt{x^2 + 1}, \qquad y = \ln(x + \sqrt{x^2 + 1}).$$

The other quadratic root is negative, so cannot equal e^y. Since $\sqrt{x^2 + 1} > |x|$, the selected root is positive for every real x.

> For $x = -3/4$, the root is $1/2$: $\sinh^{-1}(-3/4) = -\ln 2$.

Derive inverse cosh and select its root

Let $y = \cosh^{-1} x \geq 0$, so $x \geq 1$ and $t = e^y \geq 1$.

$$2x = t + t^{-1} \implies t^2 - 2xt + 1 = 0.$$

The roots are $x \pm \sqrt{x^2 - 1}$. Choose the plus sign to obtain $t \geq 1$:

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \qquad x \geq 1.$$

For $x > 1$, the minus root is below one and would give $y < 0$.

> $\cosh^{-1}(5/4) = \ln 2$, but $\cosh y = 5/4$ has both $y = \pm \ln 2$.

Derive inverse tanh and keep its open domain

Let $y = \tanh^{-1} x$. Divide the exponential definition by e^{-y}:

$$\begin{aligned} x &= \frac{e^{2y} - 1}{e^{2y} + 1}, \\ e^{2y} &= \frac{1 + x}{1 - x}, \\ \tanh^{-1} x &= \frac{1}{2} \ln \frac{1 + x}{1 - x}, \qquad |x| < 1. \end{aligned}$$

The ratio is positive on this interval. Tanh does not reach either endpoint at a finite input.

> Do not use a real logarithm formula at $x = \pm 1$ or outside the inverse's domain.

Hyperbolic practice: use signs and branches

1. Find all real solutions of $\cosh 2x = 3\sinh^2 x$.
2. Find the exact value of $\tanh^{-1}(3/5)$.
3. Explain why $\cosh^{-1}(\cosh(-\ln 3))$ is not $-\ln 3$.

Use an identity, then the appropriate logarithmic form. State the branch used.

Hyperbolic practice: worked answers

$$1 + 2\sinh^2 x = 3\sinh^2 x \implies \sinh x = \pm 1,$$
$$x = \pm \ln(1 + \sqrt{2}).$$

For the second question,

$$\tanh^{-1}(3/5) = \tfrac{1}{2} \ln \frac{1 + 3/5}{1 - 3/5} = \tfrac{1}{2} \ln 4 = \ln 2.$$

The principal inverse cosh is non-negative, so the last value is $|-\ln 3| = \ln 3$.

> An equation can have two solutions while its principal inverse returns one value.

2

2.2 Matrices

Write a system as a matrix equation

$$\begin{aligned} x+y+z&=6\\ x+2y+z&=8\\ 2x+y+3z&=13 \end{aligned} \quad\Longleftrightarrow\quad C\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\8\\13\end{pmatrix}, \qquad C=\begin{pmatrix}1&1&1\\1&2&1\\2&1&3\end{pmatrix}.$$

Here $\det C = 1 \neq 0$, so C is non-singular and there is exactly one solution. Subtracting the first equation from the second gives $y = 2$; then $x + z = 4$ and $2x + 3z = 11$.

> $(x, y, z) = (1, 2, 3)$. Check all three original equations, not just two.

A singular coefficient matrix needs a consistency check

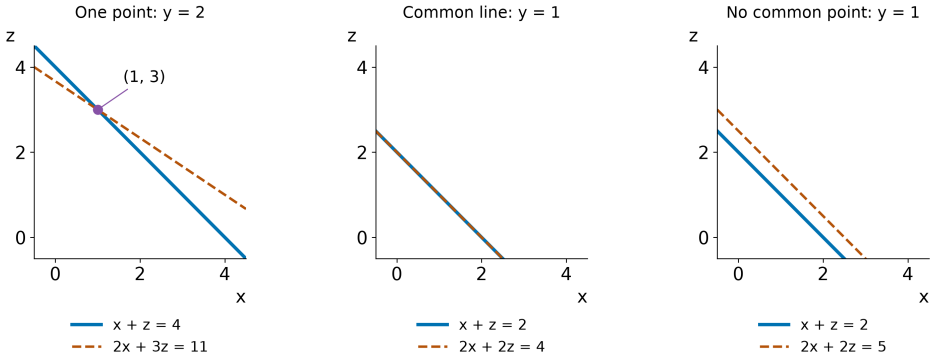
For $x+y+z=3$ and $x+2y+z=4$, subtraction gives $y=1$ and $x+z=2$. Add the equation $2x+3y+2z=7$. It adds no constraint:

$$(x,y,z)=(2-t,1,t), \qquad t\in\mathbb{R}.$$

If its right-hand side is changed to 8, subtracting the first two equations gives $0=1$: no solution.

> A zero determinant permits a consistent family or an inconsistent system. Coincident planes may leave a whole plane of solutions.

Geometry after eliminating one unknown



The panels fix the value of y first; the remaining lines show conditions on x, z.

Separate exceptional parameter values before dividing

$$\begin{aligned} x+y+z &= 3 \\ x+ay+z &= 4 \\ x+y+(a+2)z &= 3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} (a-1)y &= 1 \\ (a+1)z &= 0. \end{aligned}$$

The determinant is $(a-1)(a+1)$.

- $a \neq \pm 1$: $y = 1/(a-1)$, $z = 0$, $x = 3 - 1/(a-1)$.
- $a = 1$: $0 = 1$, so no solution.
- $a = -1$: $(x, y, z) = (7/2 - t, -1/2, t)$, a common line.

> Do not cancel a parameter factor before checking where it is zero.

An eigenvector is non-zero and stays on its line

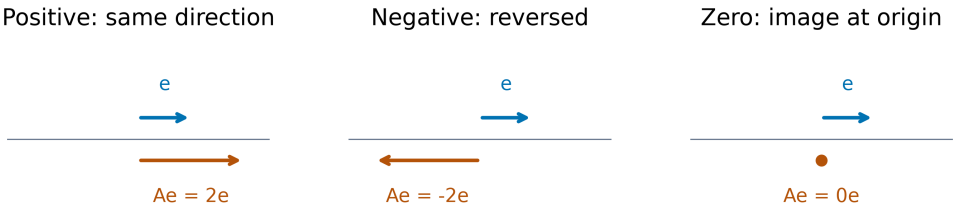
$$A\mathbf{e} = \lambda\mathbf{e}, \qquad \mathbf{e} \neq \mathbf{0}.$$

The eigenvalue λ is a scalar. A non-zero multiple of $\mathbf{e}$ is another eigenvector for the same value.

- $\lambda > 0$: it can stretch or shorten the vector, retaining its direction.
- $\lambda < 0$: the direction reverses as well.
- $\lambda = 0$: it maps to the origin, whose direction is undefined.

> The zero vector solves the equation but is never an eigenvector.

Positive, negative and zero eigenvalues



The original vector e is non-zero in every panel.

Each special vector lies on an invariant line; a zero eigenvalue collapses it to the origin.

Worked: solve the characteristic equation and check vectors

For $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$,

$$\det(\lambda I - A) = (\lambda - 2)^2 - 1 = (\lambda - 1)(\lambda - 3) = 0.$$

Solve $(A - \lambda I)\mathbf{e} = 0$ separately for each value:

$$\lambda = 1: \ \mathbf{e} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda = 3: \ \mathbf{e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Check $A(1,-1)^T = (1,-1)^T$ and $A(1,1)^T = 3(1,1)^T$.

> The eigenvalues sum to the trace 4 and multiply to the determinant 3.

Extend the method to a triangular three-by-three matrix

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \chi_B(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda + 1).$$

The diagonal entries give the roots because the characteristic matrix is triangular.

λ	conditions	one eigenvector
2	$v = w = 0$	$(1, 0, 0)^T$
3	$u = v,\ w = 0$	$(1, 1, 0)^T$
-1	$u = v = 0$	$(0, 0, 1)^T$

Do not row-reduce B to find its eigenvalues: ordinary row operations need not preserve them.

Derive the eigenvalues of powers, shifts and inverses

Start from $B\mathbf{e} = \lambda\mathbf{e}$:

$$B^2\mathbf{e} = B(\lambda\mathbf{e}) = \lambda^2\mathbf{e},$$
$$(B + cI)\mathbf{e} = (\lambda + c)\mathbf{e}.$$

Repeating gives $B^n\mathbf{e} = \lambda^n\mathbf{e}$ for positive integers n. If B is invertible, multiply the original equation by B^{-1}:

$$B^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}, \qquad \lambda \neq 0.$$

> Positive powers do not need invertibility; an inverse does. Never divide by a vector.

Diagonalisation matches columns and eigenvalues

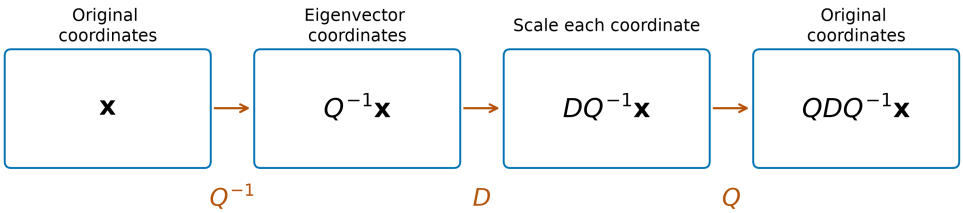
Put the eigenvectors of B in the columns of Q, in the same order as D:

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \operatorname{diag}(2, 3, -1), \quad Q^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $BQ = QD$ checks all three eigenvector equations and gives $B = QDQ^{-1}$. The columns must be linearly independent. Real, distinct eigenvalues guarantee this in the syllabus cases.

> D is a diagonal matrix. Not every matrix is diagonalisable. The order in D must match the columns of Q.

Read the three maps from right to left



Read matrix products from right to left: Q^{-1} first, then D, then Q.

Q^{-1} changes coordinates, D scales eigenvector coordinates, then Q changes back.

Cancel the middle factors to compute powers

$$B^n = (QDQ^{-1})^n = QD^nQ^{-1} = \begin{pmatrix} 2^n & 3^n - 2^n & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}.$$

The cancellations are $Q^{-1}Q = I$; this is not entry-by-entry exponentiation of B. For the earlier 2×2 matrix,

$$A^n = \frac{1}{2}\begin{pmatrix} 1 + 3^n & 3^n - 1 \\ 3^n - 1 & 1 + 3^n \end{pmatrix}.$$

> $n = 0$ gives I. Since both matrices are invertible, negative integer powers also work.

Reuse the basis for a polynomial in the matrix

If f is a polynomial, $f(B) = Qf(D)Q^{-1}$. Keep the eigenvectors and transform each eigenvalue.

$$\begin{aligned}(B+I)^2 &= Q\,\mathrm{diag}(9,16,0)\,Q^{-1},\\ (B-2I)^3 &= Q\,\mathrm{diag}(0,1,-27)\,Q^{-1},\\ B^{-1} &= Q\,\mathrm{diag}(1/2,1/3,-1)\,Q^{-1}.\end{aligned}$$

The first matrix is singular but still diagonalisable. Repeated transformed eigenvalues do not invalidate an already invertible Q.

> The eigenvalue order remains $2, 3, -1$ throughout.

Reconstruct a matrix from its eigenvectors

Suppose $(1,1)^T$ has eigenvalue 4, and $(1,-1)^T$ has eigenvalue 2.

$$R = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Check both supplied vector equations in the result. Scaling either column of Q by a non-zero number does not change QDQ^{-1}.

> The eigenvectors and their matched eigenvalues determine the linear map.

Cayley–Hamilton replaces high powers with lower ones

The characteristic polynomial of B is $\lambda^3 - 4\lambda^2 + \lambda + 6$. The Cayley–Hamilton theorem gives

$$B^3 - 4B^2 + B + 6I = 0, \qquad B^3 = 4B^2 - B - 6I.$$

Multiply by B and substitute the identity again:

$$B^4 = 4B^3 - B^2 - 6B = 15B^2 - 10B - 24I.$$

Also $(B - 2I)^3 = -2B^2 + 11B - 14I$.

> A scalar constant becomes a multiple of I, not a scalar added to a matrix.

Obtain an inverse from the characteristic identity

Since $\det B = -6 \neq 0$, multiply the Cayley–Hamilton identity by B^{-1}:

$$B^2 - 4B + I + 6B^{-1} = 0.$$

Therefore

$$B^{-1} = \frac{-B^2 + 4B - I}{6}.$$

For A, the identity $A^2 - 4A + 3I = 0$ similarly gives $A^{-1} = (4I - A)/3$.

> Check invertibility before using B^{-1}, then verify by multiplying by B.

Matrix practice: use one basis in three ways

For $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$:

1. Compute B^3 using diagonalisation.
2. Find the eigenvalues and singularity of $(B + I)^2$.
3. Express B^{-1} as a polynomial in B.

Use a second method for at least one part: direct multiplication or Cayley–Hamilton.

Matrix practice: worked answers and checks

$$B^3 = \begin{pmatrix} 8 & 19 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The changed matrix has eigenvalues $9, 16, 0$; determinant zero, so it is singular. Its existing eigenvector basis still diagonalises it.

$$B^{-1} = (-B^2 + 4B - I)/6 = \begin{pmatrix} 1/2 & -1/6 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

For the inverse, the off-diagonal product entry is $2(-1/6) + 1(1/3) = 0$.

3

2.3 Differentiation

Differentiate the six hyperbolic functions

f	f'	f	f'
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$	$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$	$\coth x$	$-\operatorname{cosech}^2 x$

For example, the quotient rule gives

$$(\tanh x)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x.$$

Cosech and coth derivatives apply on intervals excluding zero. Arguments still need the chain rule.

An inverse derivative needs its branch sign

For $y = \sin^{-1} x$, $\sin y = x$, so $\cos y\, y' = 1$. On the principal branch $\cos y \geq 0$:

$$(\sin^{-1} x)' = \frac{1}{\sqrt{1 - x^2}}, \qquad |x| < 1.$$

For $y = \cos^{-1} x$, $-\sin y\, y' = 1$ and the principal branch has $\sin y \geq 0$:

$$(\cos^{-1} x)' = -\frac{1}{\sqrt{1 - x^2}}, \qquad (\tan^{-1} x)' = \frac{1}{1 + x^2}.$$

Inverse sine/cosine have endpoint values at ± 1, but no finite derivative there.

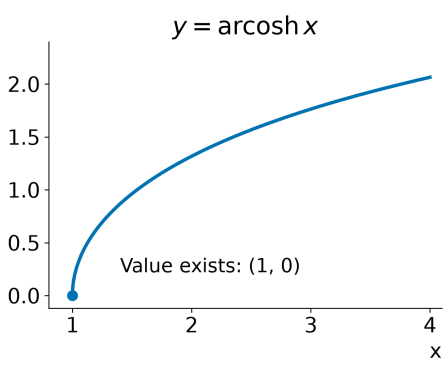
Derive inverse hyperbolic derivatives the same way

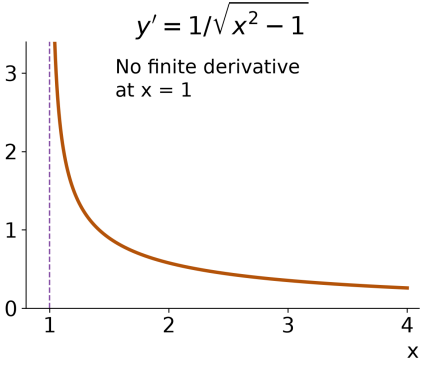
For inverse hyperbolic functions, use the defining relation. If $y = \sinh^{-1} x$, then $\cosh y\, y' = 1$ and $\cosh y = \sqrt{1 + x^2} > 0$. Similarly use the selected branch for inverse cosh and the tanh identity:

$$\begin{aligned} (\sinh^{-1} x)' &= 1/\sqrt{1 + x^2}, & x &\in \mathbb{R}, \\ (\cosh^{-1} x)' &= 1/\sqrt{x^2 - 1}, & x &> 1, \\ (\tanh^{-1} x)' &= 1/(1 - x^2), & |x| &< 1. \end{aligned}$$

A domain for the inverse's values can include a point excluded by its finite derivative.

The value exists where the derivative becomes unbounded





$\cosh^{-1} 1 = 0$, while the finite derivative formula requires $x > 1$.

Worked: retain the inner derivative and its domain

Apply $\frac{d}{dx} f(u(x)) = f'(u(x))\,u'(x)$:

$$\frac{d}{dx}\cosh^{-1}(2x) = \frac{2}{\sqrt{4x^2-1}}, \qquad x > 1/2,$$

$$\frac{d}{dx}\sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}, \qquad |x| < 1/2,$$

$$\frac{d}{dx}\tanh^{-1}(x^2) = \frac{2x}{1-x^4}, \qquad |x| < 1.$$

The first function exists at $x = 1/2$, but its derivative does not have a finite value there.

> A correct outer formula without $u'(x)$ is an incomplete chain-rule answer.

Differentiate an implicit relation without freezing y

For $x^2 + xy + y^2 = 7$, treat y as a function of x:

$$2x + y + xy' + 2yy' = 0,$$

$$y' = -\frac{2x + y}{x + 2y}, \qquad x + 2y \neq 0.$$

At $(1, 2)$, $y' = -4/5$. Differentiate the unsimplified relation once more:

$$2 + 2y' + 2(y')^2 + (x + 2y)y'' = 0.$$

> The derivative of yy' is $(y')^2 + yy''$, not just yy''.

Worked: substitute only after the second differentiation

At $(1,2)$, use $y' = -4/5$ in

$$y'' = -\frac{2 + 2y' + 2(y')^2}{x + 2y}.$$

Thus

$$y'' = -\frac{2 - 8/5 + 32/25}{5} = -\frac{42}{125}.$$

A zero denominator in a slope formula needs separate analysis; it does not prove that the original curve has no point there.

> Keep x, y, y' variable until all the required differentiations are complete.

Combine an inverse function with an implicit product

Given $y^2 - \sinh^{-1}(xy) = 1 - \ln 2$, the point $(3/4, 1)$ lies on the curve. Set $p = y'$ and $S = \sqrt{1 + x^2 y^2}$. Differentiating gives

$$2yp - \frac{y + xp}{S} = 0.$$

At the point $S = 5/4$, so $2p - (1 + 3p/4)/(5/4) = 0$, giving $p = 4/7$.

> Both variables occur inside the inverse function: $(xy)' = y + xy'$.

Carry the square-root chain rule into the second derivative

Differentiate the preceding first relation, using $S' = xy(y + xp)/S$:

$$2p^2 + 2yy' - \frac{2p + xy'}{S} + \frac{xy(y + xp)^2}{S^3} = 0.$$

At $(3/4, 1)$ with $p = 4/7$, this becomes

$$\frac{7}{5}y'' + \frac{128}{245} = 0, \qquad y'' = -\frac{128}{343}.$$

> The square-root denominator depends on y too; dropping that dependence loses a term.

Parametric derivatives: divide by dx over dt twice

For $x = t^2 - 4$, $y = e^t$ and $t > 0$,

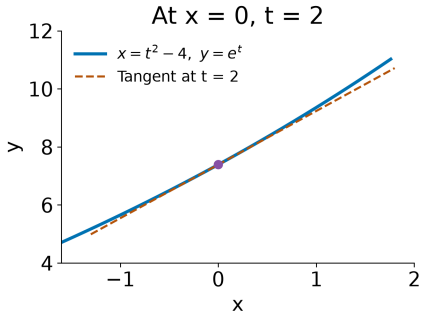
$$\frac{dy}{dx} = \frac{e^t}{2t}, \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(e^t/(2t))}{2t} = \frac{e^t(t-1)}{4t^3}.$$

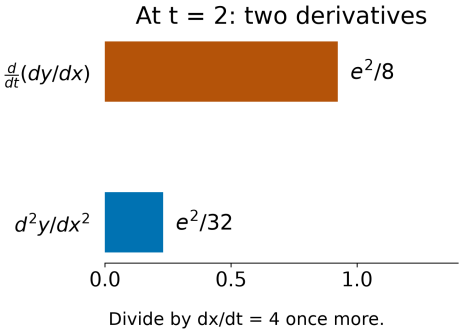
At $x = 0$, the branch gives $t = 2$:

$$y = e^2, \qquad y_x = e^2/4, \qquad y_{xx} = e^2/32.$$

Differentiating the slope with respect to t alone gives $e^2/8$, not the second x-derivative.

The parameter and the coordinate are different variables





The extra division by dx/dt changes the second derivative; identify the branch before evaluating.

Build Maclaurin coefficients from derivatives

Maclaurin's series uses derivatives evaluated at zero:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots .$$

For $f(x) = e^{1/(x+3)}$, write $E = e^{1/3}$.

$$f' = -(x+3)^{-2} f, \qquad f'' = \{2(x+3)^{-3} + (x+3)^{-4}\} f.$$

At zero the values are $E, -E/9, 7E/81$, so

$$f(x) = E - Ex/9 + 7Ex^2/162 + \cdots .$$

> The quadratic coefficient is $f''(0)/2!$, not $f''(0)$.

Use successive implicit derivatives for a tan expansion

For $y = \tan x$, start with $y' = 1 + y^2$:

$$y'' = 2yy', \qquad y^{(3)} = 2(y')^2 + 2yy''.$$

At zero, $y = 0$, $y' = 1$, $y'' = 0$ and $y^{(3)} = 2$. Thus

$$\tan x = x + \frac{2}{3!}x^3 + \cdots = x + x^3/3 + \cdots.$$

For $\ln(1 + x)$, derivative values $0, 1, -1, 2$ instead give $x - x^2/2 + x^3/3 + \cdots$.

> The derivative relation can be simpler than repeatedly differentiating a quotient.

Locate x zero before a parametric expansion

For $x = t^2 - 4$, $y = e^t$, $t > 0$, $x = 0$ means $t = 2$. Use the earlier x-derivatives:

$$y = e^2\left(1 + \frac{x}{4} + \frac{x^2}{64}\right) + \cdots .$$

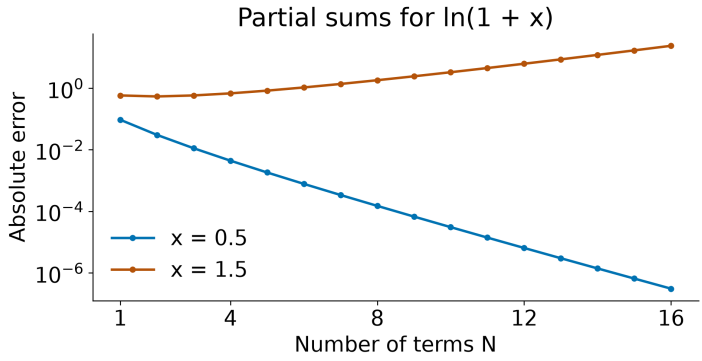
For $x = \sinh t$, $y = 2t + \cosh t$, one obtains

$$y_x = \frac{2 + \sinh t}{\cosh t}, \qquad y_{xx} = \frac{1 - 2\sinh t}{\cosh^3 t}.$$

Now $x = 0$ means $t = 0$, giving $y = 1 + 2x + x^2/2 + \cdots$.

> A parameter value equal to zero is not automatically the expansion point.

More terms are not a universal promise of improvement



Only $x = 0.5$ lies inside the logarithm series interval $-1 < x \leq 1$.

Differentiation practice: point, variable, then factorial

For $x = t^2 - 1$, $y = e^t$ and $t > 0$:

1. Find dy/dx and d^2y/dx^2.
2. Find the polynomial through x^2 in the Maclaurin expansion of y.
3. Explain why substituting $t = 0$ is invalid for this expansion.

A finite Taylor polynomial is local. An infinite series needs an interval of convergence; outside it the series can diverge.

Differentiation practice: worked answers

$$y_x = \frac{e^t}{2t}, \qquad y_{xx} = \frac{e^t(t-1)}{4t^3}.$$

At $x = 0$, use $t = 1$, not $t = 0$. The derivative values are $y = e$, $y_x = e/2$, $y_{xx} = 0$.

$$y = e + \frac{e}{2}x + 0x^2 + \cdots .$$

The point $t = 0$ has $x = -1$, outside the chosen branch's expansion point, and $dx/dt = 0$ there.

> A zero quadratic coefficient is a result, not a reason to omit the second-derivative calculation.

4

2.4 Integration

Reverse the derivatives and retain the integration constant

$$\int \sinh x \, dx = \cosh x + C, \qquad \int \cosh x \, dx = \sinh x + C,$$

$$\int \tanh x \, dx = \ln \cosh x + C, \qquad \int \coth x \, dx = \ln |\sinh x| + C,$$

$$\int \operatorname{sech} x \, dx = \tan^{-1}(\sinh x) + C,$$

$$\int \operatorname{cosech} x \, dx = \ln |\tanh(x/2)| + C.$$

The coth and cosech antiderivatives apply on intervals excluding zero.

> Check by differentiation. For an inner factor, $\int \cosh 3x \, dx = \sinh 3x/3 + C$.

Three surd forms have three different inverse answers

For $a > 0$:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) + C, \qquad |x| < a,$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}(x/a) + C, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(x/a) + C, \qquad x > a.$$

For either interval $|x| > a$, the third form can instead use $\ln|x + \sqrt{x^2 - a^2}| + C$.

> On $x < -a$, use the logarithmic form or $-\cosh^{-1}(-x/a) + C$; check its sign.

Complete the square before selecting a substitution

Completing the square exposes the standard form:

$$x^2 + 4x + 13 = (x+2)^2 + 9.$$

Thus

$$\int \frac{dx}{\sqrt{x^2+4x+13}} = \sinh^{-1}\left(\frac{x+2}{3}\right) + C.$$

For a rational denominator,

$$4x^2 + 8x + 13 = 4(x+1)^2 + 9, \qquad \int \frac{dx}{4x^2+8x+13} = \frac{1}{6}\tan^{-1}\frac{2(x+1)}{3} + C.$$

> The outside multiplier must survive both the quadratic coefficient and the inner derivative.

Worked: a hyperbolic substitution removes a positive surd

For $\sqrt{x^2+4}$ set $x = 2\sinh u$, so $dx = 2\cosh u\,du$ and the root is $2\cosh u > 0$.

$$\begin{aligned}\int \sqrt{x^2+4}\,dx &= 4\int \cosh^2 u\,du \\ &= 2u + \sinh 2u + C \\ &= 2\sinh^{-1}(x/2) + \frac{x}{2}\sqrt{x^2+4} + C.\end{aligned}$$

From $x = 0$ to $3/2$, the new limits are $u = 0, \ln 2$; the integral is $2\ln 2 + 15/8$.

> Change both limits, or return to x before using the original limits.

Worked: a trigonometric substitution needs a branch

Set $x = 2\sin\theta$. Then $\sqrt{4 - x^2} = 2|\cos\theta|$. Choose $-\pi/2 \le \theta \le \pi/2$ before replacing this by $2\cos\theta$.

$$\begin{aligned}\int_0^1 \sqrt{4 - x^2}\, dx &= 4\int_0^{\pi/6} \cos^2\theta\, d\theta \\ &= [2\theta + \sin 2\theta]_0^{\pi/6} \\ &= \pi/3 + \sqrt{3}/2.\end{aligned}$$

The square root is non-negative; a sign cannot be dropped merely to simplify the algebra.

Repeated parts can return to the original integral

Let $I = \int e^x \sin x \, dx$. Two integrations by parts give

$$I = e^x \sin x - \int e^x \cos x \, dx,$$

$$\int e^x \cos x \, dx = e^x \cos x + I.$$

Collect the repeated integral:

$$2I = e^x(\sin x - \cos x), \qquad I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

Returning to I is useful: move it to the other side instead of repeating the same work.

A half-angle substitution makes a rational integral

Set $t = \tan(x/2)$:

$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2\,dt}{1+t^2}.$$

For $0 \le x \le \pi/2$, the new limits are $0 \le t \le 1$:

$$\int_0^{\pi/2} \frac{dx}{2+\cos x} = 2\int_0^1 \frac{dt}{3+t^2} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1 = \frac{\pi}{3\sqrt{3}}.$$

> Choose a branch that does not cross an odd multiple of π.

Derive a reduction formula before using it

Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$. Use $u = \sin^{n-1} x$, $dv = \sin x\,dx$. For $n \geq 2$, the boundary term is zero:

$$\begin{aligned} I_n &= [-\sin^{n-1} x \cos x]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x\,dx \\ &= (n-1)(I_{n-2} - I_n). \end{aligned}$$

Thus $I_n = (n-1)I_{n-2}/n$.

$I_0 = \pi/2$, $I_1 = 1$; hence $I_4 = 3\pi/16$. Even and odd indices end at different bases.

A non-zero boundary term changes the recurrence

For $J_n = \int_0^1 x^n e^{2x}\,dx$, integration by parts gives, for $n \geq 1$,

$$J_n = \left[\frac{x^n e^{2x}}{2}\right]_0^1 - \frac{n}{2}J_{n-1} = \frac{e^2}{2} - \frac{n}{2}J_{n-1}.$$

Evaluate the base directly:

$$J_0 = (e^2 - 1)/2, \quad J_1 = (e^2 + 1)/4, \quad J_2 = (e^2 - 1)/4.$$

> Applying the recurrence at $n = 0$ would lose the lower-endpoint contribution.

Two parts steps can lower the index by two

For $K_n = \int_0^2 (2-x)^n \sinh x \, dx$, the first parts step gives

$$K_n = -2^n + n \int_0^2 (2-x)^{n-1} \cosh x \, dx.$$

For $n \geq 2$, the second step has zero boundary term and yields

$$K_n = -2^n + n(n-1)K_{n-2}.$$

Direct base values are $K_0 = \cosh 2 - 1$, $K_1 = \sinh 2 - 2$; hence $K_2 = 2\cosh 2 - 6$.

> State the index range and evaluate the base cases separately.

A suggested derivative can supply the recurrence

Given $H_n = \int_0^{\pi/4} \sec^n x\,dx$, differentiate the hinted expression:

$$(\tan x \sec^{n-2} x)' = (n-1)\sec^n x - (n-2)\sec^{n-2} x.$$

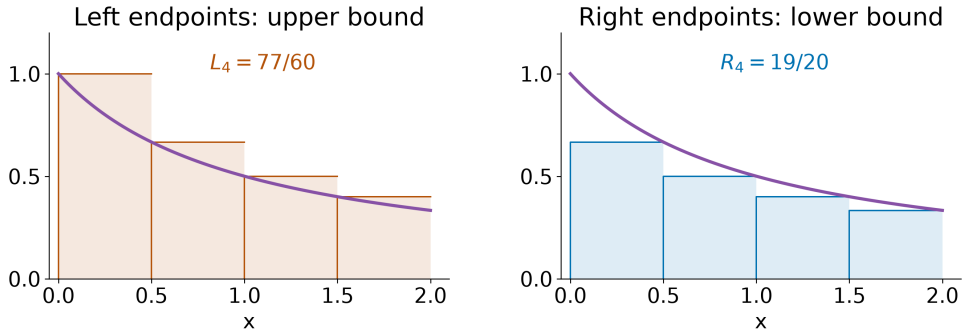
Integrate both sides with the stated limits:

$$2^{(n-2)/2} = (n-1)H_n - (n-2)H_{n-2}.$$

For $n \geq 2$,

$$H_n = \frac{2^{(n-2)/2} + (n-2)H_{n-2}}{n-1}, \qquad H_2 = 1, \quad H_4 = 4/3.$$

Rectangles compare areas, not just heights



y = 1/(1 + x), rectangle width = 1/2; integral = ln 3.

For $1/(1+x)$ on $[0, 2]$, right rectangles underestimate and left rectangles overestimate.

Keep the width when forming numerical bounds

With four rectangles on $[0,2]$, $\Delta x = 1/2$:

$$R_4 = \tfrac{1}{2}(2/3 + 1/2 + 2/5 + 1/3) = 19/20,$$
$$L_4 = \tfrac{1}{2}(1 + 2/3 + 1/2 + 2/5) = 77/60.$$

Therefore $19/20 < \ln 3 < 77/60$. For unit-width rectangles under $1/x$, integers $n \geq 2$ give

$$\ln(n+1) < \sum_{r=1}^{n} 1/r < 1 + \ln n.$$

> For an increasing function the left/right directions reverse. Check monotonicity on the actual interval.

Rewrite a sum as height times a shrinking width

$$\sum_{r=1}^{n} \frac{n}{4n^2 + r^2} = \frac{1}{n} \sum_{r=1}^{n} \frac{1}{4 + (r/n)^2}.$$

This is a right-endpoint Riemann sum for the continuous function $f(x) = 1/(4 + x^2)$ on $[0, 1]$.

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{4n^2 + r^2} = \int_0^1 \frac{dx}{4 + x^2} = \frac{1}{2} \tan^{-1}(1/2).$$

The function decreases; the finite sum is below the integral. The left-right gap is $1/(20n) \to 0$.

Cartesian arc length begins with a small displacement

From $ds^2 = dx^2 + dy^2$, for a single traversal with increasing x,

$$L = \int_a^b \sqrt{1 + (dy/dx)^2}\, dx.$$

For $y = \cosh x$ on $[0, \ln 2]$, the square root is $\cosh x > 0$, giving $L = 3/4$. For $y = x^2$ on $[0, 1]$,

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx = \frac{\sqrt{5}}{2} + \frac{\sinh^{-1} 2}{4} > \sqrt{2}.$$

> Length is not area under the curve. A curved route is longer than its chord.

Parametric arc length uses speed and a single traversal

For $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$, $0 \le t \le \pi$,

$$\sqrt{x'^2 + y'^2} = 3\sqrt{2 - 2\cos t} = 6\sin(t/2) \ge 0.$$

Thus $L = \int_0^{\pi} 6\sin(t/2)\,dt = 12$. For $x = \ln\cosh 2t$, $y = \tan^{-1}(\sinh 2t)$, the speed is

$$\sqrt{4\tanh^2 2t + 4\operatorname{sech}^2 2t} = 2.$$

On $0 \le t \le \ln 2$, its length is $2\ln 2$.

> The parameter interval must trace the required arc once.

Polar length includes radial and sideways changes

Since $ds^2 = dr^2 + r^2 d\theta^2$,

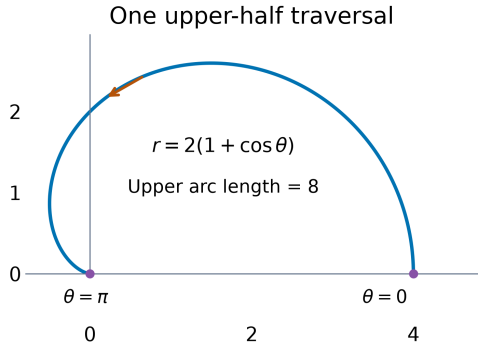
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2}\, d\theta.$$

For $r = 2(1 + \cos\theta)$ on $[0, \pi]$, the square root is $4|\cos(\theta/2)| = 4\cos(\theta/2)$.

$$L = \int_{0}^{\pi} 4\cos(\theta/2)\, d\theta = 8.$$

Use radians and the intended arc; this interval traces the upper half once.

Follow the polar arc before integrating



The upper cardioid arc starts at radius 4 and ends at the pole; its length is 8.

Surface area uses distance to the chosen axis

For a surface area of revolution, a narrow band has area $2\pi\rho\,ds$, where ρ is a non-negative distance. For $y = x + 1$, $0 \le x \le 2$, $ds = \sqrt{2}\,dx$:

$$S_x = 2\pi\sqrt{2}\int_0^2 (x+1)\,dx = 8\pi\sqrt{2},$$

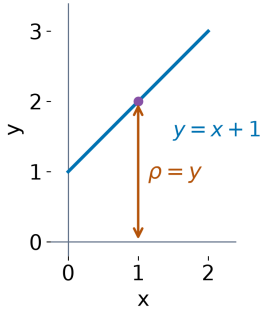
$$S_y = 2\pi\sqrt{2}\int_0^2 x\,dx = 4\pi\sqrt{2}.$$

About the x-axis use $|y|$; about the y-axis use $|x|$.

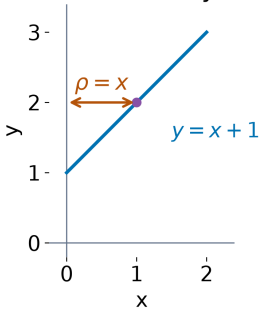
> $\pi\int y^2\,dx$ computes volume, not surface area. The required surface forms are Cartesian or parametric.

Changing the axis changes the band radius

Rotation about x-axis



Rotation about y-axis



The same arc can generate two different surface areas because the distance to the axis changes.

A parametric surface retains both radius and speed

For $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le \pi/2$, the speed is 2.

$$S_x = 2\pi \int_0^{\pi/2} (2\sin t)(2)\,dt = 8\pi,$$
$$S_y = 2\pi \int_0^{\pi/2} (2\cos t)(2)\,dt = 8\pi.$$

Each rotation generates a hemisphere of radius 2.

> Use $2\pi \int \rho(t)\sqrt{x'(t)^2 + y'(t)^2}\,dt$; decreasing coordinates do not produce negative speed.

An unfamiliar surface substitution can reverse the limits

Rotate $y = \tanh x$, $0 \leq x \leq \tanh^{-1}(1/3)$, about the x-axis:

$$S = 2\pi \int_0^{\tanh^{-1}(1/3)} \tanh x \sqrt{1 + \operatorname{sech}^4 x}\, dx.$$

Set $u = \sqrt{1 + \operatorname{sech}^4 x}$. Then

$$\frac{du}{dx} = -\frac{2 \operatorname{sech}^4 x \tanh x}{u} < 0 \quad (x > 0).$$

The endpoint values are $u = \sqrt{2}$ and $u = \sqrt{145}/9$.

> At the second endpoint, $\operatorname{sech}^2 x = 8/9$. The transformed upper x-limit is the lower u-limit.

Finish the transformed surface integral

Using $u^2 - 1 = \operatorname{sech}^4 x$ and reversing the limits gives

$$S = \pi \int_{\sqrt{145}/9}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\, du.$$

Split $u^2/(u^2 - 1) = 1 + 1/(u^2 - 1)$:

$$S = \pi \left[u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| \right]_{\sqrt{145}/9}^{\sqrt{2}}.$$

The integrand is positive on these limits, agreeing with a positive surface area.

> A correct antiderivative cannot repair the wrong radius, speed or transformed limits.

Integration practice: reduction, limit and surface

1. Use $I_n = (n-1)I_{n-2}/n$ to find I_6.
2. Find $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{9n^2 + r^2}$.
3. Rotate $y = x + 2$, $0 \leq x \leq 1$, about the y-axis. Find the surface area.

For each part, name the base value, rectangle width or rotation radius before calculating.

Integration practice: worked answers

$$I_6 = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{5\pi}{32}.$$

The limit is a Riemann sum with width $1/n$:

$$\int_0^1 \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}(1/3).$$

For the surface, $\rho = x$ and $ds = \sqrt{2}\,dx$:

$$S_y = 2\pi\sqrt{2}\int_0^1 x\,dx = \pi\sqrt{2}.$$

Changing the vertical intercept does not change this y-axis radius or the slope.

5

2.5 Complex numbers

Multiplication changes length and adds angles

The polar form writes $z = r\operatorname{cis}\theta$, $w = s\operatorname{cis}\phi$, where $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

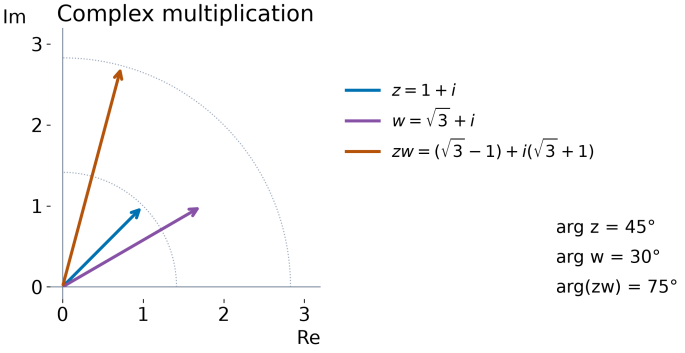
$$zw = rs\operatorname{cis}(\theta+\phi), \qquad z/w = (r/s)\operatorname{cis}(\theta-\phi), \quad w \neq 0.$$

Multiplication is a rotation alone only when the multiplier has modulus one. For $z = 1 + i$, $w = \sqrt{3} + i$, the product is

$$zw = (\sqrt{3}-1) + i(\sqrt{3}+1), \quad |zw| = 2\sqrt{2}, \quad \arg(zw) = 75^\circ.$$

> Arguments are equivalent modulo 2π; choose the required interval when reporting them.

Track the scale and rotation together



Moduli multiply: $\sqrt{2} \times 2 = 2\sqrt{2}$. Arguments add: $45° + 30° = 75°$.

Positive and negative integer powers

De Moivre's theorem gives

$$(r\operatorname{cis}\theta)^n = r^n \operatorname{cis}(n\theta)$$

for integer n; negative powers require $r > 0$. For $z = 1 + i = \sqrt{2}\operatorname{cis}(\pi/4)$,

$$z^6 = 8\operatorname{cis}(3\pi/2) = -8i, \qquad z^{-2} = \tfrac{1}{2}\operatorname{cis}(-\pi/2) = -i/2.$$

> The reciprocal reverses the angle and inverts the modulus. Zero has no negative power or argument.

Prove de Moivre for positive integer exponents

Let $u = \cos\theta + i\sin\theta$. The case $n = 1$ holds. Assume $u^k = \cos k\theta + i\sin k\theta$ for a positive integer k.

$$\begin{aligned} u^{k+1} &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\ &= \cos(k\theta + \theta) + i\sin(k\theta + \theta), \end{aligned}$$

by the angle-addition formulae. Thus the statement for k implies that for $k + 1$. Mathematical induction now proves the result for all positive integers.

For a negative exponent, use $u^{-1} = \cos\theta - i\sin\theta$ and the positive theorem.

Expand a fifth power before comparing parts

Write $c = \cos\theta$, $s = \sin\theta$. De Moivre and the binomial theorem give

$$\cos 5\theta = c^5 - 10c^3s^2 + 5cs^4 = 16c^5 - 20c^3 + 5c,$$
$$\sin 5\theta = 5c^4s - 10c^2s^3 + s^5 = 16s^5 - 20s^3 + 5s.$$

Use $s^2 = 1 - c^2$ in the real part, or $c^2 = 1 - s^2$ in the imaginary part.

> The real and imaginary equations are separate identities. Show the expansion if the question asks for a derivation.

Divide parts only where the denominators are non-zero

Divide the imaginary and real parts of $(c + is)^3$ by c^3, where $c \neq 0$:

$$\tan 3\theta = \frac{3t - t^3}{1 - 3t^2}, \qquad t = \tan\theta.$$

The same method for the fifth power gives

$$\tan 5\theta = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$

The first quotient also requires $\cos 3\theta \neq 0$; the second requires $\cos 5\theta \neq 0$.

> An identity produced by division retains the restrictions used in that division.

Turn a power into multiple angles with exponentials

Use $2i\sin\theta = e^{i\theta} - e^{-i\theta}$. Raising to the sixth power gives

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20.$$

Therefore

$$\sin^6\theta = \frac{10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta}{32}.$$

Similarly $\sin^4\theta = (3 - 4\cos 2\theta + \cos 4\theta)/8$.

> Pair opposite exponentials into cosines. Keep the minus sign from $i^6 = -1$.

Reciprocal identities need the original sine restriction

From $\sin 3\theta = 3s - 4s^3$,

$$\operatorname{cosec}^3 \theta = \frac{4}{3\sin\theta - \sin 3\theta}, \qquad \sin\theta \neq 0.$$

For fifth powers, combine the fifth- and third-angle identities:

$$\sin 5\theta - 5\sin 3\theta + 10\sin\theta = 16\sin^5\theta.$$

Thus

$$\operatorname{cosec}^5 \theta = \frac{16}{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}.$$

The denominator is $16\sin^5\theta$; it is non-zero exactly when the reciprocal is defined.

A polynomial can become a multiple-angle equation

For $32x^5 - 40x^3 + 10x - 1 = 0$, put $x = \cos\theta$, $0 \le \theta \le \pi$.

$$2\cos 5\theta - 1 = 0 \quad \Longrightarrow \quad \cos 5\theta = 1/2.$$

The five distinct roots are

$$x = \cos(q\pi/15), \qquad q \in \{1, 5, 7, 11, 13\}.$$

Cosine is one-to-one on this angle interval, and a degree-five polynomial has at most five distinct roots.

> Count distinct values, not just the number of angles written down.

Transfer the roots to a reciprocal polynomial

For $x^5 - 10x^4 + 40x^2 - 32 = 0$, zero is not a root. Divide by x^5 and put $1/x = \cos\theta$:

$$1 - 10\cos\theta + 40\cos^3\theta - 32\cos^5\theta = 0.$$

This is $1 - 2\cos 5\theta = 0$, so

$$x = \sec(q\pi/15), \qquad q \in \{1, 5, 7, 11, 13\}.$$

Each cosine value is non-zero, and the reciprocal map retains distinctness.

Rational powers require a branch convention

For $z = r\operatorname{cis}\theta \neq 0$ and a reduced fraction p/q, $q > 0$, the branches are

$$w_k = r^{p/q}\operatorname{cis}\frac{p(\theta + 2k\pi)}{q}, \qquad k = 0, \ldots, q-1.$$

For $8^{2/3}$, these give all three solutions of $w^3 = 64$, with modulus 4. A principal value selects one branch. An equation asking for all roots needs the whole set.

> Reduce p/q before counting branches; a negative exponent also excludes zero.

Find all roots before choosing principal arguments

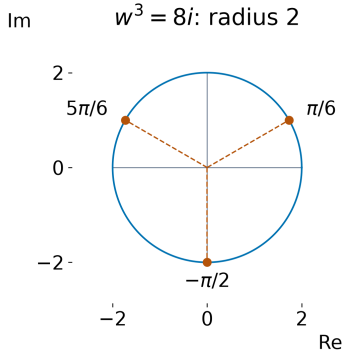
To solve $w^3 = 8i = 8\,\mathrm{cis}(\pi/2)$, use

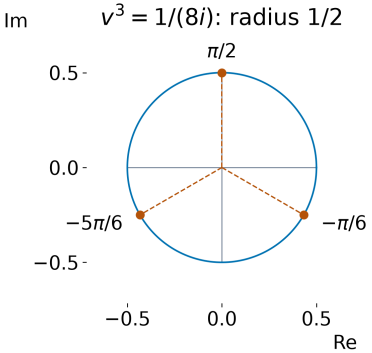
$$|w| = 2, \qquad \arg w = \pi/6 + 2k\pi/3, \quad k = 0, 1, 2.$$

In a principal interval these are $\pi/6, 5\pi/6, -\pi/2$. The reciprocals solve $v^3 = 1/(8i)$ and have modulus $1/2$ with opposite arguments.

> Changing an angle by 2π changes its representative, not the root.

Reciprocals invert radii and reverse arguments





Cube roots have radius 2; reciprocals have radius $1/2$ and opposite angles.

Roots of unity give a factorisation and a sum

The n roots of unity are $z_k = \operatorname{cis}(2k\pi/n)$, $k = 0, \ldots, n-1$. They lie on the unit circle with spacing $2\pi/n$ radians.

$$z^n - 1 = \prod_{k=0}^{n-1} (z - z_k).$$

For $n > 1$, their sum is zero. If $\omega^3 = 1$ and $\omega \neq 1$,

$$\omega^3 - 1 = (\omega - 1)(1 + \omega + \omega^2) = 0 \quad \Longrightarrow \quad 1 + \omega + \omega^2 = 0.$$

> Roots give the linear factors $z - z_k$; the roots themselves are not the factors.

A power may visit only part of the root set

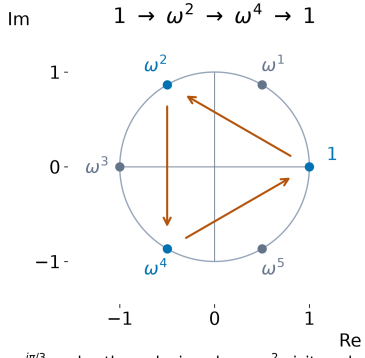
For $\omega = e^{i\pi/3}$, powers of ω visit all six roots. It is a primitive root of unity. Powers of ω^2 visit only $1, \omega^2, \omega^4$, then repeat. For any integer m,

$$\sum_{k=0}^{5} \omega^{mk} = \begin{cases} 6, & 6 \mid m, \\ 0, & 6 \nmid m. \end{cases}$$

If $6 \nmid m$, the geometric ratio is not one; its sixth power is one, making the finite numerator zero.

> When $6 \mid m$, use the original six terms, each equal to one.

A full root set can contain a shorter power cycle



$\omega = e^{i\pi/3}$ cycles through six values; ω^2 visits only three.

Powers of ω^2 visit only three of the six roots.

Combine C and iS before summing

Let $C = \sum_{r=0}^{n} \cos r\theta$ and $S = \sum_{r=0}^{n} \sin r\theta$. This geometric series has ratio $z = e^{i\theta} \neq 1$:

$$C + iS = \frac{1 - z^{n+1}}{1 - z} = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} e^{in\theta/2}.$$

Comparing parts gives

$$C = \frac{\sin((n+1)\theta/2)\cos(n\theta/2)}{\sin(\theta/2)}, \quad S = \frac{\sin((n+1)\theta/2)\sin(n\theta/2)}{\sin(\theta/2)}.$$

If θ is a multiple of 2π, evaluate directly: $C = n + 1$, $S = 0$.

Geometric weights change the complex ratio

For $C_n + iS_n = \sum_{r=1}^{n} \rho^r e^{ir\theta}$, put $q = \rho e^{i\theta}$:

$$C_n + iS_n = \frac{q(1-q^n)}{1-q}, \qquad q \neq 1.$$

Multiply by $1 - \overline{q}$ to obtain a real denominator $D = 1 - 2\rho\cos\theta + \rho^2$:

$$C_n + iS_n = \frac{q - q^{n+1} - \rho^2 + \rho^2 q^n}{D}.$$

For $q = (1+i)/2$, $q/(1-q) = i$ and $q^4 = -1/4$, so the first four terms sum to $5i/4$.

Binomial weights call for the binomial theorem

If the weights are $\binom{n}{r}$, combine the sums as

$$C + iS = \sum_{r=0}^{n} \binom{n}{r} e^{ir\theta} = (1 + e^{i\theta})^n.$$

On $-\pi < \theta < \pi$,

$$1 + e^{i\theta} = 2\cos(\theta/2)e^{i\theta/2}.$$

This binomial expansion and de Moivre give the real and imaginary parts. At $n = 6$, $\theta = \pi/3$, the result is $27e^{i\pi} = -27$.

> Starting at $r = 1$ removes the real term 1, not an imaginary term.

Justify the infinite limit before discarding the last term

If $|q| = \rho < 1$, then $q^n \to 0$, so

$$\sum_{r=1}^{\infty} q^r = \frac{q}{1-q}.$$

With $D = 1 - 2\rho\cos\theta + \rho^2$,

$$C_\infty = \frac{\rho\cos\theta - \rho^2}{D}, \qquad S_\infty = \frac{\rho\sin\theta}{D}.$$

For $q = (1+i)/2$, the infinite sum is i.

> A finite quotient alone does not justify an infinite sum when $|q| \geq 1$.

Bring down an index from a finite identity

For $q \neq 1$, apply $q\,d/dq$ to the finite geometric polynomial:

$$\sum_{r=1}^{n} rq^r = \frac{q\{1-(n+1)q^n + nq^{n+1}\}}{(1-q)^2}.$$

For $|q| < 1$, $n|q|^n \to 0$, giving

$$\sum_{r=1}^{\infty} rq^r = \frac{q}{(1-q)^2}.$$

At $q = i/2$, this is $-8/25 + 6i/25$.

> At $q = 1$, the finite sum is $n(n+1)/2$. Differentiating a finite sine sum with respect to θ gives indexed cosines.

Complex practice: choose the right sum and root set

1. Find all roots of $w^3 = -8$ with arguments in $(-\pi, \pi]$.
2. Find $\sum_{r=1}^{5} \binom{5}{r} i^r$.
3. Find $\sum_{r=1}^{\infty} r(i/3)^r$, including the convergence condition.

For the sums, combine real and imaginary parts into one complex expression first.

Complex practice: worked answers

The cube roots have modulus 2 and arguments $\pi/3, \pi, -\pi/3$.

$$\sum_{r=1}^{5} \binom{5}{r} i^r = (1+i)^5 - 1 = -5 - 4i.$$

Since $|i/3| = 1/3 < 1$,

$$\sum_{r=1}^{\infty} r(i/3)^r = \frac{i/3}{(1-i/3)^2} = -\frac{9}{50} + \frac{6}{25}i.$$

The first is a root problem, the second a finite binomial sum, and the third a convergent indexed geometric sum.

6

2.6 Differential equations

Build a differential equation from a salt balance

A well-mixed tank holds 50 litres. Each minute, 10 litres enters carrying 2 grams of salt and 10 litres leaves. If M is mass in grams and t time in minutes,

$$\text{salt out per minute} = 10(M/50) = M/5.$$

Therefore

$$M' = 2 - M/5.$$

> Constant volume, fixed rates and immediate mixing are assumptions, not conclusions of the differential equation.

Normalise before choosing an integrating factor

A first-order linear equation has the form $y' + P(x)y = Q(x)$. For $2xy' - 4y = 2x^3$, first divide by $2x$ on an interval such as $x > 0$:

$$y' - \frac{2}{x}y = x^2.$$

The required P is $-2/x$, not -4.

> The coefficient of the derivative must be one. The interval cannot cross a zero of the coefficient without a separate check.

Derive the integrating factor from a product derivative

We want $\mu y' + \mu P y = (\mu y)' = \mu y' + \mu' y$. Thus $\mu' = P\mu$, so choose

$$\mu = e^{\int P\,dx}.$$

Multiply every term, integrate, then divide:

$$(\mu y)' = \mu Q, \qquad y = \frac{\int \mu Q\,dx + C}{\mu}.$$

A non-zero constant multiple of μ gives the same family: it rescales the arbitrary constant.

Worked: finish the integrating-factor solution

For $y' - 2y/x = x^2$ on $x > 0$, choose $\mu = x^{-2}$:

$$\begin{aligned} (x^{-2}y)' &= 1, \\ x^{-2}y &= x + C, \\ y &= x^3 + Cx^2. \end{aligned}$$

If $y(1) = 2$, then $C = 1$ and $y = x^3 + x^2$.

> Substitution gives $2xy' - 4y = 2x^3$; check the original, unnormalised equation too.

A logarithmic factor can reveal an inverse-sinh integral

For $y' - (x+2)y/((x+2)^2+9) = 1$, let $D = (x+2)^2 + 9$. Since $P = -D'/(2D)$, choose $\mu = D^{-1/2}$:

$$\left(\frac{y}{\sqrt{D}}\right)' = \frac{1}{\sqrt{D}}.$$

With $y(1) = 0$,

$$y = \sqrt{D}\left[\sinh^{-1}\left(\frac{x+2}{3}\right) - \sinh^{-1} 1\right].$$

Integrating factors and the surd standard forms work together. The inverse is not a reciprocal.

A hyperbolic factor also needs a valid interval

Solve $y' + y\coth x = \cosh x$ on $x > 0$.

$$\mu = e^{\int \coth x dx} = \sinh x, \qquad (y\sinh x)' = \sinh x \cosh x.$$

Hence

$$y = \tfrac{1}{2}\sinh x + C\operatorname{cosech} x.$$

If $y(\ln 2) = 1$, use $\sinh \ln 2 = 3/4$ to find $C = 15/32$.

> The original equation is undefined at zero; the formula is used on the stated interval.

CF, PI and a particular solution mean different things

For a linear equation $L[y] = f(x)$:

- The complementary function solves $L[y] = 0$ and contains arbitrary constants.
- A particular integral is any one solution of the full equation.
- General solution = CF + PI.

For $y' + 2y = e^x$, the answer is $Ce^{-2x} + e^x/3$. If $y(0) = 1$, $C = 2/3$ gives the particular solution.

> The particular solution includes the fixed CF contribution; it is not the PI alone.

The same particular-integral trials work at first order

For $y' + 2y = f(x)$, the complementary function is Ce^{-2x}. For $f(x) = 4x$, try $y_p = ax + b$:

$$2ax + (a + 2b) = 4x \quad \Longrightarrow \quad a = 2,\ b = -1.$$

For $f(x) = 5\sin x$, try $y_p = a\sin x + b\cos x$:

$$(2a - b)\sin x + (a + 2b)\cos x = 5\sin x,$$

so $a = 2$, $b = -1$ again. The two PIs are $2x - 1$ and $2\sin x - \cos x$.

> Add Ce^{-2x} to either PI for its general solution. The method is not restricted to second-order equations.

Auxiliary roots give three second-order CF forms

Substitute $y = e^{mx}$ into $ay'' + by' + cy = 0$, $a \neq 0$, to obtain the auxiliary equation:

$$am^2 + bm + c = 0.$$

roots	complementary function
$m_1 \neq m_2$ real	$Ae^{m_1 x} + Be^{m_2 x}$
m repeated	$(A + Bx)e^{mx}$
$p \pm qi$	$e^{px}(A\cos qx + B\sin qx)$

Two copies of the same exponential are not two independent terms.

Worked: match every coefficient of a polynomial PI

For $y'' + y' - 2y = 2x^2 + 3x + 1$, try $y_p = ax^2 + bx + c$. Substitute $y_p' = 2ax + b$ and $y_p'' = 2a$:

$$-2a = 2, \qquad 2a - 2b = 3, \qquad 2a + b - 2c = 1.$$

Thus

$$a = -1, \quad b = -5/2, \quad c = -11/4, \qquad y_p = -x^2 - 5x/2 - 11/4.$$

> Include lower powers even if the forcing omits them; differentiation can create them.

Exponential forcing: substitute its two derivatives

For $y'' + 2y' + 5y = 4e^{-x}$, try $y_p = ke^{-x}$.

$$y_p' = -ke^{-x}, \quad y_p'' = ke^{-x}, \quad (1 - 2 + 5)k = 4.$$

So $k = 1$ and the general solution is

$$y = e^{-x}(A\cos 2x + B\sin 2x) + e^{-x}.$$

If $y(0) = y'(0) = 0$, then $A = -1$, $B = 0$: $y = e^{-x}(1 - \cos 2x)$.

> The derivative condition applies to the whole expression, including the PI.

Trigonometric forcing needs both sine and cosine

For $y'' + 2y' + 2y = 2\sin 2x + 6\cos 2x$, try $y_p = a\sin 2x + b\cos 2x$. Matching both coefficients gives

$$-2a - 4b = 2, \qquad 4a - 2b = 6,$$

so $a = 1$, $b = -1$.

$$y = e^{-x}(A\cos x + B\sin x) + \sin 2x - \cos 2x.$$

A first derivative mixes sine and cosine even when the forcing contains only one of them.

Resonance: repair a trial already in the CF

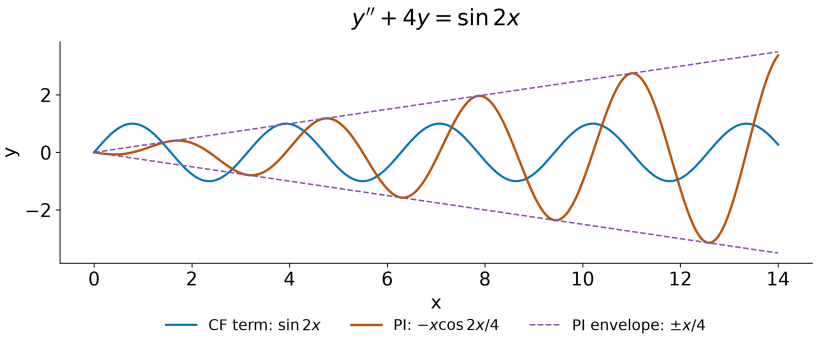
For $y'' + 4y = \sin 2x$, the trial $k \sin 2x$ gives zero on the left. Try $y_p = kx \cos 2x$ instead:

$$y_p'' + 4y_p = -4k \sin 2x.$$

Thus $k = -1/4$ and PI $= -x \cos 2x/4$. For the first-order equation $y' - 3y = 2e^{3x}$, the same idea gives PI $= 2xe^{3x}$.

> Multiply the matching trial by x until it is independent of the complementary function.

A resonant particular integral has a growing envelope



The resonant PI has a growing envelope $\pm x/4$.

Repeated overlap requires two factors of x

For $y'' - 4y' + 4y = e^{2x}$, the auxiliary root 2 is repeated. Both e^{2x} and xe^{2x} already belong to the CF. Try $y_p = kx^2e^{2x}$.

$$y_p'' - 4y_p' + 4y_p = 2ke^{2x} = e^{2x}, \qquad k = 1/2.$$

Thus

$$y = e^{2x}(A + Bx + x^2/2).$$

A single factor x is insufficient for this repeated root.

A supplied PI form is checked by direct substitution

Given that $y_p = k/x$ is a PI of $y'' + y = 2/x^3 + 1/x$ on $x > 0$:

$$y_p'' = 2k/x^3, \qquad y_p'' + y_p = 2k/x^3 + k/x.$$

Matching the forcing gives $k = 1$. Therefore

$$y = A\cos x + B\sin x + 1/x.$$

A supplied simple form can lie outside the standard polynomial, exponential and sinusoidal list. Differentiate it rather than replacing it.

Apply initial conditions after adding CF and PI

For the polynomial-forcing example,

$$y = Ae^{x} + Be^{-2x} - x^{2} - 5x/2 - 11/4.$$

If $y(0) = y'(0) = 0$, then

$$A + B = 11/4, \qquad A - 2B = 5/2,$$

so $A = 8/3$, $B = 1/12$. Two boundary conditions need not be independent:

$y'' + y = 0$, $y(0) = 0$ forces $y(\pi) = 0$ for every remaining B.

> Adding $y(\pi) = 1$ is inconsistent; adding $y(\pi) = 0$ leaves a free constant.

Explain the transient, amplitude and phase in the limit

For $y'' + 2y' + 2y = 2\sin 2x + 6\cos 2x$, the transient is $e^{-x}(A\cos x + B\sin x)$, which tends to zero. Thus at large positive x,

$$y \approx \sin 2x - \cos 2x = \sqrt{2}\sin(2x - \pi/4).$$

The amplitude is $\sqrt{2}$; the phase lag relative to $\sin 2x$ is $\pi/4$.

> A positive real part of an auxiliary root would invalidate the decay argument.

Change the independent variable with both chain-rule terms

For $x = e^t$, put $Y(t) = y(e^t)$, so $x > 0$:

$$Y_t = xy_x, \qquad Y_{tt} = xy_x + x^2 y_{xx}.$$

Therefore

$$x^2 y_{xx} = Y_{tt} - Y_t,$$

and

$$ax^2 y_{xx} + bxy_x + cy = aY_{tt} + (b - a)Y_t + cY.$$

Replacing $x^2 y_{xx}$ by Y_{tt} alone loses a chain-rule term.

Solve the transformed equation and return to x

Under $x = e^t$, $x^2y'' - 3xy' + 4y = x^2$ becomes

$$Y_{tt} - 4Y_t + 4Y = e^{2t}.$$

Using the repeated-resonance result,

$$y = x^2\{A + B\ln x + (\ln x)^2/2\}.$$

For $y(1) = 0$, $y'(1) = 1$, $A = 0$, $B = 1$.

> Return to the original variable before applying conditions stated in x and y.

Use y equals ux and retain the radical branch

Given $xy' = y + \sqrt{4x^2 + y^2}$, $y(1) = 0$, work on $x > 0$ and set $y = ux$. Then $y' = u + xu'$ and the root is $x\sqrt{4 + u^2}$:

$$xu' = \sqrt{4 + u^2}, \qquad \sinh^{-1}(u/2) = \ln x + C.$$

Integrating by separable variables and using the condition gives $C = 0$, so $u = 2\sinh \ln x = x - 1/x$ and

$$y = x^2 - 1.$$

> On another x-interval the factor is $|x|$. Check the final radical sign in the original equation.

A sum substitution can lose a constant transformed solution

For $y' = (1 + 2x + 2y)/(2x + 2y - 1)$, set $z = x + y$:

$$z' = 1 + y' = \frac{4z}{2z - 1}.$$

For $z \neq 0$, separate $(1/2 - 1/(4z))\,dz = dx$. With $y(1) = 0$, the local branch has $z > 1/2$ and

$$2(x + y) - \ln(x + y) - 4x + 2 = 0.$$

Division by z loses $z = 0$, or $y = -x$; it fails the condition. The original equation is singular at $z = 1/2$.

Interpret the salt solution in grams and minutes

For $M' + M/5 = 2$, the factor is $e^{t/5}$, giving

$$M = 10 + Ce^{-t/5}.$$

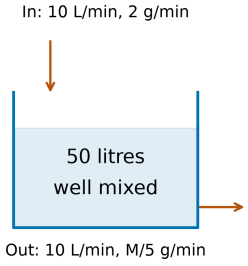
Initially fresh liquid gives $C = -10$:

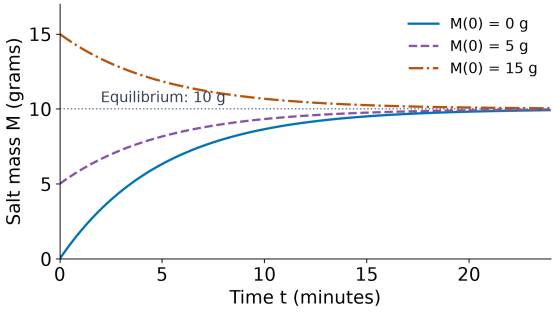
$$M = 10(1 - e^{-t/5}) \text{ grams.}$$

The mass tends to 10 grams; half this mass is reached at $t = 5\ln 2$ minutes. At equilibrium the two salt rates both equal 2 grams per minute.

> Use the model's quantities and assumptions, not just an unlabelled exponential expression.

A balance model predicts a shared limiting mass





The same salt rates give an equilibrium mass of 10 grams.

Differential-equation practice: select the matching method

1. Solve $y'' + 9y = \cos 3x$ in general form.
2. Use $x = e^t$ to solve $x^2y'' + xy' - 4y = x^3$, $x > 0$, with $y(1) = 1$, $y'(1) = 0$.

Identify the overlap in the first question and the transformed derivative in the second. Verify both final answers in their original equations.

Differential-equation practice: worked answers

For the resonant equation, try $kx\sin 3x$:

$$y_p'' + 9y_p = 6k\cos 3x, \qquad y = A\cos 3x + B\sin 3x + x\sin 3x/6.$$

The Euler-type equation becomes $Y_{tt} - 4Y = e^{3t}$, giving

$$y = Ax^2 + Bx^{-2} + x^3/5.$$

Its conditions give $A + B = 4/5$, $2A - 2B = -3/5$, so

$$y = x^2/4 + 11/(20x^2) + x^3/5.$$

Finish by checking the equation you were actually given

1. Retain the domain, branch, parameter interval and non-zero conditions.
2. Check algebraic answers by substitution or a second method.
3. For a sum, count terms and justify an infinite limit.
4. For geometry, identify the measured quantity before integrating.
5. For a differential equation, substitute the final function into the original equation and check every condition.

> A valid intermediate formula does not guarantee a valid answer to the original problem.

Exit practice: six checks across the topic

1. Evaluate $\cosh^{-1}(\cosh(-\ln 3))$.
2. Is $(B+I)^2$ invertible when B has eigenvalues $2, 3, -1$?
3. For $x = t^2 - 1$, $y = e^t$, $t > 0$, what is the quadratic Maclaurin coefficient?
4. Find $\int_0^{\pi/2} \sin^6 x\,dx$ from a reduction formula.
5. Sum $\sum_{r=1}^{\infty} r(i/3)^r$.
6. Choose a PI trial for $y'' - 2y' + y = e^x$, and find its coefficient.

Check yourself

1. $\ln 3$: the principal inverse cosh returns a non-negative value.
2. No: one transformed eigenvalue is $(-1+1)^2 = 0$.
3. Zero: at $t = 1$, $y_{xx} = e^t(t-1)/(4t^3) = 0$.
4. $I_6 = (5/6)(3/4)(1/2)(\pi/2) = 5\pi/32$.
5. $-9/50 + 6i/25$, since $|i/3| < 1$ and the sum is $q/(1-q)^2$.
6. kx^2e^x: the root 1 is repeated. Substitution gives $2k = 1$.

Check each answer against its domain, assumptions and original problem.