

Further Pure Mathematics 1

A-Level Further Mathematics ■ Topic 1

A-Level Further Mathematics

Seven sections, seven sets of methods

- 1 Roots: obtain information without solving the polynomial.
- 2 Rational graphs: establish domains, ranges and related curves.
- 3 Series: prove finite sums before taking limits.
- 4 Matrices: calculate, invert and explain transformations.
- 5 Polar curves: identify the interval and region before integrating.
- 6 Vectors: construct planes, intersections and shortest routes.
- 7 Induction: join a valid starting case to a general forward step.

1

1.1 Roots of polynomial equations

Derive the relations from factors

Start from factors, including repeated and complex roots:

$$a(x - \alpha)(x - \beta) = a\{x^2 - (\alpha + \beta)x + \alpha\beta\}.$$

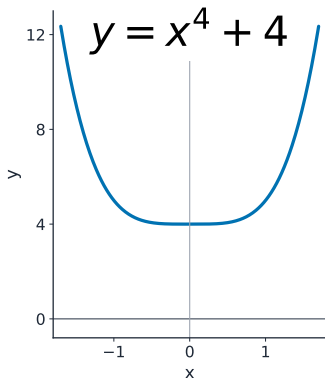
Comparison with $ax^2 + bx + c$ gives $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$. For $ax^4 + bx^3 + cx^2 + dx + e = 0$, $a \neq 0$:

$$e_1 = -b/a, \quad e_2 = c/a, \quad e_3 = -d/a, \quad e_4 = e/a.$$

Here e_j sums products of j different root positions. For a cubic the list ends at $e_3 = -d/a$.

Count multiplicities: roots 1, 1, 3 give $e_1 = 5$, $e_2 = 7$, $e_3 = 3$.

A real graph does not show every root



No x-intercepts

Four complex roots

$$1 + i, \quad 1 - i \\ -1 + i, \quad -1 - i$$

Sum = 0

Product = 4

$x^4 + 4 = 0$ has roots $1 \pm i, -1 \pm i$: sum 0, product 4. There are no real intercepts.

Worked: power sums from a quartic

Given $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$, find $S_j = \sum \alpha^j$.

$$e_1 = 2, \quad e_2 = 3, \quad e_3 = 4, \quad e_4 = 5.$$

$$S_2 = e_1^2 - 2e_2 = 2^2 - 2(3) = -2,$$

$$S_3 = e_1^3 - 3e_1e_2 + 3e_3 = 2^3 - 3(2)(3) + 3(4) = 2.$$

Each root satisfies the quartic. Add all four equations:

$$S_4 = 2S_3 - 3S_2 + 4S_1 - 4(5) = 2(2) - 3(-2) + 4(2) - 20 = -2.$$

Complex roots can give a negative sum of squares. The constant occurs four times.

Worked: recover unknown coefficients

A monic cubic has $S_1 = 2$, $S_2 = 6$ and $S_3 = 5$. Write it as $x^3 + bx^2 + cx + d = 0$.

$$b = -S_1 = -2,$$

$$c = (S_1^2 - S_2)/2 = (4 - 6)/2 = -1.$$

Add its three root equations to obtain

$$S_3 + bS_2 + cS_1 + 3d = 0.$$

Thus $5 - 2(6) - 2 + 3d = 0$, giving $d = 3$.

The equation is $x^3 - 2x^2 - x + 3 = 0$. Check all supplied sums, not individual roots.

Worked: shifted and scaled roots

The roots of $P(x) = x^3 - 3x + 1$ are α, β, γ . Find an equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1$. Set $y = 2x + 1$, so the inverse substitution is $x = (y - 1)/2$:

$$\begin{aligned} 8P((y - 1)/2) &= (y - 1)^3 - 12(y - 1) + 8, \\ &= y^3 - 3y^2 - 9y + 19. \end{aligned}$$

Set this expression equal to zero.

Check the new root sum: $2S_1 + 3 = 3$, agreeing with the coefficient -3 .

Reciprocals, forbidden values and squared roots

For $y = 1/x$, substitute $x = 1/y$ and clear y^n . A cubic $ax^3 + bx^2 + cx + d = 0$ gives

$$dy^3 + cy^2 + by + a = 0, \quad d \neq 0.$$

For squared roots, write $P(x) = E(x^2) + xO(x^2)$ and eliminate x :

$$y = x^2 \implies E(y)^2 - yO(y)^2 = 0.$$

For $P(x) = x^3 - 2x^2 - x + 2$, this is

$$(2 - 2y)^2 - y(y - 1)^2 = 0 \iff (y - 1)^2(y - 4) = 0.$$

Reciprocals require non-zero roots. Squaring can merge values: retain repeated factors.

Roots: try without solving the cubic

The roots of $x^3 - 3x + 1 = 0$ are α, β, γ .

1 Find S_2 and S_3 .

2 Obtain an equation whose roots are $1/\alpha, 1/\beta, 1/\gamma$.

3 Obtain an equation whose roots are $\alpha^3 + 1, \beta^3 + 1, \gamma^3 + 1$.

For the last part, first use the original equation to reduce $x^3 + 1$ to a linear expression in x .

Roots: worked answers

Here $e_1 = 0$, $e_2 = -3$, $e_3 = -1$.

$$S_2 = 0^2 - 2(-3) = 6, \quad S_3 = 0 - 0 + 3(-1) = -3.$$

Since the constant is non-zero, reciprocals exist. Substituting $x = 1/y$ gives

$$y^3 - 3y^2 + 1 = 0.$$

For the last part, $x^3 = 3x - 1$, so $y = x^3 + 1 = 3x$.

$$27P(y/3) = y^3 - 27y + 27 = 0.$$

The root sum is 0, also $S_3 + 3 = 0$.

2

1.2 Rational functions and graphs

Original domains survive simplification

For $f(x) = P(x)/Q(x)$, begin with $Q(x) \neq 0$.

$$f(x) = \frac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1.$$

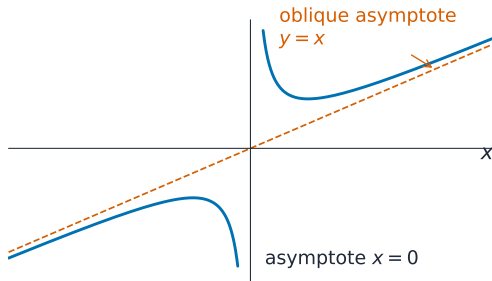
The graph is a line with a hole at $(1, 2)$, not a vertical asymptote. For a genuine slant asymptote, divide:

$$\frac{x^2 - 1}{x - 2} = x + 2 + \frac{3}{x - 2}.$$

The limiting lines are $x = 2$ and $y = x + 2$.

After cancellation: lower numerator degree gives horizontal limit 0; equal degrees give the leading-coefficient ratio.

Read both branches, not just the asymptotes



oblique asymptote when $\deg(\text{num}) = \deg(\text{den}) + 1$
(long division)

$f(x) = x + 1/x$: turning points $(-1, -2)$ and $(1, 2)$; no axis intersections. The sign of $1/x$ locates each branch relative to $y = x$.

Worked: prove a range

For $y = x + 1/x$, $x \neq 0$, rearrange to

$$x^2 - yx + 1 = 0.$$

A real input exists if and only if its discriminant is non-negative:

$$y^2 - 4 \geq 0 \quad \Longleftrightarrow \quad y \leq -2 \text{ or } y \geq 2.$$

The constant 1 excludes $x = 0$ automatically. Endpoints occur at $x = \pm 1$. For

$y = x/(x^2 + 1)$, the equation is $yx^2 - x + y = 0$. Its discriminant gives $-1/2 \leq y \leq 1/2$; at $y = 0$ it becomes linear, with the valid solution $x = 0$.

Check excluded inputs and degenerate coefficients before accepting a discriminant argument.

A sketch need not have stationary points

Consider $h(x) = (x+1)/(x^2 + 2x - 2)$. The domain excludes $x = -1 \pm \sqrt{3}$.

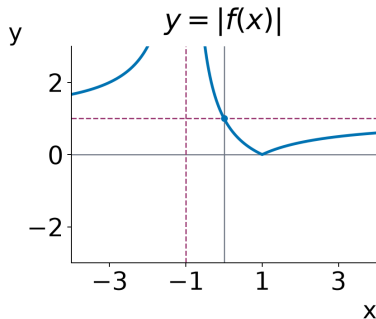
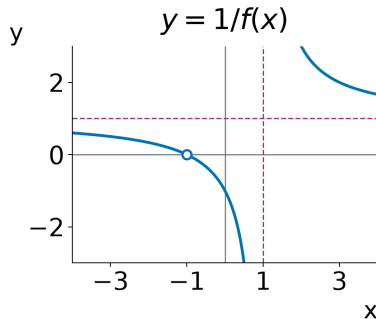
$$h'(x) = \frac{x^2 + 2x - 2 - (x+1)(2x+2)}{(x^2 + 2x - 2)^2} = -\frac{(x+1)^2 + 3}{(x^2 + 2x - 2)^2}.$$

For every allowed x , the numerator is negative and denominator positive. Thus each separate branch decreases and there are no stationary points. Intercepts: $(-1, 0)$ and $(0, -1/2)$. Horizontal asymptote: $y = 0$.

Do not connect decreasing branches across the two excluded inputs.

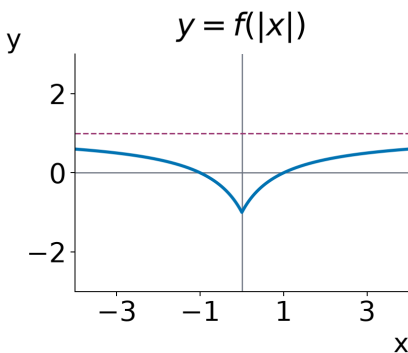
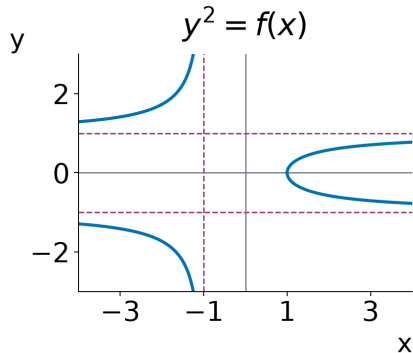
Change the output: modulus and reciprocal

$$f(x) = (x - 1)/(x + 1)$$



For $f(x) = (x - 1)/(x + 1)$, $1/f(x)$ excludes both -1 and 1 : the open point $(-1, 0)$ is not a zero.

Change the input, or square the output



$f(|x|)$ reflects the right half. For $y^2 = f(x)$, keep $f(x) \geq 0$ and draw both square-root branches.

Explain the operation before sketching

Take $f(x) = (x - 1)/(x + 1)$.

$|f(x)|$ Reflect negative outputs upwards; keep $x \neq -1$.

$1/f(x)$ Exclude original poles and zeros: $x \neq -1, 1$.

$f(|x|)$ Reflect the right half in the y -axis. Domain: all real x ; zeros ± 1 ; range $[-1, 1)$.

$y^2 = f(x)$ Require $f(x) \geq 0$, then draw both square-root branches. Inputs: $x < -1$ or $x \geq 1$.

An input-modulus join can be smooth when the right-hand derivative at zero is zero. A corner is not automatic.

Graphs: solve two inequalities

With $f(x) = (x - 1)/(x + 1)$, find the solution sets:

$$(a) \quad |f(x)| < 2, \quad (b) \quad f(|x|) < \frac{1}{2}.$$

- 1 Mark every boundary and excluded input for (a).
- 2 For (b), use $t = |x| \geq 0$ before manipulating the inequality.
- 3 Explain why multiplying directly by $x + 1$ in (a) is unsafe.

Graphs: sign and domain checks

For (a), $f = 2$ at $x = -3$, $f = -2$ at $x = -1/3$, with pole $x = -1$. Squaring is equivalent here because both sides are non-negative:

$$(x - 1)^2 < 4(x + 1)^2 \iff (3x + 1)(x + 3) > 0.$$

Thus $x < -3$ or $x > -1/3$; strict boundaries are excluded. For (b), $t + 1 > 0$, so

$$\frac{t - 1}{t + 1} < \frac{1}{2} \iff 2t - 2 < t + 1 \iff t < 3.$$

Hence $-3 < x < 3$, including zero.

$x + 1$ changes sign. An inequality reverses when multiplied by a negative quantity.

3

1.3 Summation of series

Choose a formula after fixing the index range

For a positive integer n :

$$\sum_{r=1}^n r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{r=1}^n r^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Expand a polynomial before applying these results. For example,

$$\begin{aligned} \sum_{r=1}^n (2r+1)(r+2) &= 2 \sum_{r=1}^n r^2 + 5 \sum_{r=1}^n r + 2n \\ &= \frac{n(4n^2 + 21n + 29)}{6}. \end{aligned}$$

For different bounds, subtract partial sums:

$$\sum_{r=m}^n r^2 = \frac{n(n+1)(2n+1) - (m-1)m(2m-1)}{6}.$$

Worked: show what actually cancels

Partial fractions expose a two-step difference:

$$\frac{1}{r(r+2)} = \frac{1}{2} \left(\frac{1}{r} - \frac{1}{r+2} \right).$$

Writing the first and last terms gives

$$S_n = \frac{1}{2} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{3} - \cdots - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2} \right).$$

Therefore

$$S_n = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}.$$

Both final fractions tend to zero, so $S_\infty = 3/4$.

A gap of two leaves two terms at each boundary. State the finite sum first.

A shift of two leaves two terms at each end

Index	1	2	3	4	5	6	7	8
positive	$+\frac{1}{1}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{4}$	$+\frac{1}{5}$	$+\frac{1}{6}$		
negative			$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$	$-\frac{1}{6}$	$-\frac{1}{7}$	$-\frac{1}{8}$

$$S_6 = F(1) + F(2) - F(7) - F(8)$$

For $\sum_{r=1}^n \{F(r) - F(r+2)\}$, the survivors are $F(1) + F(2) - F(n+1) - F(n+2)$.

Differences can be powers, not fractions

For $q \neq 1$, cancellation gives

$$(q - 1) \sum_{r=1}^n q^r = q^{n+1} - q,$$
$$\sum_{r=1}^n q^r = \frac{q(1 - q^n)}{1 - q}.$$

For $|q| < 1$, $q^n \rightarrow 0$, giving sum to infinity $q/(1 - q)$. Check excluded parameter cases directly: $q = 1$ gives n ; $q = -1$ gives alternating partial sums. Neither converges.

Terms tending to zero are necessary for convergence, but are not sufficient:
 $\sum 1/r$ diverges.

Series: independent boundary checks

Find each finite sum, then decide whether its limit exists:

$$(a) \quad \sum_{r=3}^n r \quad (n \geq 3), \quad (b) \quad \sum_{r=1}^n \frac{1}{(r+1)(r+2)}.$$

For (b), show the partial fractions and the uncanceled boundary terms. **Parameter check.** For which real q does $\sum_{r=0}^{\infty} q^r$ converge? Explain what fails at $q = \pm 1$.

Series: worked answers

For (a), remove the first two terms:

$$\sum_{r=3}^n r = \frac{n(n+1)}{2} - 3 \longrightarrow +\infty.$$

For (b),

$$\frac{1}{(r+1)(r+2)} = \frac{1}{r+1} - \frac{1}{r+2}, \quad S_n = \frac{1}{2} - \frac{1}{n+2} \longrightarrow \frac{1}{2}.$$

The geometric series starting at $r=0$ converges exactly for $-1 < q < 1$, with sum $1/(1-q)$. At $q=1$, partial sums increase without bound; at $q=-1$, they alternate between 1 and 0.

4

1.4 Matrices

Check the sizes before calculating

An $m \times n$ matrix times an $n \times p$ matrix gives an $m \times p$ matrix. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 2 \end{pmatrix}.$$

Multiply each row by each column:

$$AB = \begin{pmatrix} 1 & -4 \\ 9 & 1 \end{pmatrix}, \quad (AB)_{12} = 1(0) + 2(-1) + (-1)(2) = -4.$$

BA exists but is 3×3 ; it cannot equal AB .

Addition/subtraction need matching sizes. A zero matrix has only 0s; for A of size $m \times n$, $I_m A = A = A I_n$.

A determinant tests whether an inverse exists

For a 2×2 matrix,

$$T^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad - bc \neq 0.$$

For a 3×3 example, expand with positional signs $+, -, +$:

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \det M = 1(1) - 2(-2) = 5.$$

A square matrix with zero determinant is **singular** 奇异的. It has no inverse. A singular plane map may collapse to a line *or to a point*.

For a parameter-dependent determinant, identify every zero before dividing.

Worked: find every singular parameter

Consider $M_a = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ a & 0 & 1 \end{pmatrix}$. Expansion across the first row gives

$$\det M_a = 1 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 \\ a & 1 \end{vmatrix} = 1 + 2a.$$

It is singular exactly when $a = -1/2$. For every other a , the cofactor-transpose method gives an inverse with denominator $1 + 2a$.

Establish the allowed parameter values before dividing by the determinant.

Worked: cofactor, transpose, divide

For the matrix M with determinant 5, form the **cofactor** 代数余子式 matrix using the sign pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$:

$$C = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{5} C^T = \frac{1}{5} \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -1 \\ -2 & 4 & 1 \end{pmatrix}.$$

Check by multiplication. For instance, entry $(2, 1)$ is

$$(MM^{-1})_{21} = \frac{0(1) + 1(2) + 1(-2)}{5} = 0.$$

Check the other eight entries against I_3 too.

A matrix of cofactors is not yet the inverse: transpose it and divide by the determinant.

Undo operations in reverse order

For non-singular square matrices of the same size,

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (ABC)^{-1} = C^{-1}B^{-1}A^{-1}.$$

Why? Associativity allows adjacent inverse pairs to cancel:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I.$$

From $AXB = C$, multiply on the correct sides:

$$A^{-1}(AXB)B^{-1} = A^{-1}CB^{-1} \implies X = A^{-1}CB^{-1}.$$

Do not move a matrix through another matrix. Multiplication is generally non-commutative.

Columns tell you where the basis moves

For column coordinates, $\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}$. The columns of T are the images of $(1, 0)$ and $(0, 1)$.

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad H = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}.$$

R_θ : anticlockwise rotation about the origin. H : shear fixing the x -axis, taking $(0, 1)$ to $(h, 1)$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \text{reflection in the } x\text{-axis}, \quad \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}: \text{stretch parallel to the } x\text{-axis}.$$

A 2×2 linear map fixes the origin. It cannot represent a non-zero translation.

Worked: composition and orientation

Let $S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A stretch followed by a 90° anticlockwise rotation is

$$RS = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}; \quad SR = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \quad \text{reverses the order.}$$

The area scale factor is $|\det T|$, never a negative area. The sign distinguishes preserved and reversed orientation. An enlargement by k has matrix kI and determinant k^2 .

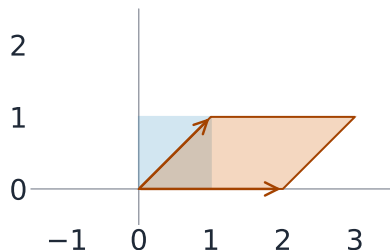
Reflection in $y = x$ swaps the basis columns and has determinant -1 .

State the transformation's axis or centre, direction and factor where appropriate.

Equal areas can have opposite orientations

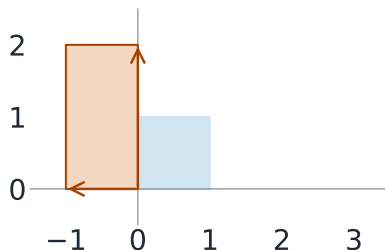
Blue: original unit square

$$T = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$



Orange: its image

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$



$\det T = 2$: area $\times 2$ $\det U = -2$: area $\times 2$, orientation reversed

Both images have area 2. Opposite determinant signs distinguish orientation, not area magnitude.

Fixed points and invariant lines are different

An **invariant point** 不变点 obeys $T\mathbf{x} = \mathbf{x}$. For $T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$:

$$2x + y = x \iff y = -x.$$

Every point on that line is fixed. An **invariant line** 不变直线 maps onto itself as a set.

For a non-singular $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, substitute $y = mx$ and impose $Y = mX$:

$$bm^2 + (a - d)m - c = 0.$$

State the resulting line equations and check $x = 0$ separately.

Eigenvector directions give lines through the origin. They do not settle every off-origin line.

Worked: a parameter giving a line of fixed points

Let $T_\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} R_\theta$, with $0 < \theta < 2\pi$. A non-zero fixed point requires

$$\det(T_\theta - I) = 2 - 2\cos\theta - \sin\theta = 0.$$

Put $t = \tan(\theta/2)$, checking $\theta = \pi$ separately:

$$\frac{4t^2 - 2t}{1 + t^2} = 0.$$

$t = 0$ gives excluded endpoints; $t = 1/2$ gives $\theta = 2 \arctan(1/2)$. At $\theta = \pi$, the determinant is 4, not 0.

$$T_\theta - I = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \implies \text{fixed line } y = 2x.$$

Test lines away from the origin

For non-singular $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, put $y = mx + q$. The image of its general point has

$$X = (a + bm)x + bq, \quad Y = (c + dm)x + dq.$$

For the whole line to remain invariant, require $Y = mX + q$ for all x . Compare the coefficient of x and the constant term:

$$bm^2 + (a - d)m - c = 0, \quad q(d - mb - 1) = 0.$$

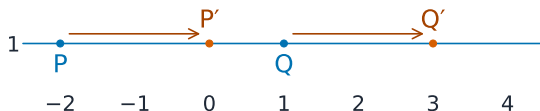
For a vertical line $x = q$, require $X = aq + by = q$ for all y : $b = 0$ and $(a - 1)q = 0$.

A nonsingular map then sends the line onto itself. Fixed points additionally require $X = x$ and $Y = y$.

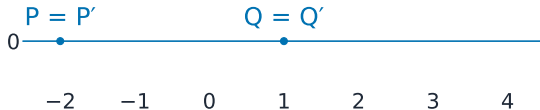
An invariant line can contain moving points

$$H = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}: (x, y) \rightarrow (x + 2y, y)$$

Invariant line: points move along it



Line of invariant points: each point stays fixed



Under $(X, Y) = (x + 2y, y)$, every $y = c$ is invariant. Only $y = 0$ consists entirely of fixed points.

Matrices: choose the right condition

For $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$:

- 1 Find all invariant points.
- 2 Find the invariant lines through the origin, including any vertical line.
- 3 Determine the image area of a triangle of area 5.

For the shear $H = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, explain why $y = 1$ is invariant although none of its points stays fixed.

Matrices: worked answers

Fixed points obey $2x = x$, $x + 3y = y$, so only $(0, 0)$ is fixed. For $y = mx$, the line condition gives $-m - 1 = 0$: $y = -x$ is invariant. Also $(0, y) \mapsto (0, 3y)$, so $x = 0$ is invariant.

$$\text{Image area} = |\det T| \times 5 = |6| \times 5 = 30.$$

For the shear, $(x, 1) \mapsto (x + 2, 1)$; every point moves two units right along the same line. The map sends that line *onto* itself.

A stretch parallel to an axis generally fixes the perpendicular axis, not the axis it stretches along.

5

1.5 Polar coordinates

Choose a radius, a quadrant and an interval

Use $r \geq 0$ and radians for calculus:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$

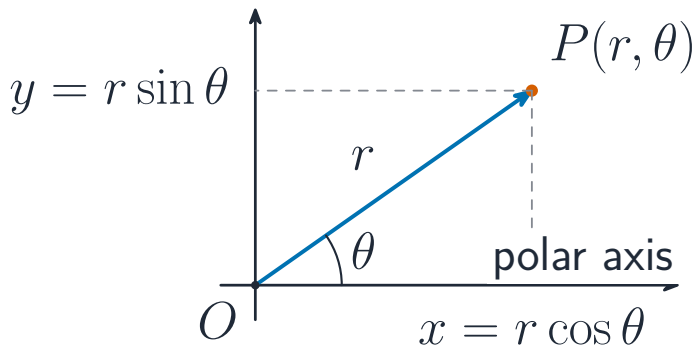
For $(-\sqrt{3}, 1)$, $r = 2$ and $\theta = 5\pi/6$ on $0 \leq \theta < 2\pi$. The value $-\pi/6$ from $\arctan(y/x)$ chooses the wrong quadrant. For $r = 4 \cos \theta$, multiply by r :

$$x^2 + y^2 = 4x \iff (x - 2)^2 + y^2 = 4.$$

The circle is traced once on $-\pi/2 \leq \theta \leq \pi/2$.

The pole has no unique angle. Multiplication or division by r can introduce or lose it: check separately.

Radius and angle determine the components

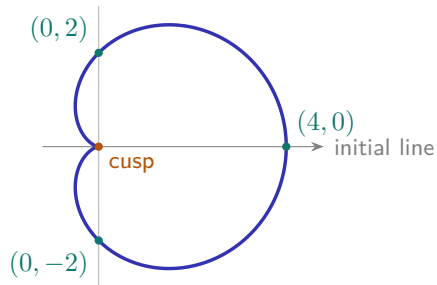


$x = r \cos \theta$, $y = r \sin \theta$. Components may be negative even though $r \geq 0$; use the quadrant to recover the angle.

Worked: sketch from features

Sketch the cardioid $r = 2(1 + \cos \theta)$, $-\pi \leq \theta \leq \pi$.

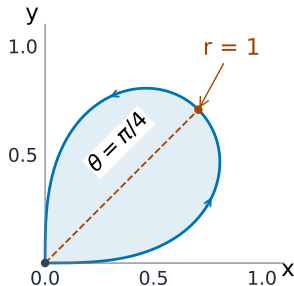
- 1 $r(-\theta) = r(\theta)$, interval symmetric: **reflect in the initial line.**
- 2 $\theta = 0$: $r = 4$, furthest point $(4, 0)$.
- 3 $\theta = \pm\pi/2$: $r = 2$, points $(0, \pm 2)$.
- 4 $\theta = \pm\pi$: $r = 0$, a **cusp** at the pole.



Test both the formula and the allowed interval for symmetry. A restricted interval can remove a whole branch.

A stated interval selects one loop

$$r^2 = \sin 2\theta, \quad 0 \leq \theta \leq \pi/2$$



Both endpoints meet at the pole (0, 0).

$r^2 = \sin 2\theta, 0 \leq \theta \leq \pi/2$: non-negative radius, two pole endpoints, symmetry about $\theta = \pi/4$.

Worked: sector area of one loop

A thin sector contributes $\frac{1}{2}r^2 d\theta$. Sweep the region once:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta.$$

For $r^2 = \sin 2\theta$ on $0 \leq \theta \leq \pi/2$,

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} \sin 2\theta d\theta \\ &= \left[-\frac{1}{4} \cos 2\theta \right]_0^{\pi/2} = \frac{1}{2}. \end{aligned}$$

The equation already gives r^2 . Do not square it again. The loop endpoints select the limits.

Worked: a region between two curves

Find the area inside $r = 4 \cos \theta$ but outside $r = 2$ in the first quadrant. Solve $4 \cos \theta = 2$: the radii meet at $\theta = \pi/3$. On $0 \leq \theta \leq \pi/3$, $4 \cos \theta$ is the outer radius.

$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/3} (16 \cos^2 \theta - 4) d\theta \\ &= [2\theta + 2 \sin 2\theta]_0^{\pi/3} = \frac{2\pi}{3} + \sqrt{3}. \end{aligned}$$

Subtract squared radii, not their squared difference. This question asks only for the upper region, so do not double it.

Furthest from the pole is not highest

For $r = 2 \sin 2\theta$, $0 \leq \theta \leq \pi/2$, the largest radius is 2 at $\theta = \pi/4$. Height above the initial line is instead

$$y = r \sin \theta = 4 \sin^2 \theta \cos \theta.$$

Set $u = \cos \theta$, with $0 \leq u \leq 1$:

$$y = 4u(1 - u^2), \quad \frac{dy}{du} = 4 - 12u^2 = 0 \quad \implies \quad u = 1/\sqrt{3}.$$

Thus $y_{\max} = 8/(3\sqrt{3})$; both endpoints give zero.

Write the distance being maximised first: r , x and y answer different questions.

An unfamiliar curve still uses the sector formula

For $r = \theta e^{\theta/2}$, $0 \leq \theta \leq \pi$, the radius increases while the curve turns anticlockwise through half a turn.

$$A = \frac{1}{2} \int_0^{\pi} \theta^2 e^{\theta} d\theta.$$

Integrate by parts twice:

$$\begin{aligned} \int \theta^2 e^{\theta} d\theta &= \theta^2 e^{\theta} - 2 \int \theta e^{\theta} d\theta \\ &= e^{\theta}(\theta^2 - 2\theta + 2) + C. \end{aligned}$$

Therefore $A = \frac{1}{2} e^{\pi}(\pi^2 - 2\pi + 2) - 1$.

The largest radius does not turn this region into a circle.

Polar curves: interval and area practice

- 1 Convert $x = 2$ to polar form with $r \geq 0$. Give an interval that covers the whole line.
- 2 Find the total area of $r = 2(1 + \cos \theta)$ on $-\pi \leq \theta \leq \pi$.
- 3 A student squares $r^2 = \sin 2\theta$ again before integrating its loop area. Explain the error.

For the second part, use $\cos^2 \theta = (1 + \cos 2\theta)/2$.

Polar curves: worked answers

The line satisfies $r \cos \theta = 2$, hence

$$r = 2 \sec \theta, \quad -\pi/2 < \theta < \pi/2.$$

The endpoints are excluded; $r = 0$ is not on this line. For the cardioid,

$$\begin{aligned} A &= \frac{1}{2} \int_{-\pi}^{\pi} 4(1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= 2(2\pi + 0 + \pi) = 6\pi. \end{aligned}$$

The loop equation already specifies r^2 . Its area integrand is $\frac{1}{2} \sin 2\theta$, not $\frac{1}{2} \sin^2 2\theta$.

6

1.6 Vectors

Choose scalar or vector product

For non-zero vectors, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$. A zero scalar product means perpendicular directions.

$$\mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}.$$

Its magnitude is the parallelogram area $|\mathbf{u}||\mathbf{v}| \sin \theta$; a triangle has half that area. For $\mathbf{u} = (1, 1, 0)$ and $\mathbf{v} = (0, 1, 1)$:

$$\mathbf{u} \times \mathbf{v} = (1, -1, 1), \quad \mathbf{u} \cdot \mathbf{v} = 1.$$

The vector product is anti-commutative: reversing the order changes its sign. Parallel inputs give zero, not a usable normal.

A unit normal gives the geometric vector product

For non-parallel $\mathbf{u}, \mathbf{v}$, let $\hat{\mathbf{n}}$ be the unit normal in the right-hand-rule direction:

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin \theta \hat{\mathbf{n}}, \quad |\hat{\mathbf{n}}| = 1.$$

For $(1, 1, 0)$ and $(0, 1, 1)$, the component calculation gave $(1, -1, 1)$, hence

$$\hat{\mathbf{n}} = \frac{(1, -1, 1)}{\sqrt{3}}, \quad \text{parallelogram area} = \sqrt{3}, \quad \text{triangle area} = \frac{\sqrt{3}}{2}.$$

The product is perpendicular to both inputs. Its sign depends on their order; its magnitude is an area.

Worked: convert a plane equation

The plane is $\mathbf{r} = (1, 0, 2) + \lambda(1, 1, 0) + \mu(0, 1, 1)$. Its independent directions give the normal

$$\mathbf{n} = (1, 1, 0) \times (0, 1, 1) = (1, -1, 1).$$

Use its known point to obtain the scalar-product form:

$$\mathbf{r} \cdot \mathbf{n} = (1, 0, 2) \cdot (1, -1, 1) = 3.$$

Thus the Cartesian equation is $x - y + z = 3$. Conversely, choose $x = 1 + \lambda$, $z = 2 + \mu$; then $y = \lambda + \mu$, recovering the parametric form.

A normal must be non-zero. Two parallel directions describe a line, not a plane.

Classify a line and a plane before dividing

Substitute $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ into $\mathbf{n} \cdot \mathbf{r} = d$:

$$\mathbf{n} \cdot \mathbf{a} + t(\mathbf{n} \cdot \mathbf{u}) = d.$$

For the plane $x - y + z = 3$:

One intersection: $\mathbf{r} = t(1, 0, 1)$ gives $2t = 3$, hence $(3/2, 0, 3/2)$.

Contained line: $(1, 0, 2) + t(1, 1, 0)$ gives the identity $3 = 3$.

Parallel, separate: $\mathbf{r} = t(1, 1, 0)$ would require $0 = 3$.

When $\mathbf{n} \cdot \mathbf{u} = 0$, test the base point. Do not use a formula with a zero denominator.

Worked: foot and point-to-plane distance

Find the nearest point on $x - y + z = 3$ to $P = (3, 0, 3)$. Write the normal line through P :

$$\mathbf{h} = (3, 0, 3) - t(1, -1, 1).$$

Substitute into the plane: $6 - 3t = 3$, giving $t = 1$ and $H = (2, 1, 2)$. Check $2 - 1 + 2 = 3$ and $\overrightarrow{PH} = (-1, 1, -1)$ is normal.

$$D = \frac{|\mathbf{n} \cdot \mathbf{p} - d|}{|\mathbf{n}|} = \frac{|6 - 3|}{\sqrt{3}} = \sqrt{3} = |PH|.$$

The plane constant d is not generally its distance from the origin: divide $|d|$ by $|\mathbf{n}|$.

Worked: plane angles and a line–plane angle

For $x - y + z = 3$ and $x + y + z = 5$, use normals $\mathbf{n}_1 = (1, -1, 1)$, $\mathbf{n}_2 = (1, 1, 1)$:

$$\cos \alpha = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1}{3}, \quad \alpha \approx 70.5^\circ.$$

For a line with direction $\mathbf{u} = (1, 0, 1)$ and the first plane:

$$\sin \phi = \frac{|\mathbf{u} \cdot \mathbf{n}_1|}{|\mathbf{u}||\mathbf{n}_1|} = \frac{2}{\sqrt{6}}, \quad \phi \approx 54.7^\circ.$$

The line–plane angle is complementary to the angle with the normal. Use non-zero directions and normals.

Worked: intersection of two planes

For $x - y + z = 3$ and $x + y + z = 5$, subtract to get $y = 1$. Then $x + z = 4$, so $(4, 1, 0)$ is a common point.

$$\mathbf{n}_1 \times \mathbf{n}_2 = (-2, 0, 2).$$

A full intersection line is therefore

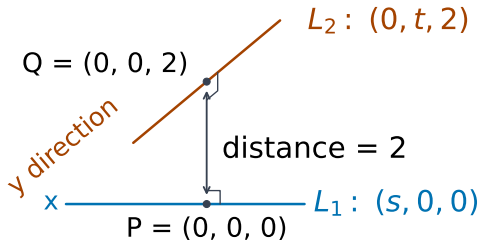
$$\mathbf{r} = (4, 1, 0) + t(-1, 0, 1).$$

Substitution gives $x - y + z = 3$ and $x + y + z = 5$ for every t .

Parallel normals give zero cross product. Compare the plane constants at the same scale to distinguish identical and separate planes.

The shortest connector must meet both lines

Two skew lines: different heights and different directions



PQ is perpendicular to both lines in three-dimensional space.

This schematic uses the x -axis and a y -parallel line at height 2. The common perpendicular joins their nearest points.

Worked: prove skewness and find a distance

Let $L_1 : \mathbf{r} = (1, 0, 1) + s(1, 1, 0)$ and $L_2 : \mathbf{r} = (2, 2, 3) + t(0, 1, 1)$. Their directions give $\mathbf{u} \times \mathbf{v} = (1, -1, 1) \neq \mathbf{0}$.

$$(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v}) = (1, 2, 2) \cdot (1, -1, 1) = 1 \neq 0.$$

The non-parallel lines are not coplanar, so they are skew. Project the base-point displacement onto the common normal:

$$D = \frac{|(\mathbf{b} - \mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|} = \frac{1}{\sqrt{3}}.$$

This formula gives a length. It does not yet locate the nearest points.

Worked: locate the common perpendicular

Take $P = (1 + s, s, 1)$ on L_1 and $Q = (2, 2 + t, 3 + t)$ on L_2 . Then
 $\overrightarrow{PQ} = (1 - s, 2 + t - s, 2 + t)$. Orthogonality to both directions gives

$$2s - t = 3, \quad s - 2t = 4 \quad \implies \quad s = 2/3, \quad t = -5/3.$$

Thus $P = (5/3, 2/3, 1)$ and $Q = (2, 1/3, 4/3)$. The common perpendicular is

$$\mathbf{r} = (5/3, 2/3, 1) + k(1, -1, 1).$$

At $k = 1/3$ it reaches Q ; $|PQ| = 1/\sqrt{3}$, agreeing with the distance formula.

Vectors: test the conditions

- 1 Find the plane through $(1, 0, 2)$ with normal $(2, -1, 3)$ and its distance from the origin.
- 2 Classify the line $(1, 0, 2) + t(1, 1, 0)$ relative to $x - y + z = 3$.
- 3 Why does the skew-distance formula fail for parallel lines? Give an appropriate distance expression instead.

A correct distance expression must have a non-zero denominator.

Vectors: worked answers

The plane constant is $2(1) - 0 + 3(2) = 8$:

$$2x - y + 3z = 8, \quad D_{\text{origin}} = \frac{8}{\sqrt{14}}.$$

The line lies in $x - y + z = 3$: its direction has zero scalar product with the normal, and its base point is in the plane. For parallel lines, $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. Use their shared non-zero direction $\mathbf{u}$ instead:

$$D = \frac{|(\mathbf{b} - \mathbf{a}) \times \mathbf{u}|}{|\mathbf{u}|}.$$

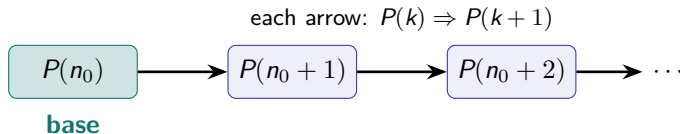
The numerator is a parallelogram area; dividing by its base length gives the perpendicular height.

7

1.7 Proof by induction

A repeatable rule must start somewhere

Write $P(n)$ for the statement, with range $n \geq n_0$.



- 1 Prove $P(n_0)$ directly.
- 2 Assume $P(k)$ for an arbitrary integer $k \geq n_0$.
- 3 Use this hypothesis to establish $P(k+1)$.
- 4 Conclude that $P(n)$ holds for all integers $n \geq n_0$.

The hypothesis is conditional, not an assumption that every case is already true.
The starting integer need not be 1.

Worked: add the next term to a sum

Prove $\sum_{r=1}^n r(r+1) = n(n+1)(n+2)/3$ for $n \geq 1$. At $n = 1$, both sides equal 2.

Assume the result at $n = k$. Separate the next term:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= (k+1)(k+2) \left(\frac{k}{3} + 1 \right) \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

This is the $k+1$ formula. The base and forward step prove it for every positive integer by induction.

Worked: isolate an integer multiple

Prove $6 \mid (10^n - 4^n)$ for every positive integer n . At $n = 1$, the expression is 6. Assume $10^k - 4^k = 6q$ for some integer q . Then

$$\begin{aligned}10^{k+1} - 4^{k+1} &= 10(10^k - 4^k) + 6 \cdot 4^k \\ &= 6(10q + 4^k).\end{aligned}$$

The bracket is an integer, so the next expression is divisible by 6. The result follows for all positive integers by induction.

Expose the assumed divisible expression, then account for every remaining term.

Worked: prove a recurrence formula

Suppose $u_1 = 2$ and $u_{n+1} = 3u_n + 2$. Prove $u_n = 3^n - 1$. The base case is $u_1 = 2 = 3^1 - 1$. For arbitrary $k \geq 1$, assume $u_k = 3^k - 1$ and start from the given recurrence:

$$u_{k+1} = 3u_k + 2 = 3(3^k - 1) + 2 = 3^{k+1} - 1.$$

This establishes the next case. Therefore $u_n = 3^n - 1$ for all positive integers by induction.

Checking 2, 8, 26 suggests a pattern. It does not replace the arbitrary- k step.

Worked: preserve two bounds

Let $u_1 = 1$ and $u_{n+1} = (4u_n + 3)/(u_n + 2)$. Prove $0 < u_n < 3$. The first term satisfies both inequalities. Assume $0 < u_k < 3$. The next numerator and denominator are positive, so $u_{k+1} > 0$. Also,

$$3 - u_{k+1} = \frac{3(u_k + 2) - (4u_k + 3)}{u_k + 2} = \frac{3 - u_k}{u_k + 2} > 0.$$

Thus $0 < u_{k+1} < 3$, and both bounds hold for all positive integers by induction. The established bounds then give

$$u_{n+1} - u_n = \frac{(3 - u_n)(u_n + 1)}{u_n + 2} > 0.$$

The sequence increases. This deduction uses the proven bound; it does not replace that proof.

Worked: induct on matrix powers

For $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, prove $M^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix}$ for $n \geq 1$. The formula at $n = 1$ gives M itself. Assume it holds at $n = k$. Multiply matrices, not individual entries raised to powers:

$$\begin{aligned} M^{k+1} &= M^k M = \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This is the required next case, so induction proves the formula for all positive integers.

Conjecture a derivative pattern, then prove it

For $f(x) = xe^{2x}$, the first derivatives are

$$e^{2x}(2x + 1), \quad e^{2x}(4x + 4), \quad e^{2x}(8x + 12).$$

Conjecture $f^{(n)}(x) = 2^{n-1}e^{2x}(2x + n)$ for $n \geq 1$. The first derivative verifies the base. Assuming the k formula, differentiate using the product rule:

$$f^{(k+1)} = 2^{k-1}e^{2x}\{2(2x + k) + 2\} = 2^k e^{2x}(2x + k + 1).$$

This is the next formula, proving the conjecture for all positive integers by induction.

Even derivative orders need two differentiations

For $f = x \sin x$, claim $f^{(2n)} = (-1)^n (x \sin x - 2n \cos x)$ for $n \geq 1$. The base is $f' = 2 \cos x - x \sin x$. Assume the $2k$ formula. Differentiate *twice*:

$$\begin{aligned} f^{(2k+1)} &= (-1)^k \{x \cos x + (2k+1) \sin x\}, \\ f^{(2k+2)} &= (-1)^k \{-x \sin x + (2k+2) \cos x\} \\ &= (-1)^{k+1} \{x \sin x - 2(k+1) \cos x\}. \end{aligned}$$

Thus the formula holds for all positive integers n by induction.

Increasing n by one increases the claimed derivative order from $2n$ to $2n+2$.

Worked: turn factorial trials into a proof

For $S_n = \sum_{r=1}^n r r!$, the first sums are 1, 5, 23, 119. Adding 1 suggests $S_n = (n+1)! - 1$. At $n = 1$, both sides equal 1. Assume the formula at $n = k$:

$$\begin{aligned} S_{k+1} &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+2)(k+1)! - 1 = (k+2)! - 1. \end{aligned}$$

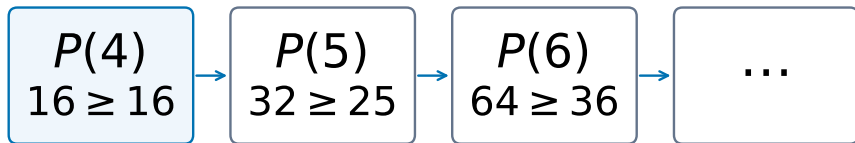
Therefore the conjecture holds for all positive integers by induction.

The identity $r r! = (r+1)! - r!$ explains the pattern. A list of successful trials alone cannot prove it.

The starting value is part of the claim

Claim: $2^n \geq n^2$ for integers $n \geq 4$

For every $k \geq 4$: $P(k) \Rightarrow P(k+1)$



Base case Reached by the proved implication
Checked values alone do not justify the arrows.

$2^n \geq n^2$ is true for all integers $n \geq 4$. The claim fails at $n = 3$: $8 < 9$.

Worked: justify the inequality in the step

Prove $2^n \geq n^2$ for integers $n \geq 4$. At $n = 4$, $16 = 16$. Assume $2^k \geq k^2$ for arbitrary $k \geq 4$. Then

$$2^{k+1} \geq 2k^2.$$

To reach $(k+1)^2$, compare rather than assume:

$$2k^2 - (k+1)^2 = (k-1)^2 - 2 > 0 \quad (k \geq 4).$$

Hence $2^{k+1} \geq (k+1)^2$. The base at 4 and general forward link prove the claim for all integers $n \geq 4$.

Induction: detect the gap

Decide which arguments establish their claimed range. Repair or refute each invalid one.

- 1 $P(n)$ is $n \leq 5$. $P(1)$ holds, and $P(k+1) \Rightarrow P(k)$. Therefore $P(n)$ holds for all $n \geq 1$.
- 2 $P(1)$ holds, and $P(k) \Rightarrow P(k+2)$. Therefore $P(n)$ holds for all $n \geq 1$.
- 3 The formula for $f^{(2n)}$ is checked at $n = 1$, then differentiated once to prove the next n case.

Audit the implication and the conclusion

- 1 The implication is backward. Although it is true, $P(6)$ is false. Prove $P(k) \Rightarrow P(k+1)$ for a valid claim instead.
- 2 The link from one base reaches only odd indices. To cover all positive integers with a $+2$ link, establish both $P(1)$ and $P(2)$.
- 3 One differentiation reaches order $2k+1$, not $2(k+1)$. Differentiate twice and verify the required expression.

State the domain, base, forward implication and conclusion. Their logical fit matters as much as the algebra.

8

Mixed practice

Connect the methods

- 1 $T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Use its power formula to find $T^4(0, 1)^T$ and its area scale factor.
- 2 Explain why the line $y = -x$ consists of invariant points of T^4 .
- 3 Find the triangle area spanned by $(1, 1, 0)$ and $(0, 1, 1)$.
- 4 Explain the shared boundary-checking habit in a rational graph, a polar area and an induction proof.

Mixed practice: worked answers

$$T^4 = \begin{pmatrix} 16 & 15 \\ 0 & 1 \end{pmatrix}, \quad T^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 15 \\ 1 \end{pmatrix}, \quad |\det T^4| = 16.$$

For $(x, -x)$, $T^4(x, -x)^T = (x, -x)^T$, so every point stays fixed. The cross product $(1, -1, 1)$ has magnitude $\sqrt{3}$; the triangle area is $\sqrt{3}/2$. A graph needs excluded inputs and branch endpoints; a polar region needs valid angles and bounding rays; an induction proof needs its first allowed integer and a link that reaches the whole claimed range.

Before applying a method, establish where it is defined and what it must prove.

Check yourself

Before leaving a solution, ask:

- 1 What is the domain or allowed starting value?
- 2 Which algebraic or geometric condition justifies this method?
- 3 Does the result satisfy the original equation or claim?

Quick retrieval: For $r^2 = \sin 2\theta$, what is the area integrand? For a matrix T , what scales area? From one induction base, what does a $+2$ link miss?

Answers: $\frac{1}{2} \sin 2\theta$; $|\det T|$; the other parity.