

Further Mathematics is marked overwhelmingly on method, so each solution here names the technique in its first line and then carries it out in full. Where a step is one examiners routinely find missing – an induction conclusion, a checked endpoint, a stated pivot – it is called out. 本科目按方法给分, 故每题先写明所用方法, 再完整展开。

Complex numbers and roots

I Find the three cube roots of $8i$, giving each in the form $a + bi$.

$\sqrt{3} + i$, $-\sqrt{3} + i$, and $-2i$.

Write $8i$ in modulus-argument form: modulus 8, argument $\pi/2$, so $8i = 8e^{i\pi/2}$. By de Moivre the cube roots have modulus $8^{1/3} = 2$ and arguments $(\pi/2 + 2k\pi)/3$ for $k = 0, 1, 2$, that is $\pi/6$, $5\pi/6$ and $3\pi/2$. Converting: $2(\cos \pi/6 + i \sin \pi/6) = \sqrt{3} + i$; $2(\cos 5\pi/6 + i \sin 5\pi/6) = -\sqrt{3} + i$; and $2(\cos 3\pi/2 + i \sin 3\pi/2) = -2i$. The roots are spaced $2\pi/3$ apart on a circle of radius 2, which is the check to state.

I The equation $x^3 - 5x^2 + 7x - 3 = 0$ has roots α, β, γ . Find the equation whose roots are $\alpha + 2, \beta + 2$ and $\gamma + 2$.

$$y^3 - 11y^2 + 39y - 45 = 0.$$

Do NOT find the roots. Substitute: if $y = x + 2$ then $x = y - 2$, so replace x in the original equation. $(y-2)^3 - 5(y-2)^2 + 7(y-2) - 3 = 0$. Expanding: $(y^3 - 6y^2 + 12y - 8) - 5(y^2 - 4y + 4) + 7y - 14 - 3 = y^3 - 11y^2 + 39y - 45 = 0$. The substitution method is what is being examined; solving the cubic and adding 2 to each root scores far less even when the answer is right.

Matrices

I Find the eigenvalues and corresponding eigenvectors of the matrix with rows $(4, 1)$ and $(2, 3)$.

$\lambda = 5$ with eigenvector $(1, 1)$; $\lambda = 2$ with eigenvector $(1, -2)$.

Solve $\det(A - \lambda I) = 0$: $(4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$, giving $\lambda = 5$ and $\lambda = 2$. For $\lambda = 5$, $(A - 5I)x = 0$ gives $-x + y = 0$, so $y = x$ and an eigenvector is $(1, 1)$. For $\lambda = 2$, $2x + y = 0$, so $y = -2x$ and an eigenvector is $(1, -2)$. Any non-zero multiple is acceptable, and saying so avoids a needless normalisation. Check by multiplying: $A(1,1) = (5,5) = 5(1,1)$.

I Using the result above, find A^5 applied to the vector $(3, 0)$, without computing A^5 .

$$(2 \cdot 5^5 + 2^5, 2 \cdot 5^5 - 2 \cdot 2^5) = (6282, 6186).$$

Write the vector in the eigenbasis: $(3, 0) = a(1, 1) + b(1, -2)$. From the components, $a + b = 3$ and $a - 2b = 0$, so $b = 1$ and $a = 2$. Applying A five times scales each eigenvector by its eigenvalue to the fifth power: $A^5(3,0) = 2 \cdot 5^5(1,1) + 1 \cdot 2^5(1,-2) = 6250(1,1) + 32(1,-2) = (6282, 6186)$. This is the point of diagonalisation – the matrix itself is never multiplied out.

Calculus and differential equations

| Given I_n as the integral of $x^n e^{-x}$ from 0 to infinity, show that $I_n = n \cdot I_{n-1}$ and hence find I_4 .

$$I_4 = 24.$$

Integrate by parts with $u = x^n$ and $dv = e^{-x}dx$: $I_n = [-x^n e^{-x}]_0^\infty + n \int x^{n-1} e^{-x} dx$. The boundary term vanishes at both ends – at infinity because the exponential dominates any power, and at zero because x^n is zero for $n \geq 1$ – so $I_n = n \cdot I_{n-1}$. The base case must be stated: $I_0 = \int_0^\infty e^{-x} dx = 1$. Then $I_4 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot I_0 = 24$. The base case is the step most often omitted and it is worth a mark on its own.

| Solve $\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$, given $y(0) = 0$ and $y'(0) = 1$.

$$y = (x + \frac{1}{2}x^2)e^{2x}.$$

The auxiliary equation $m^2 - 4m + 4 = 0$ has the repeated root $m = 2$, so the complementary function is $(A + Bx)e^{2x}$. The right-hand side e^{2x} already appears in the complementary function, and so does xe^{2x} , so the particular integral must be tried as Cx^2e^{2x} – this escalation is where the marks are. Differentiating and substituting gives $2C = 1$, so $C = \frac{1}{2}$. The general solution is $y = (A + Bx + \frac{1}{2}x^2)e^{2x}$. Applying $y(0) = 0$ gives $A = 0$; differentiating and applying $y'(0) = 1$ gives $B = 1$. Hence $y = (x + \frac{1}{2}x^2)e^{2x}$.

Proof and vectors

| Prove by induction that the sum of $r \cdot r!$ from $r = 1$ to n equals $(n+1)! - 1$ for all positive integers n .

Proved.

Base case: for $n = 1$ the left side is $1 \cdot 1! = 1$ and the right side is $2! - 1 = 1$, so the statement holds. Inductive step: assume the sum to k equals $(k+1)! - 1$. Then the sum to $k+1$ is $(k+1)! - 1 + (k+1)(k+1)! = (k+1)![1 + (k+1)] - 1 = (k+1)!(k+2) - 1 = (k+2)! - 1$, which is the statement with $n = k+1$. Conclusion: the statement is true for $n = 1$, and its truth for $n = k$ implies its truth for $n = k+1$, so by induction it holds for all positive integers n . That final sentence is a mark in its own right and is the one most often left out.

| Find the shortest distance between the lines $\mathbf{r} = (1,0,2) + s(2,1,-1)$ and $\mathbf{r} = (0,3,1) + t(1,-1,2)$.

$$\sqrt{6} / 2, \text{ about } 1.22.$$

The lines are skew, so use the common perpendicular. Its direction is the vector product of the two direction vectors: $(2,1,-1) \times (1,-1,2) = (2 \cdot 2 - (-1)(-1), (-1)(1) - 2 \cdot 2, 2(-1) - 1 \cdot 1) = (1, -5, -3)$. The vector joining a point on each line is $(0,3,1) - (1,0,2) = (-1, 3, -1)$. The distance is the magnitude of the projection of that joining vector onto the common perpendicular: $|(-1)(1) + 3(-5) + (-1)(-3)| / \sqrt{1 + 25 + 9} = |-1 - 15 + 3| / \sqrt{35} = 13 / \sqrt{35} = 2.20$. State the method first – it is the same three lines whatever the numbers, and naming it secures the method marks before any arithmetic.