Course Companion

A-Level Further Mathematics

Every topic handout in one volume

全部专题讲义合集

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Further Pure Mathematics 1

A-Level Further Mathematics

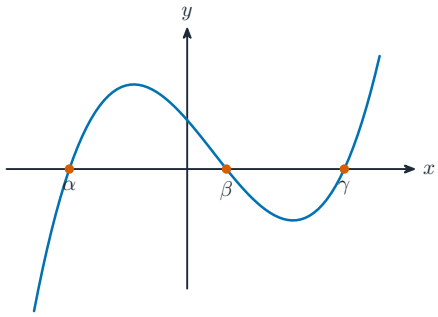
This handout covers Topic 1: **Further Pure Mathematics** 进阶纯数学 1. It adds new algebra, matrices, polar coordinates 极坐标, more vectors 向量, and proof by induction.

Roots of polynomial equations

Before trusting a program's calculated roots 根, check their sum and product against its stored polynomial 多项式 **coefficients** 系数. These relations also give useful sums when finding the individual roots would be difficult. A **root** makes the polynomial zero. Count repeated roots with their multiplicities and include complex roots; a real graph shows only the real ones. The relations here apply to degrees 2, 3 and 4, always with a non-zero leading coefficient.

Derive the relations from factors

Expand $a(x-\alpha)(x-\beta)$ and compare coefficients with ax^2+bx+c. This gives



$$a(x-\alpha)(x-\beta)(x-\gamma) = ax^3+bx^2+cx+d$$

$$\alpha+\beta+\gamma = -b/a$$

$$\alpha\beta+\beta\gamma+\gamma\alpha = c/a$$

$$\alpha\beta\gamma = -d/a$$

Roots from factors

$$\alpha+\beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

For $ax^3+bx^2+cx+d=0$ with roots α,β,γ, expansion similarly gives

$$e_1 = \alpha+\beta+\gamma = -\frac{b}{a},$$

$$e_2 = \alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}, \qquad e_3 = \alpha\beta\gamma = -\frac{d}{a}.$$

The signs alternate because each factor contains a minus sign. They do not say that the roots themselves are negative. For $(x-1)^2(x-3)=0$, use the list $1,1,3$, giving

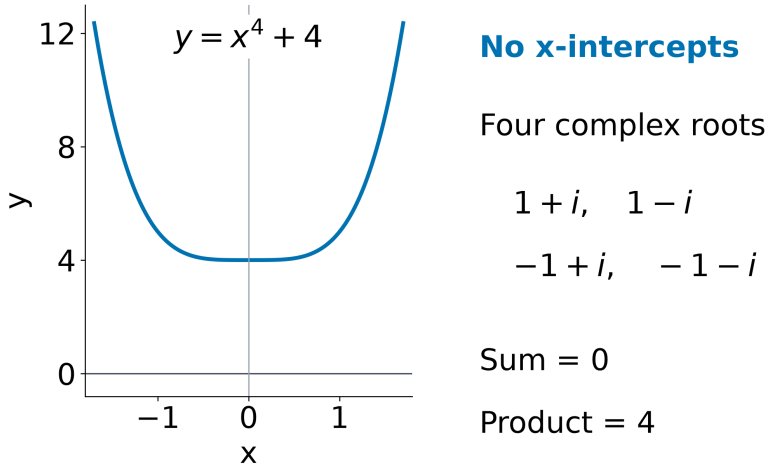
$e_1 = 5, e_2 = 7, e_3 = 3$. Counting only the two distinct values breaks the coefficient relations.

Include quartics and complex roots

For a **quartic equation** 四次方程 $ax^4+bx^3+cx^2+dx+e=0$, define e_1 as the sum of its four roots, e_2 as the sum of all six different pairs, e_3 as the sum of all four different triples, and e_4 as their product. Then

$$e_1 = -\frac{b}{a}, \qquad e_2 = \frac{c}{a}, \qquad e_3 = -\frac{d}{a}, \qquad e_4 = \frac{e}{a}.$$

The six pairs are $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$: count each once. These are **symmetric functions** 对称函数, unchanged when the roots exchange names.



The four roots of $x^4+4=0$ are $1+i, 1-i, -1+i, -1-i$. Their sum is 0 and their product is 4, although the real graph has no x-intercepts.

Find sums of powers without solving the equation

Write $S_k = \alpha^k + \beta^k + \cdots$, including each root with its multiplicity. Expanding the square and cube of the root sum gives

$$S_2 = e_1^2 - 2e_2, \qquad S_3 = e_1^3 - 3e_1e_2 + 3e_3.$$

The first formula works for degrees 2, 3 and 4; the second is written here for degrees 3 and 4. For a higher sum, use the equation satisfied by each root. If $x^4+bx^3+cx^2+dx+e=0$, add its four root equations:

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$$S_4 + bS_3 + cS_2 + dS_1 + 4e = 0.$$

> **Worked example.** For $x^4-2x^3+3x^2-4x+5=0$, the symmetric sums are $2, 3, 4, 5$. Therefore $S_2 = 4-6 = -2$ and $S_3 = 8-18+12 = 2$. The original equation then gives $S_4 = 2(2) - 3(-2) + 4(2) - 4(5) = -2$. Negative sums of squares are possible when the roots are complex.

Recover unknown coefficients from given sums

For a cubic $x^3+bx^2+cx+d=0$ with leading coefficient 1, first use $b=-S_1$ and $c=(S_1^2-S_2)/2$. To connect the remaining coefficient to higher sums, add the three original root equations:

$$S_3 + bS_2 + cS_1 + 3d = 0.$$

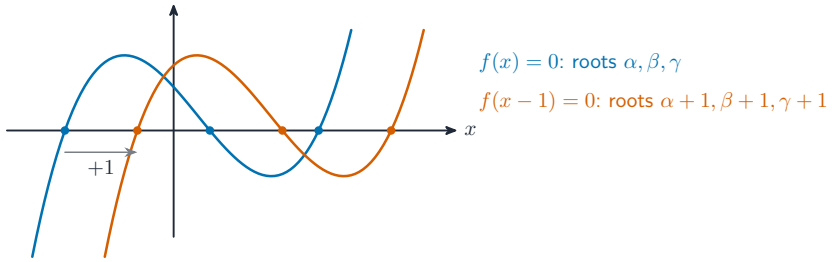
Multiply each root equation by its own root and add again:

$$S_4 + bS_3 + cS_2 + dS_1 = 0.$$

The constant contributes $3d$ in the first equation but dS_1 in the second. Substitute the known sums, solve the resulting simultaneous equations and check in both relations.

Shift or scale all roots

To find an equation with roots $y = mx+k$, where $m \neq 0$, use the inverse **substitution** 代换 $x = (y-k)/m$. Substitute into the original polynomial, clear denominators 分母 and collect powers of y. A non-zero constant multiple of an equation has the same roots.



Substitute x minus 1, shift every root

> **Worked example.** Let the original polynomial be $P(x) = x^3 - 3x + 1$. For roots $2\alpha+1, 2\beta+1, 2\gamma+1$, put $x = (y-1)/2$ and multiply by 8:

$$(y-1)^3 - 12(y-1) + 8 = 0,$$

$$y^3 - 3y^2 - 9y + 19 = 0.$$

Its root sum is 3, also $2S_1 + 3(1) = 3$. For a degree-n polynomial, the corresponding sum check is $mS_1 + nk$. Do not substitute $2y+1$ when the original variable should be $(y-1)/2$.

Reciprocals and forbidden values

For new roots $y = 1/x$, put $x = 1/y$ and multiply by y^n. Check the original constant term is non-zero: a zero root has no reciprocal. A cubic $ax^3+bx^2+cx+d = 0$ becomes $dy^3 + cy^2 + by + a = 0$.

For example, $2x^3-3x^2+4x-5 = 0$ gives reciprocal-root equation $5y^3-4y^2+3y-2 = 0$. Its root sum is $4/5$, also e_2/e_3. The new constant term excludes $y = 0$.

For $y = 1/(x-k)$, use $x = k + 1/y$ and check $P(k) \neq 0$ before claiming that every original root has an image. Clearing denominators does not define a reciprocal of zero.

Square the roots and preserve multiplicities

Separate a polynomial into its even and odd powers: $P(x) = E(x^2) + xO(x^2)$. If $y = x^2$, eliminate x by writing

$$E(y)^2 - yO(y)^2 = 0.$$

This combines the two possible signs of a square root: one of them solves the original equation. If different original roots have the same square, keep their repeated factors in the new equation.

> **Worked example.** For $x^3 - 2x^2 - x + 2 = 0$, $E(y) = 2 - 2y$ and $O(y) = y - 1$. Thus $(2-2y)^2 - y(y-1)^2 = 0$, which simplifies, up to a non-zero constant factor, to $(y-1)^2(y-4) = 0$. Original roots $1, -1, 2$ become $1, 1, 4$. Removing the repeated factor would lose one transformed root.

Simplify a higher-power transformation first

If $x^3 + 3x - 2 = 0$, each root satisfies $x^3 = 2 - 3x$. A new root $y = x^3 + 1$ is therefore $3 - 3x$. Use $x = (3-y)/3$ in the original equation; multiplying by 27 and collecting terms gives $y^3 - 9y^2 + 54y - 54 = 0$. Now use coefficient relations on this new equation to find sums of transformed powers.

Rational functions and graphs

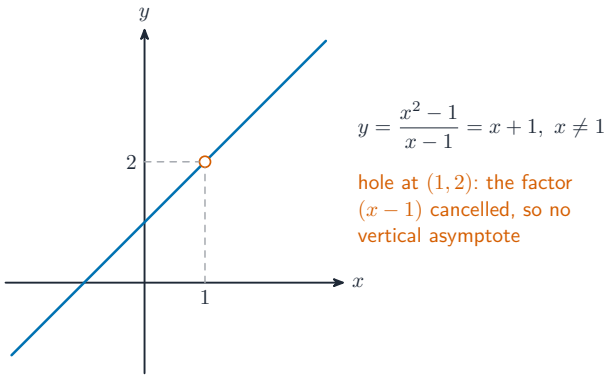
Check a fitted curve before trusting it

A program may fit a ratio of two **polynomials** to data. Its graph can look smooth on a screen while hiding an excluded input or a separate branch. For a **rational function** 有理函数 $f(x) = P(x)/Q(x)$, begin with the original **denominator**: require $Q(x) \neq 0$. Then find the features that a sketch must show.

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An excluded input can be a hole

A denominator zero gives a **vertical asymptote** 垂直渐近线 only when the remaining expression becomes unbounded there. Factor first; cancellation may instead leave a missing point. For $f(x) = (x^2-1)/(x-1) = x+1$, the original restriction $x \neq 1$ remains. Draw the line with an open point at $(1, 2)$, not a vertical asymptote. A zero of the numerator is an intercept only if the original denominator is non-zero there.



A cancelled factor leaves a hole

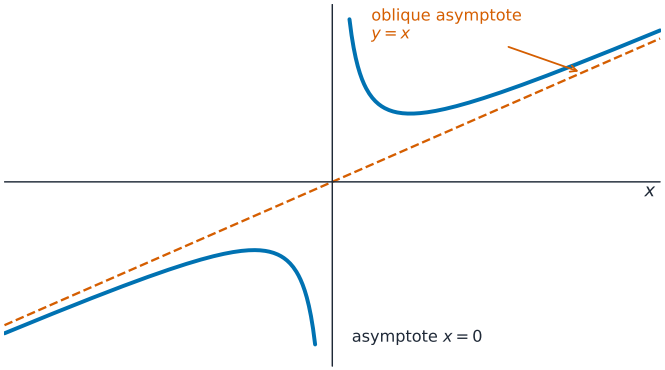
Asymptotes from degree and division

After cancellation, compare degrees: a lower-degree numerator gives horizontal limit 0; equal degrees give the ratio of leading coefficients. An **oblique asymptote** 斜渐近线 is a sloping line, so the numerator degree must be exactly one higher. Divide to obtain the **quotient** 商 plus the **remainder** 余数 over the denominator. For $(x^2-1)/(x-2)$, this is $x + 2 + 3/(x-2)$. Thus $y = x + 2$ is the slant limit as $x \to \pm\infty$; $x = 2$ is vertical.

A horizontal or slant asymptote describes distant behaviour; it may be crossed at a finite input. For $f(x) = (x^2+x)/(x^2+1)$, the horizontal asymptote is $y = 1$, and $f(1) = 1$.

Turning points 极值点 and a complete sketch

For $f(x) = x + 1/x$, $x \neq 0$, $f'(x) = 1 - 1/x^2$. The **turning points** are $(-1, -2)$, a local maximum, and $(1, 2)$, a local minimum, as the derivative changes sign. There are no axis intersections. Draw both branches with vertical asymptote $x = 0$ and slant asymptote $y = x$. The left branch lies below the slant line and the right above it. Near zero, the left branch falls to $-\infty$ and the right rises to $+\infty$.



oblique asymptote when deg(num) = deg(den) + 1 (long division)

For this curve the limiting lines, branch sides and two turning points determine the main shape; detailed plotting is unnecessary.

Check whether stationary points exist

For $h(x) = (x+1)/(x^2+2x-2)$, the quotient rule gives

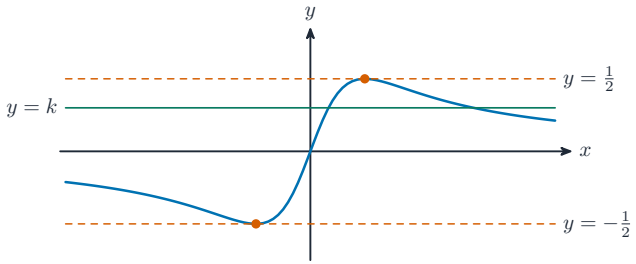
$$h'(x) = -\frac{(x+1)^2+3}{(x^2+2x-2)^2}.$$

The numerator inside the minus sign is positive, and the denominator is positive at every allowed input. Therefore the derivative never vanishes. There are no stationary points, and each separate domain interval has a decreasing branch; do not join branches across a pole 极点.

Prove the range with a discriminant 判别式

Rearrange $y = x + 1/x$ as $x^2 - yx + 1 = 0$. A real input exists exactly when its **discriminant** satisfies $y^2 - 4 \geq 0$. The range is $y \leq -2$ or $y \geq 2$, including both turning values. Check exclusions and any case where the equation stops being quadratic. For $y = x/(x^2+1)$, rearrangement gives $yx^2 - x + y = 0$. At $y = 0$ this is the valid linear equation $x = 0$; do not discard that range value.

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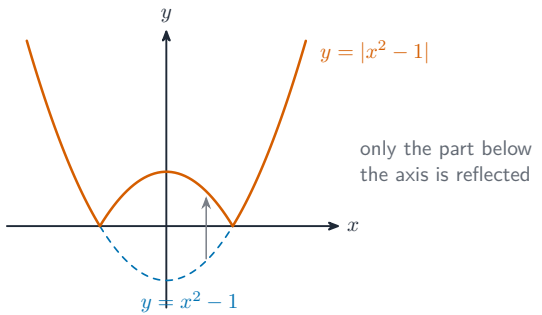
$kx^2 - x + k = 0$ has real roots iff $1 - 4k^2 \geq 0$, so $-\frac{1}{2} \leq y \leq \frac{1}{2}$

The discriminant finds the range

For the second function, $1 - 4y^2 \geq 0$ gives $-1/2 \leq y \leq 1/2$. Its denominator is never zero for real x, so all these values occur, including 0 and the two endpoints.

Change the output: modulus and reciprocal

Take $f(x) = (x-1)/(x+1)$. For $y = |f(x)|$, keep nonnegative pieces and reflect negative pieces above the x-axis. The exclusion $x = -1$ remains. Its only zero is $x = 1$. For $y = 1/f(x) = (x+1)/(x-1)$, exclude both $x = -1$ and $x = 1$: the first was never in the original domain and the second makes the reciprocal undefined. There is a hole at $(-1, 0)$ and a vertical asymptote at $x = 1$; the hole is not a zero.

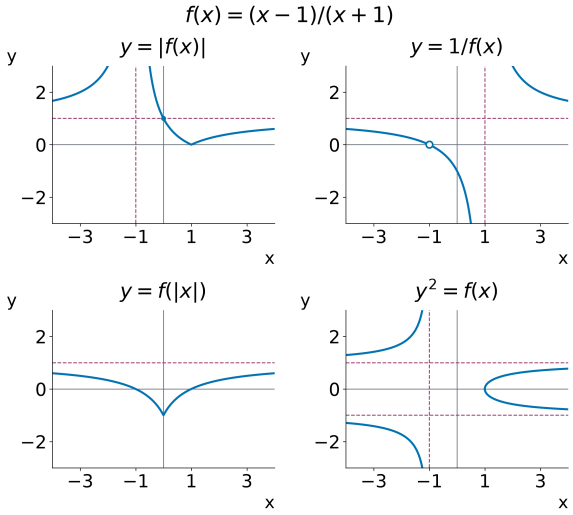


Modulus of the output reflects the negative part

Change the input, or square the output

For $f(|x|)$, keep the original right half and reflect it across the y-axis. Here the denominator $|x| + 1$ never vanishes; the range is $[-1, 1)$, with zeros at $x = \pm 1$ and a corner at $(0, -1)$. Such a corner is not automatic: if the original right-hand slope at zero is zero, the join can be smooth. For $y^2 = f(x)$, keep only original inputs where $f(x) \geq 0$, then draw both $y = \sqrt{f(x)}$ and $y = -\sqrt{f(x)}$. Here $x < -1$ or $x \geq 1$; the

two branches meet at $(1, 0)$, and no graph exists for $-1 < x < 1$.



The open point belongs to neither reciprocal branch. Taking a modulus of the output and taking a modulus of the input give different domains and different zeros.

For a contrasting smooth join, take $h(x) = (x^2 + 4x + 8)/(x + 2)$. Since $h(0) = 4$ and $h'(0) = 0$, reflecting the right half gives matching slopes 0 on both sides: $h(|x|)$ is smooth at $(0, 4)$.

Solve inequalities from boundaries and branches

To solve $|(x - 1)/(x + 1)| < 2$, find where $f = 2$ and $f = -2$: the boundary inputs are -3 and $-1/3$. Include the pole -1 in the sign or branch diagram. Test the intervals separated by these values. The answer is $x < -3$ or $x > -1/3$, with strict endpoints excluded. Do not multiply an inequality by $x + 1$ without first deciding its sign; multiplying by $(x + 1)^2$ is safe only after excluding $x = -1$.

Use symmetry to solve an input-modulus inequality

For $f(|x|) < 1/2$, put $t = |x| \geq 0$. Then $(t - 1)/(t + 1) < 1/2$ has a positive denominator and gives $t < 3$. Translate back to $-3 < x < 3$. The value zero is included. If a quadratic calculation produces a negative boundary for t, it cannot be a value of $|x|$; inspect the allowed half-line before reflecting the result.

Build the sketch from evidence

Record the domain, intercepts, asymptotes, stationary coordinates and one-sided behaviour. Use a range calculation to confirm which vertical gaps are real; a horizontal asymptote need not be an excluded output value. For a related graph, state the new domain before drawing. For an inequality, mark all boundaries and poles, test

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branches, and preserve strict or inclusive endpoints.

Summation of series

Count work without adding every stage

An algorithm performs r^2 operations at stage r. To estimate its total work after thousands of stages, we need one expression for the whole sum. **Summation** 求和 can combine standard formulas or cancel shifted terms. An infinite total needs a separate check: a finite formula alone does not prove convergence.



Nested dolls: a series adds term after term, and the method of differences or a standard result gives the sum in closed form

Standard sums and their bounds

For a positive integer 正整数 n, the standard results are:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \qquad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

The limits matter: $\sum_{r=1}^{n} 1 = n$, but $\sum_{r=0}^{n} 1 = n + 1$. Check the number of terms before applying a formula. The cube identity uses consecutive integers from 1; it is not true for arbitrary lists of numbers.

Expand first, then sum each part

For $\sum_{r=1}^{n}(2r+1)(r+2)$, expand to $2r^2+5r+2$. The result is $2\sum r^2+5\sum r+2n$. Substitute the standard results and simplify:

$$\sum_{r=1}^{n}(2r+1)(r+2)=\frac{n(4n^2+21n+29)}{6}.$$

At $n=2$, the original terms are 3×3 and 5×4, totalling 29. The formula also gives $2(16+42+29)/6=29$. A small-n check catches missing constants and index errors.

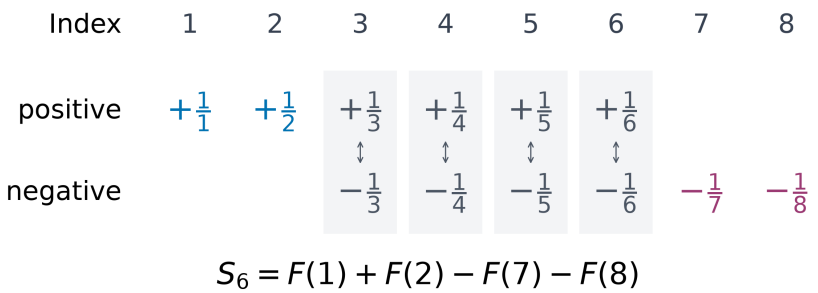
Change the starting or ending index

If $F(m)=\sum_{r=1}^{m}u_r$, then $\sum_{r=p}^{q}u_r=F(q)-F(p-1)$. The last removed term is $p-1$, not p. For $\sum_{r=0}^{n}(r+1)^2$, put $k=r+1$. The new bounds are 1 and $n+1$, so the sum is $(n+1)(n+2)(2n+3)/6$. Change the limits as well as the expression.

The method of differences 差分法

Write a term as $F(r)-F(r+1)$, then show the first few terms and the last one. The middle terms **telescope** 裂项相消, leaving the boundary values $F(1)-F(n+1)$. A wider shift leaves more boundary terms. For $F(r)-F(r+2)$, the two starting terms and two ending terms survive:

$$\sum_{r=1}^{n}(F(r)-F(r+2))=F(1)+F(2)-F(n+1)-F(n+2).$$

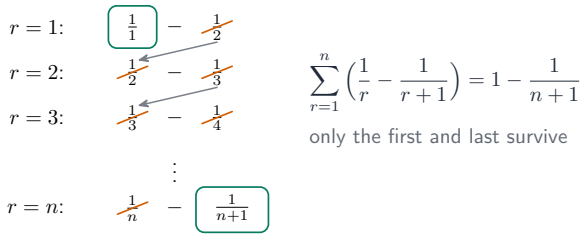


With $F(r)=1/r$, the terms at indices 3 through 6 cancel in this six-term example. Both ending terms must remain.

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Partial fractions 部分分式 make cancellation visible

For $1/[r(r+2)]$, solve $1=A(r+2)+Br$: $A=1/2, B=-1/2$. Thus each term is $\frac{1}{2}(1/r-1/(r+2))$. Use the two-step cancellation, keeping its factor:



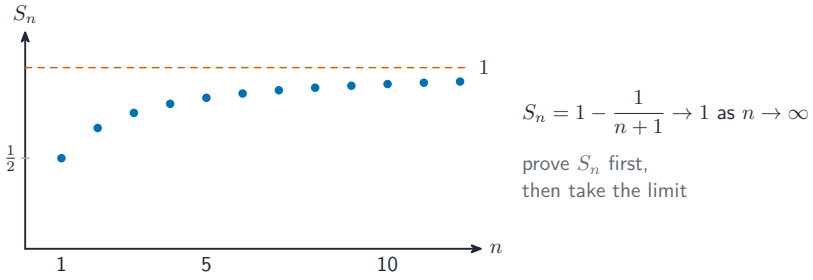
Inner terms cancel in pairs

$$S_n=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}.$$

A decomposition is not yet a sum. Show enough first and last terms to establish the cancellation, especially when three partial fractions or an index shift are involved. Check the formula at $n=1$: it gives $1/3$, the original first term.

Take the limit of the finite sum

A series is **convergent** 收敛 when its partial sums approach a finite real limit. In the last example the two ending fractions tend to zero, so its **sum to infinity** 无穷和 is 3/4. The remaining tail after n terms is $S_\infty-S_n$. Here it is $\dfrac{1}{2(n+1)}+\dfrac{1}{2(n+2)}$, which can be used to choose how many terms give a required accuracy.



Finite sums approach the limit

Terms tending to zero is not enough. For $u_r=\sqrt{r+1}-\sqrt{r}$, the terms tend to zero, but $S_n=\sqrt{n+1}-1$ grows without bound. A finite partial-sum limit is the deciding evidence.

Differences need not be partial fractions

Let $P_r = r(r+1)(r+2)$. Factor the common part to obtain $P_{r+1} - P_r = 3(r+1)(r+2)$. Hence a sum of $(r+1)(r+2)$ is $(P_{n+1} - P_1)/3$. Another route is to put two fractions over one denominator. For example, $1/r^2 - 1/(r+1)^2 = (2r+1)/[r^2(r+1)^2]$. Recognising the difference avoids trying to use a formula for $\sum 1/r^2$.

Check parameter endpoints separately

Let $u_r = t^{r(r+1)} - t^{(r+1)(r+2)}$. The finite sum is $S_n = t^2 - t^{(n+1)(n+2)}$. For $|t| < 1$, its limit is t^2. At $t = 1$ or $t = -1$, every exponent is even and every term is zero: the infinite sum is 0, not 1. For $|t| > 1$, the ending power grows without bound. The convergence set is $[-1, 1]$, but the sum needs separate endpoint values.

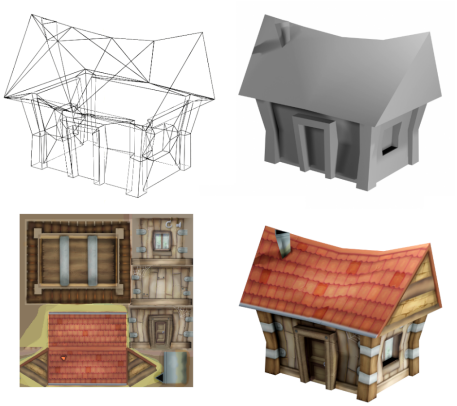
Prove the sum before taking infinity

Choose standard sums after expanding, or differences after rearranging the term. State the index limits, show the surviving boundaries and check a small value of n. For an infinite series, inspect the limit of the exact partial sum. Check special parameter values in the original terms; use the remaining tail, not just the next term, for an error bound.

Matrices

Undo an image edit

A graphics program can store a rotation 旋转 or stretch 拉伸 as a **matrix** 矩阵. Its columns show where the two unit directions move; multiplying maps every point in the image. To undo the edit, an **inverse** must exist. To combine edits, their order matters. A correct answer should explain both the calculation and what happens to the shape.



A 3-D model built from vertices: graphics software moves and resizes it by multiplying every vertex by a transformation matrix

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Check the sizes before calculating

A matrix with m rows and n columns has size $m \times n$. Add or subtract only matrices of the same size, entry by entry; the **zero matrix** 零矩阵 has every entry zero. If A is $m \times n$ and B is $n \times p$, then AB is $m \times p$. Match the inner sizes, then multiply each row by each column. A product in the reverse order may have a different size or may not exist.

For the non-square matrices

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 2 \end{pmatrix},$$

$$AB = \begin{pmatrix} 1 & -4 \\ 9 & 1 \end{pmatrix}.$$

For example, entry $(1, 2)$ is $1(0) + 2(-1) + (-1)(2) = -4$. The product BA exists but is 3×3, so it cannot equal AB.

Identity, order and singular matrices

The **identity matrix** 单位矩阵 I_n is $n \times n$, with ones on its main diagonal and zeros elsewhere. For A of size $m \times n$, $I_m A = A = A I_n$. Usually $AB \neq BA$; matrix multiplication is not commutative. A square matrix is **singular** when its **determinant** 行列式 is zero; it has no inverse. A non-singular square matrix has a unique inverse with $AA^{-1} = A^{-1}A = I$. A singular plane 平面 map may collapse the plane to a line or to a single point.

For $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$,

$$\det A = 3(4) - 1(2) = 10, \qquad A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

For any 2×2 matrix, swap the two diagonal entries, negate the other two entries, then divide by $ad - bc$, provided this determinant is non-zero.

Expand a three by three determinant

To find a 3×3 determinant, choose a row and delete the row and column of each chosen entry. Each remaining 2×2 determinant is a **minor** 余子式. Across the first row the signs are $+, -, +$. For a parameter, factor the determinant and state every value making it zero. Do not divide by a parameter expression before checking whether it can vanish.

$$M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Expanding across the first row,

$$\det M = 1\begin{vmatrix}1 & 1\\ 0 & 1\end{vmatrix} - 2\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} + 0 = 1 - 2(-2) = 5.$$

The middle minus sign belongs to the position, even if the minor itself is negative.

Find and check a three by three inverse

A **cofactor** 代数余子式 is a minor multiplied by its position sign. Use the sign pattern $\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$, then transpose the cofactor matrix: rows become columns. Divide that transpose by the non-zero determinant. Check the result by multiplication. Computing one cofactor incorrectly or forgetting the transpose can give a plausible-looking matrix that is not an inverse.

For the matrix M above, the cofactor matrix and inverse are

$$C = \begin{pmatrix}1 & 2 & -2\\ -2 & 1 & 4\\ 2 & -1 & 1\end{pmatrix},$$

$$M^{-1} = \frac{1}{5}C^{\mathsf{T}} = \frac{1}{5}\begin{pmatrix}1 & -2 & 2\\ 2 & 1 & -1\\ -2 & 4 & 1\end{pmatrix}.$$

For example, the second row of M times the first column of M^{-1} gives $(0+2-2)/5 = 0$, as required for entry $(2,1)$ of I_3. Check all nine entries.

Undo several operations in reverse order

For non-singular square matrices of the same size, $(AB)^{-1} = B^{-1}A^{-1}$. Multiplication is associative, so $ABB^{-1}A^{-1} = I$. The reverse order removes the last operation first. Extend this to three factors: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. In a matrix equation, multiply on the correct side: from $AXB = C$, obtain $X = A^{-1}CB^{-1}$ when both inverses exist.

There is no general cancellation rule that moves a matrix through another matrix. Also, AB can be square and singular even when A and B are non-square: calculate the product before trying to take its determinant.

Build a plane transformation from columns

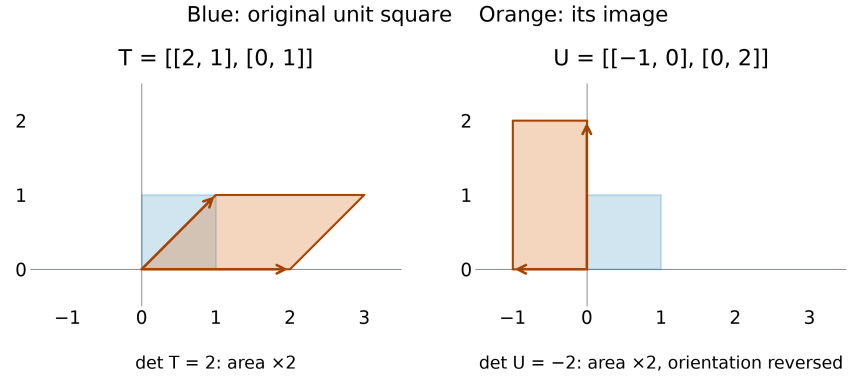
For column coordinates, $\begin{pmatrix}X\\ Y\end{pmatrix} = T\begin{pmatrix}x\\ y\end{pmatrix}$. The first column is the image of $(1,0)$ and the second is the image of $(0,1)$. These determine a linear **transformation**; the origin stays fixed. Recognise rotation, reflection 反射, **enlargement** 放大, stretch and shear 切变. State the centre or axis, direction and factor as appropriate. A translation with a non-zero displacement cannot be represented by a 2×2 matrix acting on these column coordinates.

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Transformation	Matrix
Anticlockwise rotation through θ about the origin	$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$
Reflection in the x-axis	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Reflection in $y = x$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
Enlargement, centre origin, scale factor k	$\begin{pmatrix}k & 0\\ 0 & k\end{pmatrix}$
Stretch parallel to the x-axis, factor k	$\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$
Shear fixing the x-axis, $(0,1)\mapsto(h,1)$	$\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$

Compose edits and interpret area

The product AB applies B first, then A. A stretch $S = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ followed by a 90° anticlockwise rotation R gives $RS = \begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}$. Reversing them gives $SR = \begin{pmatrix}0 & -2\\ 1 & 0\end{pmatrix}$. The area scale factor is $|\det T|$. The sign of $\det T$ tells whether orientation is reversed, not whether an area is negative. A negative enlargement factor reverses both coordinate directions, so its determinant k^2 is positive.



Both images have area 2. Their determinants have opposite signs: area magnitude and orientation are different properties.

Find points that stay fixed

An **invariant point** keeps its own coordinates: $T\binom{x}{y} = \binom{x}{y}$. Solve $(T-I)\binom{x}{y} = \binom{0}{0}$. The origin is always a solution; a line of fixed points requires more solutions than the origin alone. For $T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, the equations reduce to $x + y = 0$. Every point on $y = -x$ stays fixed. If $\det(T-I) \neq 0$, the origin is the only invariant point; if it is zero, solve the equations to find the actual set.

For $T_\theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} R_\theta$, a non-zero fixed point requires

$$\det(T_\theta - I) = 2 - 2\cos\theta - \sin\theta = 0.$$

On $0 < \theta < 2\pi$, put $t = \tan(\theta/2)$, checking $\theta = \pi$ separately. Then $4t^2 - 2t = 0$, giving $\theta = 2\arctan(1/2)$. Substitution gives $T_\theta - I = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, so the fixed line is $y = 2x$. The determinant condition alone does not give its equation.

A line can stay invariant while its points move

An **invariant line** maps onto itself as a set. Its individual points may move along it. For the non-singular map $\binom{X}{Y} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\binom{x}{y}$, put $y = mx$ and require $Y = mX$ for every point on that line. This gives $bm^2 + (a-d)m - c = 0$. State equations of lines, not just slopes, and test the vertical line $x = 0$ separately: assuming a finite gradient can miss it.

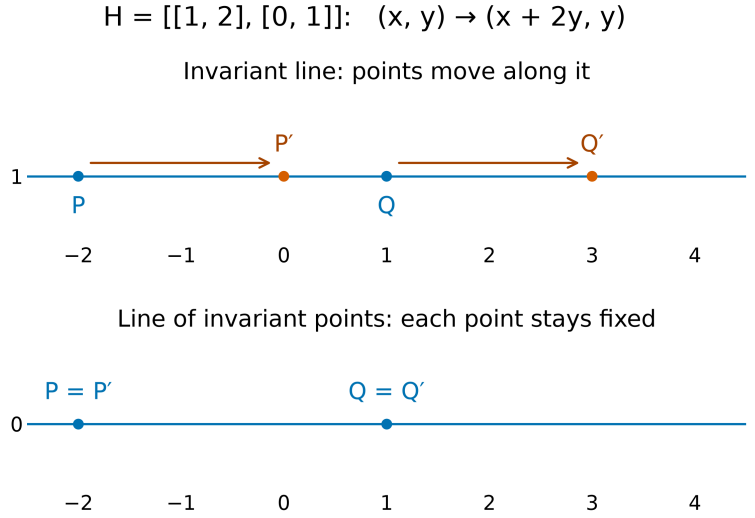
For $T = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ the finite-gradient equation is $-m - 1 = 0$, so $y = -x$ is invariant. The vertical line $x = 0$ is also invariant, because $(0, y) \mapsto (0, 3y)$.

No non-zero point on $x = 0$ stays fixed. For this matrix, solving $(T-I)(x,y)^\mathsf{T} = 0$ gives only the origin: invariant lines 不变直线 and lines of invariant points 不变点 answer different questions.

Check lines away from the origin

For a line $y = mx + c$, substitute its general point into (X, Y) and require $Y = mX + c$ for all x. Compare the coefficient of x and the constant term. Test vertical lines $x = c$ separately. For the shear $H = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, every horizontal line $y = c$ is invariant: $(x, c) \mapsto (x + 2c, c)$. Only $y = 0$ is a line of invariant points. On $y = 1$, every point moves two units right.

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The two marked points move along y=1, so the line stays invariant. Every point on y=0 stays in its own position.

Choose the condition before the algebra

First identify whether the task asks for a product, an inverse, a transformation, an invariant point or an invariant line. State sizes and restrictions before calculating, and describe geometric transformations 几何变换 fully. Verify an inverse by a product, a fixed point by $T\mathbf{v} = \mathbf{v}$, and an invariant line by its equation after transformation. A numerical check of one point cannot prove that an entire line is invariant.

The identity matrix is also called the **unit matrix**. An **inverse matrix** 逆矩阵 undoes a non-singular map; a **singular matrix** 奇异矩阵 has no inverse. When describing a **geometric transformation**, distinguish an invariant point from an invariant line.

Polar coordinates

Polar curves 极坐标曲线 are described by a radius and angle, together with an allowed interval.



A spiral staircase: spirals are described naturally in polar coordinates (r, theta)

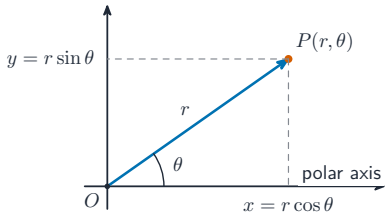
Choose coordinates that fit the measurement

A radar measures distance and direction. **Polar coordinates** store these as (r, θ), rather than the horizontal and vertical distances used by **Cartesian coordinates** 直角坐标. A circle centred at the radar has the simple equation $r = a$. Other circles need different equations: the choice of origin still matters. Even with **rotational symmetry** 旋转对称, the permitted interval may show only part of a curve; read the interval too.

Locate a point and choose its angle

The **pole** is the origin; the **initial line** 极轴 points along the positive x-axis. Use $r \geq 0$ in this course. Angles are measured anticlockwise from the initial line; use radians for calculus. Convert with $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$. In the reverse direction, use $r = \sqrt{x^2 + y^2}$ and select an angle in the stated interval using the quadrant, not just $\arctan(y/x)$.

For $(-\sqrt{3}, 1)$, $r = 2$. An angle in $0 \leq \theta < 2\pi$ is $5\pi/6$, not $-\pi/6$: the point lies in quadrant II. At the pole $r = 0$, all angle descriptions name the same point, so the angle is not unique.



The radius is a distance. Its horizontal and vertical components may be negative even though r is non-negative.

Convert equations and keep their domains

Substitute the coordinate relations into an equation, simplify, then check any step that multiplied or divided by r. Dividing by r can lose the pole; multiplying by r can introduce it. For $r = 4\cos\theta$, multiplying by r gives $x^2 + y^2 = 4x$, so $(x-2)^2 + y^2 = 4$. The circle has centre $(2, 0)$ and radius 2, traced once on $-\pi/2 \leq \theta \leq \pi/2$. A negative value from $4\cos\theta$ is not permitted under $r \geq 0$.

The line $x = 2$ becomes $r\cos\theta = 2$, or $r = 2\sec\theta$ with $-\pi/2 < \theta < \pi/2$. Multiplying by r would give $r^2\cos\theta = 2r$, which also admits $r = 0$: that extra point is not on $x = 2$.

Build a sketch from features

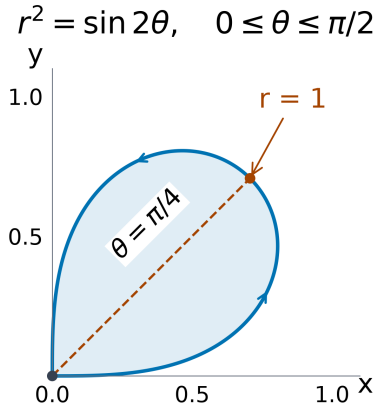
Mark the pole and initial line, the stated angle interval, points on the initial line, zero and greatest/least radii, and the shape near the pole. Use a few special angles to place the curve; detailed point-by-point plotting is unnecessary. Test symmetry using both the equation and the interval. Replacing θ by $-\theta$ tests reflection in the initial line; reflection in a line at angle α replaces θ by $2\alpha - \theta$. A symmetric formula on an asymmetric interval need not give a symmetric whole sketch.

For $r = 2(1 + \cos\theta)$ on $-\pi \leq \theta \leq \pi$, the curve is a **cardioid** 心脏线. It is symmetric about the initial line, reaches greatest radius 4 at $\theta = 0$, crosses the y-axis at $(0, \pm 2)$ and has a cusp at the pole when $\theta = \pm\pi$. It is one closed curve, not a circle.

A restricted interval can select one loop

If r^2 is given, take the non-negative square root and check where it is real. For $r^2 = \sin 2\theta$ on $0 \leq \theta \leq \pi/2$, the curve starts and ends at the pole, stays in the first quadrant and reaches $r = 1$ at $\theta = \pi/4$. Since $\sin(2(\pi/2 - \theta)) = \sin 2\theta$, this loop is symmetric about $\theta = \pi/4$. Near $\theta = 0$, $r^2 \approx 2\theta$, so it leaves the pole along the initial line; near the other endpoint it returns along the positive y-direction.

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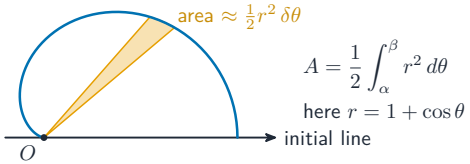


Both endpoints meet at the pole (0, 0).

Both endpoints are the same pole. The greatest radius lies on the symmetry line, not on either coordinate axis.

Find an area from radial sectors 扇形

A thin **sector** has area approximately $\frac{1}{2}r^2\,\Delta\theta$. Adding sectors gives the exact area below, when the selected region is swept once. Square r, retain the factor $1/2$ and use radian limits. For the loop above, r^2 is already $\sin 2\theta$: do not square it again. The endpoints give the bounds, not a memorised full turn.



Area as thin sectors

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta.$$

$$A = \frac{1}{2}\int_0^{\pi/2} \sin 2\theta\,d\theta = \left[-\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{2}.$$

For a complete cardioid $r = a(1+\cos\theta)$, use $(1+\cos\theta)^2$ and $\cos^2\theta = (1+\cos 2\theta)/2$. Its total area is $3\pi a^2/2$.

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Bound a region between two curves

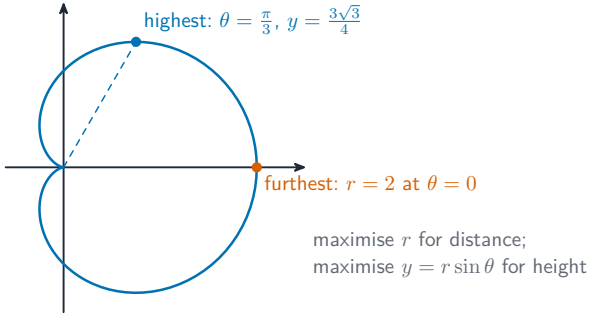
Find their common rays by solving their radii equal, and check the pole separately. Identify which radius is outer throughout each interval. The radial strip has area $\frac{1}{2}(r_{\text{outer}}^2 - r_{\text{inner}}^2)\,d\theta$; it is not the square of the difference of radii. In the first quadrant, the region inside $r = 4\cos\theta$ but outside $r = 2$ lies on $0 \le \theta \le \pi/3$. The whole region also has a reflected lower half; include it only if the question asks for it.

$$A_{\text{upper}} = \frac{1}{2}\int_0^{\pi/3} (16\cos^2\theta - 4)\,d\theta = [2\theta + 2\sin 2\theta]_0^{\pi/3} = \frac{2\pi}{3} + \sqrt{3}.$$

The pole is not a common point: $r = 2$ never reaches it. A point on one curve is not an intersection unless it lies on the other.

Furthest from the pole is not highest

Distance from the pole is r. To find its extrema, compare the allowed endpoints and stationary values of r, or of r^2 when $r \ge 0$. A point with largest r need not have largest x or y. Distance from the initial line for a first-quadrant point is $y = r\sin\theta$. Write the required distance before differentiating, then check its endpoints. For other regions, account for signs and whether the question names a full line or a half-line.



Furthest is not highest

For $r = 2\sin 2\theta$, $0 \le \theta \le \pi/2$, the largest radius is 2 at $\theta = \pi/4$. However,

$$y = 4\sin^2\theta\cos\theta.$$

Writing $u = \cos\theta$ gives $y = 4u(1-u^2)$ for $0 \le u \le 1$. Its maximum occurs at $u = 1/\sqrt{3}$, giving $y_{\max} = 8/(3\sqrt{3})$ at a different angle. The endpoints both give $y = 0$.

Integrate an unfamiliar polar curve

A polar equation need not be a named circle or cardioid. For $r = \theta e^{\theta/2}$ on $0 \le \theta \le \pi$, the radius increases from 0 to $\pi e^{\pi/2}$. The curve starts at the pole, turns anticlockwise through half a turn and ends on the negative x-axis. The region swept from the pole is bounded by the curve and the endpoint rays. Its area is $\frac{1}{2}\int_0^{\pi} \theta^2 e^{\theta}\,d\theta$. Integration by parts twice gives an exact value; do not replace it with the area of a circle having the largest radius.

$$\int \theta^2 e^{\theta}\,d\theta = e^{\theta}(\theta^2 - 2\theta + 2) + C,$$

$$A = \frac{1}{2}e^{\pi}(\pi^2 - 2\pi + 2) - 1.$$

Read the region before choosing limits

State the non-negative-radius convention and allowed angles, sketch the important features, and keep any lost or introduced pole under review when converting equations. For area, identify the region and sweep it once. For a greatest distance, decide whether the expression to optimise is r, x, y or a distance to the specified line. Check endpoints as well as stationary values.

Vectors

Vectors describe both magnitude and direction.



Forces like wind and water are vectors, with both size and direction

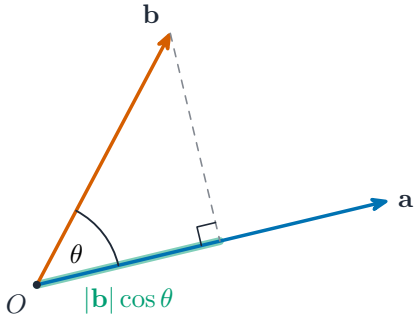
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Find the shortest safe route in space

A drone must approach a flat roof by the shortest route. A cable must join two rails that do not meet. These are different geometry problems: first identify the points, lines and **plane** involved. A direction alone is not a complete line. A distance alone does not locate its endpoints. **Vector products** 向量积 and scalar products 数量积 let you construct the geometry and then check it.

Choose the product that answers the question

The scalar product is $\mathbf{u}\cdot\mathbf{v} = u_1v_1+u_2v_2+u_3v_3$. For non-zero vectors it equals $|\mathbf{u}||\mathbf{v}|\cos\theta$, so a zero scalar product identifies **perpendicular** 垂直 directions. The zero vector has no direction or defined angle. The vector product is **anti-commutative** 反交换: swapping the order reverses its sign. Its magnitude is $|\mathbf{u}||\mathbf{v}|\sin\theta$, the area of the **parallelogram** 平行四边形 spanned by the vectors. A triangle has half this area. For non-parallel inputs, $\hat{\mathbf{n}}$ is the unit normal 单位法向量 in the ordered product direction.



The scalar product picks out $|\mathbf{b}|\cos\theta$, how far $\mathbf{b}$ reaches along $\mathbf{a}$.

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}||\mathbf{v}|\sin\theta\,\hat{\mathbf{n}}, \qquad |\hat{\mathbf{n}}| = 1,$$

$$\mathbf{u}\times\mathbf{v} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

For $\mathbf{u} = (1,1,0)$ and $\mathbf{v} = (0,1,1)$,

$$\mathbf{u}\times\mathbf{v} = (1,-1,1), \qquad \mathbf{u}\cdot\mathbf{v} = 1.$$

The cross product has zero scalar product with each input and magnitude $\sqrt{3}$. Parallel vectors have zero cross product, so they cannot supply a normal direction to a unique plane. For comparison, $(1,1,0)$ and $(1,-1,0)$ have zero scalar product and are perpendicular.

Read the three forms of a plane

A **normal vector** 法向量 is a non-zero vector perpendicular to every direction in the plane. The Cartesian form $ax + by + cz = d$ is the scalar-product form $\mathbf{r} \cdot \mathbf{n} = d$, with $\mathbf{n} = (a, b, c)$. A unit normal is $\mathbf{n}/|\mathbf{n}|$. The number d is not generally the distance from the origin unless the normal is a unit vector. In $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\mathbf{a}$ locates a point and the non-parallel vectors $\mathbf{b}, \mathbf{c}$ give directions in the plane. The parameters vary independently. Parallel direction vectors describe only a line, not a plane.

A plane with normal $(2, -1, 3)$ through $(1, 0, 2)$ has

$$2x - y + 3z = 2(1) - 0 + 3(2) = 8.$$

Multiplying every coefficient, including the right side, by the same non-zero constant gives the same plane.

Convert a parametric plane to Cartesian form

Take the cross product of its two direction vectors to find a normal. Then use a known point to find the constant: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{a}$. For three given points, first form two displacement vectors from one of them. To return to a parametric equation, choose two coordinates as free parameters, solve for the third and separate a fixed point from the two direction terms. Check both directions have zero scalar product with the normal.

For

$$\mathbf{r} = (1, 0, 2) + \lambda(1, 1, 0) + \mu(0, 1, 1),$$

the normal is $(1, -1, 1)$, giving $x - y + z = 3$. Conversely, set $x = 1 + \lambda$ and $z = 2 + \mu$. The Cartesian equation gives $y = \lambda + \mu$, recovering the original parametric form.

Classify a line and a plane before solving

Substitute $\mathbf{r} = \mathbf{a} + t\mathbf{u}$ into $\mathbf{n} \cdot \mathbf{r} = d$. If $\mathbf{n} \cdot \mathbf{u} \neq 0$, there is one intersection, found from the parameter below. Substitute back to give the actual point. If $\mathbf{n} \cdot \mathbf{u} = 0$, test the base point. When $\mathbf{n} \cdot \mathbf{a} = d$, the whole line lies in the plane; otherwise it is parallel and separate. Dividing by zero would hide this distinction.

$$t = \frac{d - \mathbf{n} \cdot \mathbf{a}}{\mathbf{n} \cdot \mathbf{u}} \quad \text{when } \mathbf{n} \cdot \mathbf{u} \neq 0.$$

For the plane $x - y + z = 3$, the line $\mathbf{r} = t(1, 0, 1)$ meets it when $2t = 3$, at $(3/2, 0, 3/2)$. The line $(1, 0, 2) + t(1, 1, 0)$ lies in the plane. The line $t(1, 1, 0)$ is parallel and separate, because its points satisfy $x - y + z = 0$ instead.

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Find the foot of a perpendicular 垂足

The foot H from a point P to a plane lies on the normal line through P. Write $\mathbf{h} = \mathbf{p} - t\mathbf{n}$, substitute into the plane equation, then solve for t. The sign determines which way to travel along the normal. Verify both conditions: H lies in the plane and $\overrightarrow{PH}$ is parallel to its normal. Merely finding some point in the plane does not make it the nearest point.

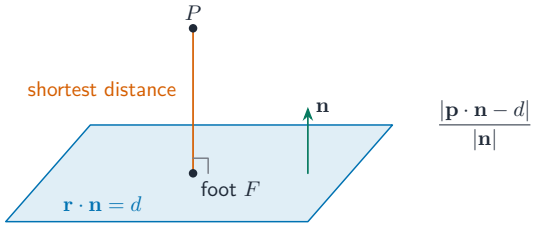
From $P = (3, 0, 3)$ to $x - y + z = 3$, use $\mathbf{n} = (1, -1, 1)$:

$$\mathbf{n} \cdot (\mathbf{p} - t\mathbf{n}) = 3 \quad \Rightarrow \quad 6 - 3t = 3.$$

Thus $t = 1$ and $H = (2, 1, 2)$. It satisfies $2 - 1 + 2 = 3$, and $\overrightarrow{PH} = (-1, 1, -1)$ is parallel to the normal.

Measure a point-to-plane distance

The shortest distance is the component of the displacement in the normal direction. For a non-zero normal, use the absolute numerator to obtain a non-negative distance; do not confuse the plane constant with the distance symbol. Multiplying a plane equation by a non-zero constant changes numerator and denominator by the same magnitude, leaving the distance unchanged. A zero numerator means the point already lies in the plane.



Distance along the normal

$$D = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|\mathbf{n} \cdot \mathbf{p} - d|}{|\mathbf{n}|}.$$

For $P = (3, 0, 3)$ and $x - y + z = 3$, $D = |6 - 3|/\sqrt{3} = \sqrt{3}$. This agrees with the length of $\overrightarrow{PH}$ above. The plane's distance from the origin is also $|3|/\sqrt{3} = \sqrt{3}$, although these are different point-to-plane problems.

Use normals for plane angles

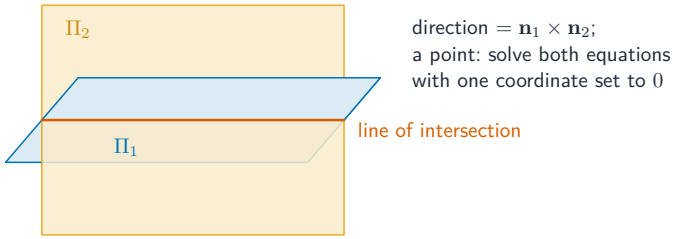
The smaller angle between two planes is the smaller angle between their normal directions, with $0 \le \alpha \le \pi/2$. The absolute scalar product removes the choice of normal direction: $\mathbf{n}$ and $-\mathbf{n}$ describe the same plane. The angle ϕ between a line and a plane is complementary to the angle between its direction and the plane normal. Use sine for the line–plane angle, with $0 \le \phi \le \pi/2$, and check both direction vectors are non-zero.

$$\cos\alpha = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}, \qquad \sin\phi = \frac{|\mathbf{u} \cdot \mathbf{n}|}{|\mathbf{u}||\mathbf{n}|}.$$

The planes $x - y + z = 3$ and $x + y + z = 5$ have normals with scalar product 1 and lengths $\sqrt{3}$, so $\alpha = \arccos(1/3) \approx 70.5°$. A line with direction $(1, 0, 1)$ makes angle $\phi = \arcsin(2/\sqrt{6}) \approx 54.7°$ with the first plane. Using cosine would give its angle with the normal instead.

Find the intersection of two planes

For non-parallel normals, the intersection direction is $\mathbf{n}_1 \times \mathbf{n}_2$. Find a common point by solving both plane equations, then give a full line equation: point plus parameter times direction. If the normals are parallel, compare the constants using the same scale factor: the planes are either identical or parallel and separate. A zero cross product cannot provide an intersection direction.



Two planes, one line

Subtract $x-y+z = 3$ from $x+y+z = 5$ to get $y = 1$, then $x+z = 4$. A common point is $(4, 1, 0)$. The cross product of the normals is $(-2, 0, 2)$, so an equivalent intersection line is

$$\mathbf{r} = (4, 1, 0) + t(-1, 0, 1).$$

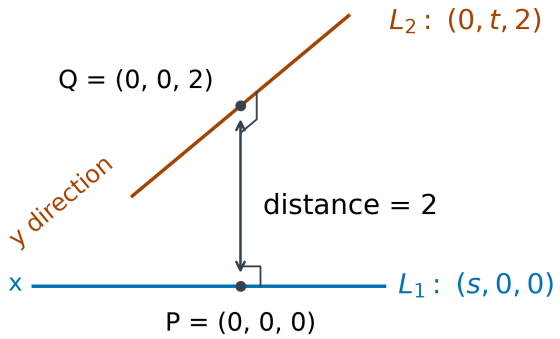
Every point of this line satisfies both plane equations; there is no need to use the unsimplified direction vector.

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Decide whether two lines are skew

Skew lines 异面直线 are neither parallel nor intersecting. For $\mathbf{r} = \mathbf{a}+s\mathbf{u}$ and $\mathbf{r} = \mathbf{b}+t\mathbf{v}$, compare their directions and, if non-parallel, solve all three coordinate equations for an intersection. Two different line parameters must be used. When $\mathbf{u} \times \mathbf{v} \neq 0$, the lines are coplanar exactly when $(\mathbf{b}-\mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v}) = 0$. A non-zero value proves they are skew. Do not apply the skew-distance denominator to parallel lines.

Two skew lines: different heights and different directions



PQ is perpendicular to both lines in three-dimensional space.

This schematic uses the x-axis and a y-parallel line at height 2. Their nearest points have the same x and y coordinates, so their connector has length 2.

Calculate the shortest skew-line distance

For non-parallel lines, $\mathbf{u}\times\mathbf{v}$ is perpendicular to both. Project the displacement between their base points onto this direction. Any non-zero multiple of the cross product gives the same distance. The formula below gives a distance, not the location of the shortest segment. For parallel lines use $|(\mathbf{b}-\mathbf{a}) \times \mathbf{u}|/|\mathbf{u}|$ instead; for intersecting non-parallel lines the numerator is zero.

$$D = \frac{|(\mathbf{b}-\mathbf{a}) \cdot (\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

Let

$$L_1 : \ \mathbf{r} = (1, 0, 1) + s(1, 1, 0), \qquad L_2 : \ \mathbf{r} = (2, 2, 3) + t(0, 1, 1).$$

Their cross product is $(1, -1, 1)$ and their base-point displacement is $(1, 2, 2)$. Its scalar product with the cross product is 1, so they are skew and $D = 1/\sqrt{3}$.

Locate the common perpendicular 公垂线

Let $P = \mathbf{a} + s\mathbf{u}$ and $Q = \mathbf{b} + t\mathbf{v}$. The shortest connector obeys $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{u} = 0$ and $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{v} = 0$. Solve these two simultaneous equations for the parameters. Give the common perpendicular as a line through P in direction $\mathbf{u} \times \mathbf{v}$, or through Q in the same direction. Check it meets both original lines and that the segment length agrees with the distance formula. A perpendicular direction through an arbitrary point is insufficient.

For the two lines above,

$$\mathbf{q} - \mathbf{p} = (1 - s,\ 2 + t - s,\ 2 + t).$$

Orthogonality gives $2s - t = 3$ and $s - 2t = 4$, hence $s = 2/3$ and $t = -5/3$. Therefore

$$P = (5/3, 2/3, 1), \qquad Q = (2, 1/3, 4/3),$$

$$\mathbf{r} = (5/3, 2/3, 1) + k(1, -1, 1)$$

is the common perpendicular. At $k = 1/3$ it reaches Q, and $|PQ| = \sqrt{3}/3$ agrees with $1/\sqrt{3}$.

Check the geometric conditions

A normal must be non-zero, a plane needs two independent directions, and the skew-distance denominator requires non-parallel lines. State parameter restrictions before squaring distance equations; absolute values can give more than one valid value. Substitute a proposed point into its plane, check an intersection line in both planes, and verify both orthogonality equations for a common perpendicular. These checks explain why the result solves the geometry.

Proof by induction

A proof using this method is an **inductive proof** 归纳证明.

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Falling dominoes: prove the first case, prove each case knocks over the next, and every case follows by induction

A repeatable rule must start somewhere

Checking a pattern for the first few **positive integers** cannot establish infinitely many cases. **Mathematical induction** 数学归纳法 joins a first true statement to a rule that reaches every next statement. The **base case** 基础情形 starts the chain. The **inductive step** 归纳步骤 proves a conditional link for an arbitrary allowed integer, not just the next number in a short table.

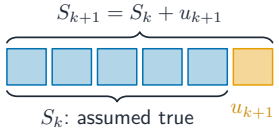
State exactly what is being proved

Write the claim $P(n)$ and its integer range, starting at n_0. Check $P(n_0)$ directly, using a given starting condition if appropriate. Then assume $P(k)$ for an arbitrary integer $k \geq n_0$ and use that assumption to derive $P(k + 1)$. The **inductive hypothesis** 归纳假设 is temporary and conditional: it does not assume the result for every integer. Conclude that $P(n)$ holds for all integers $n \geq n_0$, because the verified base case and the general link together reach them.

Do not assume the $k+1$ case while trying to prove it. Showing only $P(k+1) \Rightarrow P(k)$ gives the wrong implication. Backward algebra can help you plan, but a proof needs a valid forward argument, or a justified chain of equivalent statements.

Add the new term in a sum

For a finite sum, separate the last term of the $k+1$ sum. Replace only its first k terms using the hypothesis, then simplify to the formula with $n = k + 1$. Show the starting case, the exact hypothesis and the final conclusion. A line that merely quotes the sum formula for $k + 1$ has not established the link.



substitute the formula for S_k, add the new term, then reach the formula with $k+1$ in place of k

The inductive step adds one term

Prove $\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)/3$ for positive integers n.
At $n = 1$, both sides equal 2. Assume the formula holds for $n = k$. Then

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\ &= \frac{(k+1)(k+2)(k+3)}{3}.\end{aligned}$$

This is the required $k+1$ formula, so the result follows for all positive integers by induction.

■ Expose an integer multiple for divisibility

To prove an expression is **divisible** 整除 by a non-zero integer m, write it as m times an integer. In the inductive step, isolate the assumed divisible expression and show every remaining term is also a multiple of m. Both a convenient multiple of the k expression and the difference between consecutive expressions can help. Name the integer multiplier; "it is divisible by inspection" can hide a missing use of the hypothesis.

Let $F_n = 10^n - 4^n$. At $n = 1$, $F_1 = 6$. Assume $F_k = 6q$ for an integer q. Then

$$\begin{aligned}F_{k+1} &= 10(10^k - 4^k) + 6 \cdot 4^k \\ &= 6(10q + 4^k).\end{aligned}$$

The bracket is an integer. Thus 6 divides F_{k+1}, and induction proves the result for every positive integer n.

■ Use the recurrence to move forward

A recurrence gives a legal way to pass from u_k to u_{k+1}. Check that a proposed closed form has the correct first term, substitute its k version into the recurrence and simplify. Do not insert the proposed $k+1$ formula into the recurrence and call it a proof unless you establish a reversible equivalence with the known k formula. The clearest route starts from the given recurrence.

Suppose $u_1 = 2$ and $u_{n+1} = 3u_n + 2$. The claim is $u_n = 3^n - 1$. It is true at $n = 1$. Assuming $u_k = 3^k - 1$,

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$$u_{k+1} = 3(3^k - 1) + 2 = 3^{k+1} - 1.$$

This proves the claimed closed form for all positive integers, not just the first computed terms $2, 8, 26$.

■ Preserve a bound and check denominator signs

For an inequality involving a recurrence, substitute the hypothesis into an expression whose sign can be controlled. State why its denominator is positive before using it to preserve an inequality. Sometimes the needed hypothesis contains two bounds. Prove both survive the step. Once a bound is established for every term, it can justify a later monotonicity argument without assuming the answer.

Let $u_1 = 1$ and $u_{n+1} = (4u_n + 3)/(u_n + 2)$. Prove $0 < u_n < 3$.

The first term satisfies both inequalities. If $0 < u_k < 3$, numerator and denominator of u_{k+1} are positive. Also,

$$3 - u_{k+1} = \frac{3 - u_k}{u_k + 2} > 0.$$

Thus $0 < u_{k+1} < 3$. By induction, $0 < u_n < 3$ for every positive integer n. The established bounds then give

$$u_{n+1} - u_n = \frac{(3 - u_n)(u_n + 1)}{u_n + 2} > 0,$$

so the sequence is increasing. This last deduction uses the proven bound; it does not replace that proof.

■ Prove a matrix-power formula by multiplication

For powers of one matrix, $M^{k+1} = M^k M$. Substitute the proposed expression for M^k, multiply row by column and show every entry has the $k+1$ form. A matrix power is not found by raising each entry to a power. Also, a formula checked at M and M^2 is only evidence until the arbitrary-k step has been proved.

For $M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, claim

$$M^n = \begin{pmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{pmatrix} \quad (n \geq 1).$$

The $n = 1$ case is M itself. Using the k hypothesis,

$$\begin{aligned}M^{k+1} &= \begin{pmatrix} 2^k & 2^k - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2^{k+1} & 2^{k+1} - 1 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Therefore the formula holds for all positive integers by induction. The separate $n = 0$ case is $M^0 = I$, which also fits this formula.

Conjecture 猜想 and prove a derivative pattern

A **conjecture** is a statement suggested by evidence, not an established theorem. Compute a few derivatives, look for a pattern and express it precisely with its starting value of n. For an nth-derivative formula, differentiate the assumed kth derivative once. A correct pattern needs an actual differentiation and simplification in the inductive step.

For $f(x) = xe^{2x}$, the first derivatives are $e^{2x}(2x + 1)$, $e^{2x}(4x + 4)$ and $e^{2x}(8x + 12)$. Conjecture

$$f^{(n)}(x) = 2^{n-1}e^{2x}(2x + n), \qquad n \geq 1.$$

The first derivative gives the base case. Differentiating the k formula yields

$$2^{k-1}e^{2x}\{2(2x + k) + 2\} = 2^k e^{2x}(2x + k + 1),$$

which is the required $k + 1$ derivative. Hence the conjecture is true for every positive integer n.

Match the derivative order in the statement

If the claimed formula is for derivative order $2n$, increasing n by one increases the derivative order by two. The inductive step must differentiate twice. Count the derivative order before starting the algebra. Track the alternating sign and the coefficient depending on n separately. A statement about selected even orders is not automatically a statement about every derivative order.

For $f(x) = x \sin x$, claim

$$f^{(2n)}(x) = (-1)^n(x \sin x - 2n \cos x), \qquad n \geq 1.$$

At $n = 1$, $f''(x) = 2\cos x - x \sin x$, as required. Differentiating the assumed $2k$ formula once gives $(-1)^k\{x \cos x + (2k + 1) \sin x\}$. A second differentiation gives

$$(-1)^k\{-x \sin x + (2k + 2) \cos x\} = (-1)^{k+1}\{x \sin x - 2(k + 1) \cos x\},$$

completing the step to derivative order $2(k + 1)$. The formula therefore holds for all positive integers n by induction.

Turn a short trial into a factorial 阶乘 conjecture

A **factorial** is $n! = 1 \cdot 2 \cdots n$ for a positive integer n, with $0! = 1$. A few partial sums may suggest a shifted factorial, but the pattern must then survive a general induction step. Compute the first terms carefully and look for a simple expression after adding or subtracting a constant. When a question asks for induction, present that proof even if a direct telescoping argument is also available.

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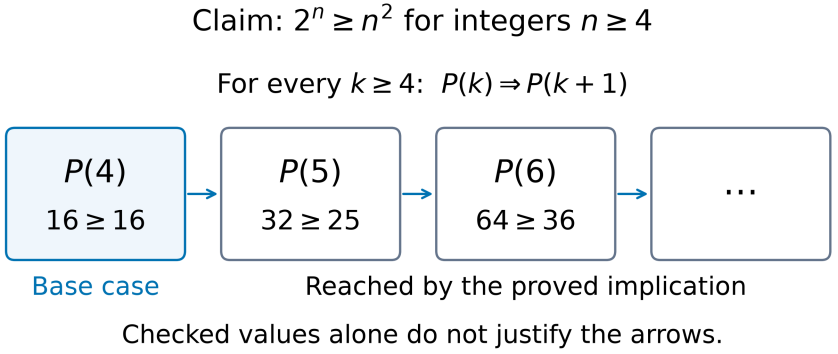
For $S_n = \sum_{r=1}^{n} r\, r!$, the first sums are $1, 5, 23, 119$. Adding 1 suggests $S_n = (n+1)! - 1$. At $n = 1$, both sides equal 1. Assuming the formula for k,

$$\begin{aligned} S_{k+1} &= (k + 1)! - 1 + (k + 1)(k + 1)! \\ &= (k + 2)(k + 1)! - 1 = (k + 2)! - 1. \end{aligned}$$

Thus the conjecture holds for all positive integers by induction. The identity $r\, r! = (r + 1)! - r!$ also explains why the sums telescope, but listing early values alone is not a proof.

Start at the first value the claim actually covers

A claim may start at $n = 4$ or another stated integer. Prove that starting case and assume k lies in the allowed range; do not silently replace the domain by all positive integers. For $2^n \geq n^2$ with $n \geq 4$, the base case is $16 = 16$. Assuming $2^k \geq k^2$ for $k \geq 4$, we have $2^{k+1} \geq 2k^2 \geq (k + 1)^2$, because $2k^2 - (k + 1)^2 = (k - 1)^2 - 2 > 0$ in that range. Hence induction proves the stated claim for all integers $n \geq 4$.



The chain proves the stated range beginning at 4. In fact n=3 fails, since 8 is less than 9; the starting point is part of the claim.

Audit the implication and the conclusion

Check that the hypothesis is used to prove the next case in the forward direction, with all expressions defined. A checked base and a backward implication alone do not establish the claim for later integers. Match the conclusion to the link: $P(k) \Rightarrow P(k + 1)$ reaches every later integer, while a link from k to $k + 2$ reaches only one parity from a single base. To cover both parities with that link, two starting cases are needed.

The statements $P(n) : n \leq 5$ give a warning. $P(1)$ is true and $P(k + 1) \Rightarrow P(k)$ is true for positive integers, yet $P(6)$ is false. The reversed implication cannot replace the required forward step.

Common mistakes.

- Sign errors in the root–coefficient relations: for $ax^3+bx^2+cx+d=0$, $\sum\alpha=-\frac{b}{a}$, $\sum\alpha\beta=\frac{c}{a}$, $\alpha\beta\gamma=-\frac{d}{a}$.
- Substituting the new root into the equation instead of the old one: for roots 2α, put $x=\frac{y}{2}$, not $x=2y$.
- Missing the oblique asymptote when the numerator's degree is one more than the denominator's.
- In the method of differences, cancelling every term: write out the first few and the last few terms to see which survive.
- Multiplying transformation matrices in the wrong order: **BA** means apply **A** first.
- Forgetting that $|\det\mathbf{M}|$ is the area scale factor, and that a negative determinant means a reflection is involved.
- Using cos for the angle between a line and a plane; that formula gives the angle with the normal.
- Ending an induction proof without the closing statement, or using the $n=k+1$ result inside its own proof.

Practice

Answer each before reading the answers below.

1. The roots of $2x^3-3x^2+4x-5=0$ are α, β and γ. Find $\alpha+\beta+\gamma$, $\alpha\beta+\beta\gamma+\gamma\alpha$ and $\alpha\beta\gamma$. Hence find $\alpha^2+\beta^2+\gamma^2$ and explain what its sign tells you about the roots.
2. For the same equation, find a cubic equation with integer coefficients whose roots are 2α, 2β and 2γ.
3. For $y=\dfrac{x^2+x-2}{x-3}$: (a) state the vertical asymptote and find the oblique asymptote; (b) show that y cannot take any value strictly between $7-2\sqrt{10}$ and $7+2\sqrt{10}$.
4. Use standard results to show that $\displaystyle\sum_{r=1}^{n} r(r+1)=\tfrac{1}{3}n(n+1)(n+2)$.
5. Use the method of differences to find $\displaystyle\sum_{r=1}^{n}\frac{1}{r(r+1)}$, and deduce the sum to infinity.
6. (a) Find the inverse of $\mathbf{A}=\begin{pmatrix}3&1\\5&2\end{pmatrix}$. (b) Describe the transformation given by $\mathbf{R}=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, find $\mathbf{R}^2$ and describe it, and find the invariant points of $\mathbf{R}$.
7. Sketch the curve $r=2(1+\cos\theta)$ for $0\le\theta<2\pi$ and find the area it encloses.
8. The line l is $\mathbf{r}=\begin{pmatrix}1\\0\\2\end{pmatrix}+t\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and the plane Π is $x+y+z=9$. Find the point where l meets Π and the acute angle between l and Π.
9. Prove by induction that $\displaystyle\sum_{r=1}^{n} r^3=\tfrac{1}{4}n^2(n+1)^2$ for all positive integers n.
10. Prove by induction that 7^n-1 is divisible by 6 for all positive integers n.

Answers

1. $\sum\alpha=\frac{3}{2}$, $\sum\alpha\beta=\frac{4}{2}=2$, $\alpha\beta\gamma=\frac{5}{2}$. $\sum\alpha^2=\left(\frac{3}{2}\right)^2-2(2)=-\frac{7}{4}$. A sum of squares of real numbers cannot be negative, so the roots are not all real: there is one real root and a complex-conjugate pair.
2. Put $y=2x$, so $x=\frac{y}{2}$: $2\cdot\frac{y^3}{8}-3\cdot\frac{y^2}{4}+4\cdot\frac{y}{2}-5=0$. Multiply by 4: $y^3-3y^2+8y-20=0$.
3. (a) $x=3$. Division gives $x^2+x-2=(x-3)(x+4)+10$, so $y=x+4+\dfrac{10}{x-3}$ and the oblique asymptote is $y=x+4$. (b) $y(x-3)=x^2+x-2$ gives $x^2+(1-y)x+(3y-2)=0$. Real x needs $(1-y)^2-4(3y-2)\ge0$, that is $y^2-14y+9\ge0$. The roots of $y^2-14y+9=0$ are $7\pm2\sqrt{10}$, so $y\le7-2\sqrt{10}$ or $y\ge7+2\sqrt{10}$.
4. $\sum r^2+\sum r=\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)=\frac{1}{6}n(n+1)\big[(2n+1)+3\big]=\frac{1}{6}n(n+1)(2n+4)=\frac{1}{3}n(n+1)(n+2)$.
5. $\dfrac{1}{r(r+1)}=\dfrac{1}{r}-\dfrac{1}{r+1}$. Writing out the terms, everything cancels except the first and the last: $\displaystyle\sum_{r=1}^{n}=1-\frac{1}{n+1}=\frac{n}{n+1}$. As $n\to\infty$, $\dfrac{1}{n+1}\to0$, so the sum to infinity is 1.
6. (a) $\det\mathbf{A}=6-5=1$, so $\mathbf{A}^{-1}=\begin{pmatrix}2&-1\\-5&3\end{pmatrix}$. (b) $\mathbf{R}$ is a rotation through $90°$ anticlockwise about the origin. $\mathbf{R}^2=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$, a rotation through $180°$. $\mathbf{R}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\y\end{pmatrix}$ gives $-y=x$ and $x=y$, so only the origin is invariant.
7. The cardioid has $r=4$ at $\theta=0$, $r=2$ at $\theta=\frac{\pi}{2}$ and a cusp at the pole when $\theta=\pi$. Area $=\frac{1}{2}\displaystyle\int_0^{2\pi}4(1+\cos\theta)^2\,d\theta=2\int_0^{2\pi}\left(1+2\cos\theta+\tfrac{1}{2}+\tfrac{1}{2}\cos2\theta\right)d\theta=2\left(\frac{3}{2}\cdot2\pi\right)=6\pi$.
8. Substitute: $(1+t)+2t+(2-t)=9$, so $3+2t=9$ and $t=3$; the point is $(4,6,-1)$. With $\mathbf{d}=(1,2,-1)$ and $\mathbf{n}=(1,1,1)$: $\sin\theta=\dfrac{|\mathbf{d}\cdot\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}=\dfrac{2}{\sqrt{6}\sqrt{3}}=\dfrac{\sqrt{2}}{3}$, so $\theta\approx28.1°$. The angle with the plane uses sin, because $\mathbf{n}$ is perpendicular to the plane.
9. Base case $n=1$: both sides equal 1. Assume $\sum_{r=1}^{k}r^3=\frac{1}{4}k^2(k+1)^2$. Then $\sum_{r=1}^{k+1}r^3=\frac{1}{4}k^2(k+1)^2+(k+1)^3=\frac{1}{4}(k+1)^2\big(k^2+4k+4\big)=\frac{1}{4}(k+1)^2(k+2)^2$, which is the formula with $n=k+1$. The result is true for $n=1$, and true for $n=k+1$ whenever it is true for $n=k$, so by induction it is true for all positive integers n.

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10. $n = 1$: $7 - 1 = 6$. Assume $7^k - 1 = 6m$ for an integer m. Then $7^{k+1} - 1 = 7 \cdot 7^k - 1 = 7(7^k - 1) + 6 = 42m + 6 = 6(7m + 1)$, which is divisible by 6. True for $n = 1$ and true for $k + 1$ whenever true for k, so true for all positive integers n.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

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Further Pure Mathematics 2

A-Level Further Mathematics

This handout covers Topic 2: **Further Pure Mathematics** 进阶纯数学 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

Hyperbolic functions

A cable needs the right curve

A freely hanging, flexible chain with uniform weight per unit length forms a **catenary** 悬链线 under gravity. Its shape uses **hyperbolic functions** 双曲函数. A shifted and scaled model is $y = a\cosh((x - h)/a) + k$, with $a > 0$. Its lowest point is $(h, a + k)$. The graph of $\cosh x$ is the normalised example; a general hanging chain need not pass through $(0, 1)$.



A chain hanging between two posts sags into a catenary, the curve y = a cosh(x/a)

Build all six functions from exponentials

An **exponential** 指数 expression uses $e^x > 0$. Define

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

The three **reciprocals** 倒数 are

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}, \qquad \operatorname{cosech} x = \frac{2}{e^x - e^{-x}},$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

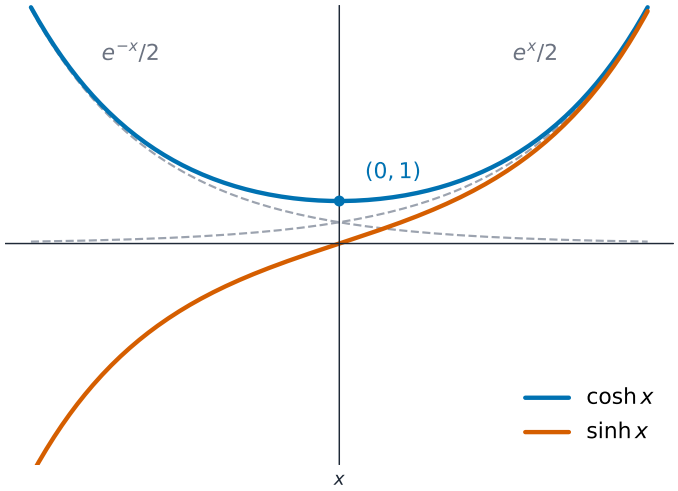
The sum $e^x + e^{-x}$ is always positive. The difference is zero only at $x = 0$. Thus sinh, cosh, tanh and sech are defined for every real input; cosech and coth exclude zero.

> **Worked example.** At $x = \ln 2$, the two exponentials are 2 and $1/2$. Hence $\sinh x = 3/4$, $\cosh x = 5/4$ and $\tanh x = 3/5$. The reciprocal values are $\operatorname{cosech} x = 4/3$, $\operatorname{sech} x = 4/5$ and $\coth x = 5/3$. At zero, sinh and tanh are zero; cosh and sech are one; cosech and coth are undefined.

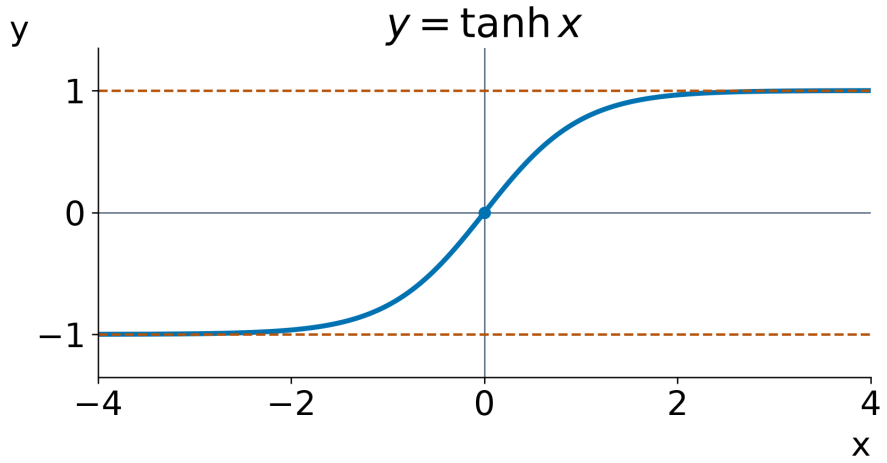
Sketch domains, ranges and limiting lines

The **domain** 定义域 lists allowed inputs; the **range** 值域 lists outputs actually reached. Use intercepts, symmetry and **asymptotes** 渐近线 to draw a complete sketch.

- Sinh is an **odd function** 奇函数 and increases through $(0, 0)$. Its range is all real numbers. It has no horizontal asymptote.
- Cosh is an **even function** 偶函数. It decreases up to $(0, 1)$, then increases. Its minimum gives range $[1, \infty)$; it has no x-intercept or horizontal asymptote.
- Tanh is odd and increases through $(0, 0)$. Its range is $(-1, 1)$. The horizontal asymptotes $y = -1$ and $y = 1$ are not attained.



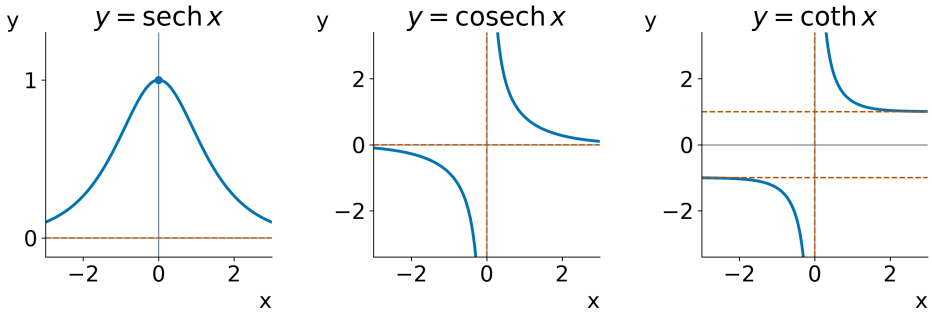
Changing the sign of x leaves cosh unchanged and reverses the sign of sinh.



As x tends to positive infinity, dividing by e^x gives $\tanh x = (1 - e^{-2x})/(1 + e^{-2x}) \to 1$.

For the reciprocal graphs:

- Sech is even, positive and defined for every real input. Its maximum is $(0, 1)$, its range is $(0, 1]$, and its horizontal asymptote is $y = 0$.
- Cosech is odd, with vertical asymptote $x = 0$ and horizontal asymptote $y = 0$. Its range is $\mathbb{R} \setminus \{0\}$. Each separate branch decreases.
- Coth is odd, with vertical asymptote $x = 0$. It approaches 1 as $x \to \infty$ and -1 as $x \to -\infty$. Its range is $(-\infty, -1) \cup (1, \infty)$; each branch decreases.



Dashed limiting lines are not parts of the graphs. Neither cosech nor coth has an axis intercept.

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Prove an identity from the requested starting point

An **identity** 恒等式 is true for every allowed value of its variable. Expanding the exponential squares gives

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = 1.$$

Dividing by $\cosh^2 x > 0$ gives $1 - \tanh^2 x = \operatorname{sech}^2 x$. Dividing by $\sinh^2 x$ gives $\coth^2 x - \operatorname{cosech}^2 x = 1$ for $x \neq 0$. The restriction remains part of the identity.

If a question requires exponential definitions, start with those definitions rather than quoting a derived identity. Work from one side and keep the same variable throughout.

Worked example. Starting from the definitions,

$$\begin{aligned} 1 - \tanh^2 u &= \frac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} \\ &= \frac{4}{(e^u + e^{-u})^2} = \operatorname{sech}^2 u. \end{aligned}$$

The denominator is non-zero for every real u. Quoting the result to prove itself would be circular.

Derive addition and double-angle results

Multiplying the exponential definitions gives

$$\begin{aligned} \sinh(u + v) &= \sinh u \cosh v + \cosh u \sinh v, \\ \cosh(u + v) &= \cosh u \cosh v + \sinh u \sinh v. \end{aligned}$$

For example, the right side of the cosh formula is

$$\frac{(e^u + e^{-u})(e^v + e^{-v}) + (e^u - e^{-u})(e^v - e^{-v})}{4} = \frac{e^{u+v} + e^{-u-v}}{2}.$$

This is the definition of $\cosh(u+v)$. The mixed terms cancel. Replace v by $-v$, using odd sinh and even cosh, for the subtraction formulas.

Putting both arguments equal to x gives

$$\sinh 2x = 2 \sinh x \cosh x,$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2 \sinh^2 x = 2 \cosh^2 x - 1.$$

Unlike the cosine addition formula, the cosh addition formula has a plus sign between the two products.

Solve equations and check both intersections

Use an identity to reduce an equation to a polynomial in sinh or cosh. Alternatively put $t = e^x > 0$, clear the non-zero denominator and solve for t. Reject roots outside the substitution's range.

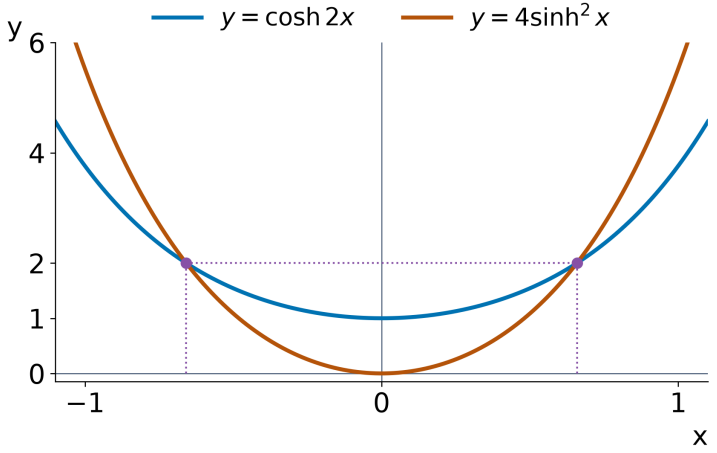
Worked example. Solve $\cosh 2x = 4 \sinh^2 x$. Since $\cosh 2x = 1 + 2 \sinh^2 x$,

$$\sinh^2 x = \frac{1}{2} \quad \Longrightarrow \quad \sinh x = \pm \frac{1}{\sqrt{2}}.$$

Both signs are valid. Using the inverse formula below gives

$$x = \pm \ln\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right).$$

Both curves are even, and both intersections have height 2. Their minima are different: $(0, 1)$ for $\cosh 2x$ and $(0, 0)$ for $4 \sinh^2 x$. This helps check a shared-axis sketch.



Taking a positive square root alone would lose the left-hand intersection.

A different substitution. For $2 \cosh x + 3 \sinh x = 1$, put $t = e^x > 0$. The equation becomes

$$\frac{5t - t^{-1}}{2} = 1 \quad \Longrightarrow \quad 5t^2 - 2t - 1 = 0.$$

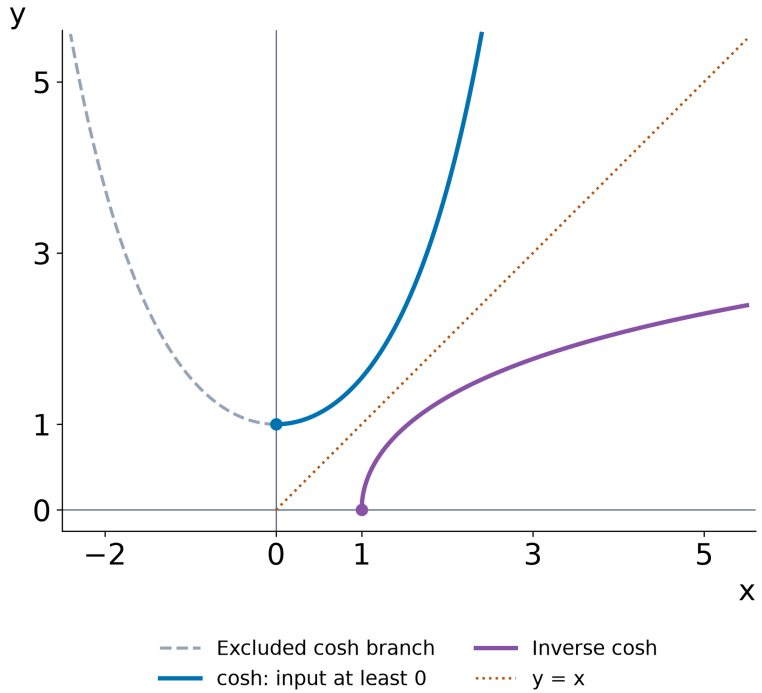
The roots are $(1 \pm \sqrt{6})/5$, but only the plus choice is positive. Thus $x = \ln((1+\sqrt{6})/5)$. A negative quadratic root cannot equal e^x for real x.

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Define an inverse before deriving its formula

An **inverse function** 反函数 reverses a one-to-one function. Its graph is reflected in $y = x$, exchanging domain and range. **Inverse hyperbolic functions** 反双曲函数 can be written in **logarithmic form** 对数形式.

Sinh is one-to-one on all real inputs. Tanh is one-to-one, with range $(-1, 1)$. Cosh needs a **principal branch** 主值分支: choose original inputs at least zero. Therefore $\cosh^{-1} x$ takes inputs $x \geq 1$ and gives outputs ≥ 0.



The faded negative-input cosh branch is not used to define the principal inverse.

The notation $\sinh^{-1} x$ means inverse sinh, not $1/\sinh x$. Also, $\cosh^{-1}(\cosh x) = |x|$ for real x; a negative original input cannot be returned by the nonnegative principal inverse.

Derive inverse sinh through a positive exponential

Let $y = \sinh^{-1} x$. Then $x = (e^y - e^{-y})/2$. Put $t = e^y > 0$:

$$t^2 - 2xt - 1 = 0 \quad \Longrightarrow \quad t = x \pm \sqrt{x^2 + 1}.$$

Since $\sqrt{x^2 + 1} > |x|$, only the plus root is positive. Therefore

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$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right), \qquad x \in \mathbb{R}.$$

The logarithm argument is positive even when $x < 0$. For example, $\sinh^{-1}(-3/4) = \ln(1/2) = -\ln 2$. The inverse sinh range is all real numbers.

Derive inverse cosh with the branch condition

Let $y = \cosh^{-1} x \geq 0$. Then $x \geq 1$ and $t = e^y \geq 1$. The quadratic is

$$t^2 - 2xt + 1 = 0 \quad \Longrightarrow \quad t = x \pm \sqrt{x^2 - 1}.$$

For $x > 1$, both roots are positive, but the minus root is below 1. It would give negative y, outside the chosen branch. At $x = 1$ both roots equal 1. Hence

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), \qquad x \geq 1.$$

For $x = 5/4$, the inverse value is $\ln 2$. In contrast, the full equation $\cosh y = 5/4$ has $y = \pm \ln 2$. More generally, $\cosh y = k$ has no real solutions for $k < 1$, one for $k = 1$, and two for $k > 1$.

Derive inverse tanh on its open interval

Let $y = \tanh^{-1} x$, with $-1 < x < 1$. Multiplying its exponential definition by e^y/e^y gives

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \quad \Longrightarrow \quad e^{2y} = \frac{1 + x}{1 - x}.$$

Both $1 + x$ and $1 - x$ are positive on this interval. Taking logarithms gives

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right), \qquad |x| < 1.$$

For example, $\tanh^{-1}(3/5) = \frac{1}{2} \ln 4 = \ln 2$. The inverse has all real outputs but excludes inputs ± 1. An unattained horizontal asymptote is not a finite inverse input.

Use inverses of reciprocal functions carefully

For the principal inverse sech, choose the original branch $y \geq 0$:

$$\operatorname{sech}^{-1} x = \cosh^{-1}(1/x) = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), \qquad 0 < x \leq 1.$$

Cosech is one-to-one across its two sign-separated branches, giving

$$\operatorname{cosech}^{-1} x = \sinh^{-1}(1/x) = \ln\left(\frac{1}{x} + \sqrt{1 + \frac{1}{x^2}}\right), \qquad x \neq 0.$$

For coth, the reciprocal input must lie inside $(-1, 1)$:

$$\coth^{-1} x = \tanh^{-1}(1/x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), \qquad |x| > 1.$$

For instance, $\operatorname{sech}^{-1}(3/5) = \ln 3$, but $\operatorname{sech} y = 3/5$ has both signs $y = \pm \ln 3$. Also, $\operatorname{cosech}^{-1}(-3/4) = -\ln 3$ and $\coth^{-1} 2 = \frac{1}{2}\ln 3$. A reciprocal of a function and its inverse function perform different operations.

Check the original restrictions

State the real domain and chosen branch before using an inverse. Keep both signs when the full equation allows them. Check every polynomial root against a substitution such as $t = e^x > 0$.

For proofs, begin from the requested definitions and show the algebra. For sketches, mark actual intercepts, symmetry, excluded inputs and limiting lines. Exact logarithmic answers should still satisfy the original hyperbolic equation.

Matrices

One map, many repeated moves

A matrix can describe one change in a model. Repeating the change needs a power such as A^{20}; multiplying twenty times hides the structure. This lesson joins three tools: solve a **matrix equation** 矩阵方程, find special vectors, then simplify powers and inverses. Keep track of when a solution or inverse exists.

Put three equations into one matrix

Put the coefficients of x, y, z in that order in every row. The right-hand side is a column, not another row of the coefficient matrix. To read the matrix equation back, multiply each row by the unknown column. A missing variable has coefficient zero.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + z &= 8 \\ 2x + y + 3z &= 13 \end{aligned} \iff \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}}_{C} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 13 \end{pmatrix}.$$

Find and check the unique point

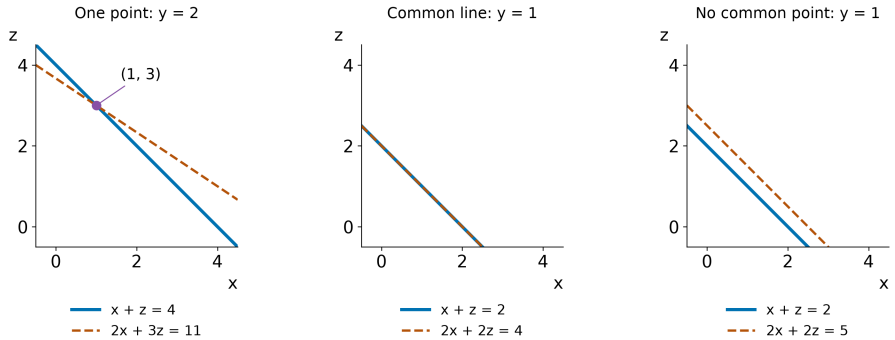
Here $\det C = 1 \neq 0$, so C is **non-singular** 非奇异: there is exactly one solution. Inverse multiplication is valid, but elimination is shorter here. Subtract the first equation from the second to get $y = 2$. The first then gives $x + z = 4$, and the third gives $2x + 3z = 11$; hence $(x, y, z) = (1, 2, 3)$.

Check all three original equations: $1 + 2 + 3 = 6$, $1 + 4 + 3 = 8$ and $2 + 2 + 9 = 13$. Geometrically, the three planes meet at one point. Checking only two planes is not enough.

A zero determinant needs another check

A **singular** 奇异 coefficient matrix has determinant zero. The equations may be **consistent** 相容 (at least one solution) or **inconsistent** 不相容 (no solution); singular does not mean no solution. For $x + y + z = 3$ and $x + 2y + z = 4$, subtraction gives $y = 1$, $x + z = 2$. A third equation $2x + 3y + 2z = 7$ adds no condition: the common line is $(x, y, z) = (2 - t, 1, t)$, $t \in \mathbb{R}$.

Change only the third right-hand side to 8. Subtracting the first two equations from the third gives $0 = 1$, so no point lies on all three planes. Pairwise intersections do not prove a common intersection. Coincident planes can also occur; three identical equations leave a whole plane of solutions.



Each panel uses the value of y forced by the first two equations. The remaining lines represent the conditions on x and z.

Separate exceptional parameter values

For the system below, subtract row 1 from rows 2 and 3. The determinant is $(a - 1)(a + 1)$, so only $a = 1, -1$ need a separate consistency check. If $a \neq \pm 1$, then $y = 1/(a - 1)$, $z = 0$ and $x = 3 - 1/(a - 1)$. Never divide by $a - 1$ or $a + 1$ before separating the zero cases.

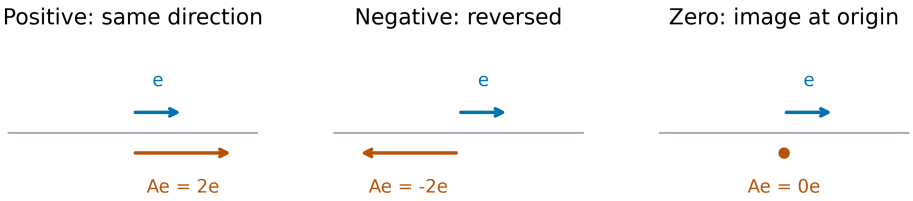
$$\begin{aligned} x + y + z &= 3 \\ x + ay + z &= 4 \\ x + y + (a+2)z &= 3 \end{aligned} \implies \begin{aligned} (a-1)y &= 1 \\ (a+1)z &= 0. \end{aligned}$$

At $a = 1$, $0 = 1$ is impossible. At $a = -1$, $y = -1/2$ but $z = t$ is free, giving the common line $(x, y, z) = (7/2 - t, -1/2, t)$. Both coefficient matrices are singular; only the second system is consistent.

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A special vector stays on its line

An **eigenvector** 特征向量 is a non-zero vector $\mathbf{e}$ with $A\mathbf{e} = \lambda\mathbf{e}$. Its **eigenvalue** 特征值 is the **scalar** 标量 λ. Every non-zero multiple of $\mathbf{e}$ is another eigenvector for the same value. A positive eigenvalue can **stretch** 拉伸 or shorten a vector. A negative value also reverses it; zero sends it to the origin, where direction is undefined. The zero vector is never an eigenvector, although it solves the equation.

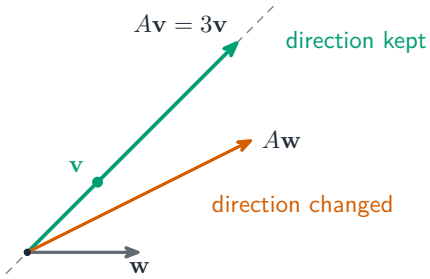


The original vector e is non-zero in every panel.

Find two eigenvalues and their vectors

A non-zero solution of $(A - \lambda I)\mathbf{e} = \mathbf{0}$ exists when $\det(A - \lambda I) = 0$. This is the **characteristic equation** 特征方程; using $\det(\lambda I - A) = 0$ gives the same roots. For the matrix below, $\lambda = 1$ gives $u + v = 0$, so choose $(1, -1)^T$. For $\lambda = 3$, $u - v = 0$, so choose $(1, 1)^T$. Check each by multiplying by A.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad \det(\lambda I - A) = (\lambda - 2)^2 - 1 = (\lambda - 1)(\lambda - 3).$$



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Extend the method to three dimensions

A triangular matrix has the diagonal entries as eigenvalues: its characteristic determinant is the product of the diagonal factors. Do not row-reduce the original matrix to find its eigenvalues; ordinary row operations need not preserve them. For each root below, solve $(B - \lambda I)\mathbf{e} = 0$. At $\lambda = 3$, the first row gives $-u + v = 0$ and the third gives $w = 0$, hence $(1, 1, 0)^T$.

$$B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \chi_B(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda + 1).$$

λ	conditions	one eigenvector
2	$v = w = 0$	$(1, 0, 0)^T$
3	$u = v,\ w = 0$	$(1, 1, 0)^T$
-1	$u = v = 0$	$(0, 0, 1)^T$

Prove what happens under powers

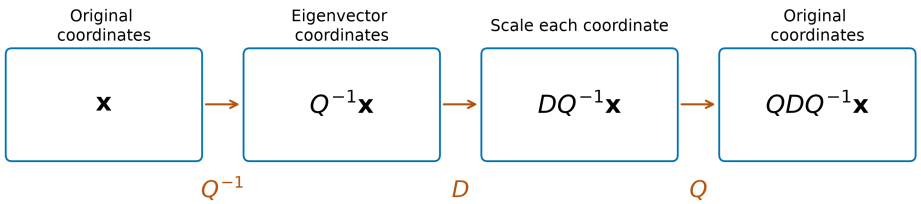
Start with $A\mathbf{e} = \lambda\mathbf{e}$, not with an inverse of the vector. Then $A^2\mathbf{e} = A(\lambda\mathbf{e}) = \lambda^2\mathbf{e}$; repeating gives $A^n\mathbf{e} = \lambda^n\mathbf{e}$ for positive integers n. Also $(A + cI)\mathbf{e} = (\lambda + c)\mathbf{e}$. If A is invertible, $\lambda \neq 0$ and multiplication by A^{-1} gives $A^{-1}\mathbf{e} = \lambda^{-1}\mathbf{e}$. Positive powers do not require invertibility.

For B, the vector $(0, 0, 1)^T$ has eigenvalue -1. Under B^2 its eigenvalue is 1, under B^3 it is -1, and under B^{-1} it is -1. The vector remains non-zero in all these cases.

Match each column to its eigenvalue

Diagonalisation 对角化 writes $B = QDQ^{-1}$, where D is a **diagonal matrix** 对角矩阵. Put eigenvectors in the columns of Q, and put their eigenvalues in the same order down D. $BQ = QD$ checks all three eigenvector equations at once. The columns must be **linearly independent** 线性无关 so that Q^{-1} exists. Real, distinct eigenvalues guarantee this in the syllabus cases; do not assume every matrix is diagonalisable.

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad Q^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$



Read matrix products from right to left: Q^{-1} first, then D, then Q.

Compute powers without repeated multiplication

In $(QDQ^{-1})(QDQ^{-1})$, the middle $Q^{-1}Q$ cancels. Thus $B^n = QD^nQ^{-1}$: raise only the diagonal entries of D to the power. For $n = 0$ this gives I; because B is invertible, the same formula holds for negative integers too. Its upper-right block shows why powers of individual entries of B would be wrong.

$$B^n = \begin{pmatrix} 2^n & 3^n - 2^n & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}.$$

For the earlier 2×2 matrix A, choose $Q_A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $D_A = \operatorname{diag}(1, 3)$. Then

$$A^n = \frac{1}{2}\begin{pmatrix} 1 + 3^n & 3^n - 1 \\ 3^n - 1 & 1 + 3^n \end{pmatrix}.$$

Reuse the vectors for a changed matrix

A polynomial $f(B)$ has the same eigenvectors with eigenvalues $f(\lambda)$. Therefore $f(B) = Qf(D)Q^{-1}$; do not change Q merely because the question asks about a shifted or squared matrix. For $(B + I)^2$, use $D' = \operatorname{diag}(9, 16, 0)$ with the same Q. It is singular but still diagonalisable. Repeated transformed eigenvalues also do not invalidate an already invertible Q.

For $(B - 2I)^3$, the diagonal becomes $\operatorname{diag}(0, 1, -27)$. For B^{-1} it becomes $\operatorname{diag}(1/2, 1/3, -1)$. In every case, keep the order $2, 3, -1$ used to build Q.

Reconstruct a matrix from its eigenvectors

Given the eigenvectors first, form Q and D, then calculate QDQ^{-1}. The columns may be scaled separately by non-zero numbers without changing the resulting matrix. If $(1, 1)^T$ has eigenvalue 4 and $(1, -1)^T$ has eigenvalue 2, the calculation below gives R. Check $R(1, 1)^T = 4(1, 1)^T$ and $R(1, -1)^T = 2(1, -1)^T$.

$$R = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

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A matrix satisfies its own equation

The **Cayley–Hamilton theorem** 凯莱 – 哈密顿定理 states that a square matrix satisfies its characteristic equation. Replace the variable by the matrix and each scalar constant by that constant times I. For B, expanding the three factors gives $\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$. Therefore $B^3 - 4B^2 + B + 6I = 0$, or $B^3 = 4B^2 - B - 6I$.

Use the characteristic equation of the matrix named in the question. Writing $B^3 - 4B^2 + B + 6 = 0$ omits the identity matrix and adds a scalar to a matrix. The theorem does not say that the whole matrix is an eigenvalue.

Reduce higher powers to lower ones

Multiply the identity for B^3 by B, then replace B^3 again. The answer becomes a combination of B^2, B, I, without diagonalising again. A shifted cube can be expanded because B and I commute. After expanding, reduce every power of degree three or more using Cayley–Hamilton.

$$\begin{aligned} B^4 &= 4B^3 - B^2 - 6B \\ &= 4(4B^2 - B - 6I) - B^2 - 6B \\ &= 15B^2 - 10B - 24I, \\ (B - 2I)^3 &= B^3 - 6B^2 + 12B - 8I \\ &= -2B^2 + 11B - 14I. \end{aligned}$$

Obtain an inverse from Cayley–Hamilton

$\det B = -6 \neq 0$, so multiply the characteristic identity by B^{-1} and collect it. This gives an inverse as a polynomial in B; do not multiply by an inverse before checking it exists. For the 2×2 matrix A, $A^2 - 4A + 3I = 0$ gives $A^{-1} = (4I - A)/3$. Cayley–Hamilton works for both matrix sizes.

$$B^2 - 4B + I + 6B^{-1} = 0,$$

$$B^{-1} = \frac{-B^2 + 4B - I}{6} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An inverse square can simplify further. Multiply the identity for B^3 by B^{-2}: $B - 4I + B^{-1} + 6B^{-2} = 0$. Substitute the expression for B^{-1} to obtain $B^{-2} = (B^2 - 10B + 25I)/36$.

Choose and justify the method

For three equations: form the coefficient matrix, separate singular parameter values, then solve or prove inconsistency and describe the geometry. For eigenvectors: state non-zero, solve the characteristic equation and match each vector to its root. For powers or inverses: choose diagonalisation or Cayley–Hamilton, show the intermediate

identity, and check the result. A determinant, a list of roots or a final matrix alone may leave most of the method unexplained.

$C\mathbf{x} = \mathbf{b}$ describes simultaneous equations. $A\mathbf{e} = \lambda\mathbf{e}$, $\mathbf{e} \neq 0$, describes a special direction. $A = QDQ^{-1}$ requires an invertible eigenvector matrix. Cayley–Hamilton reduces matrix powers and, when the matrix is non-singular, gives its inverse.

Differentiation

Follow a curve recorded by time

A moving tool may follow a curve given by $x(t)$ and $y(t)$. Its slope is a change in y per change in x, not simply per change in time t. **Differentiation** 微分 gives the slope and how it changes. Near $x = 0$, a short **polynomial** 多项式 can match the curve's value, slope and bending without describing the whole curve.

Differentiate the six hyperbolic functions

Differentiate the exponential definitions to obtain $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. Unlike ordinary cosine, hyperbolic cosine has no minus sign here. Use the quotient or reciprocal rule for the other four **hyperbolic** 双曲 functions. The real domains still exclude zeros of a denominator.

Function	Derivative
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$
$\coth x$	$-\operatorname{cosech}^2 x$

For example, $(\tanh x)' = (\cosh^2 x - \sinh^2 x)/\cosh^2 x = \operatorname{sech}^2 x$. The derivatives of cosech and coth apply on $x \neq 0$.

Derive inverse trigonometric derivatives

In $y = \sin^{-1} x$, write $\sin y = x$ and differentiate: $\cos y\, y' = 1$. The principal branch has $\cos y \geq 0$, so $y' = 1/\sqrt{1-x^2}$ for $-1 < x < 1$. For $y = \cos^{-1} x$, $-\sin y\, y' = 1$, and $\sin y \geq 0$ on its principal branch. The derivative is negative. Also retain the familiar inverse-tangent derivative.

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, \quad |x| < 1.$$

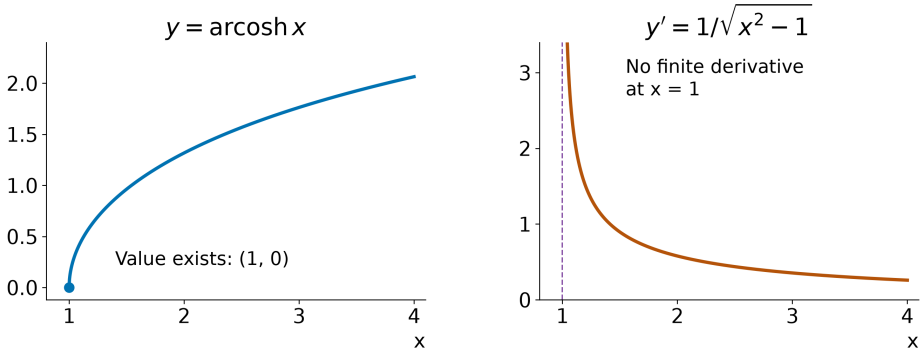
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

Here the superscript -1 means an inverse function, not a reciprocal. The inverse sine and cosine functions have endpoint values at ± 1, but no finite derivative there.

Derive inverse hyperbolic derivatives

For $y = \sinh^{-1} x$, differentiation of $\sinh y = x$ gives $\cosh y\, y' = 1$. Since $\cosh y > 0$, use $\cosh y = \sqrt{1+x^2}$. For inverse cosh choose $y \geq 0$, so $\sinh y = \sqrt{x^2-1}$. For inverse tanh use $1 - \tanh^2 y = 1 - x^2$. Record the open intervals where finite derivatives exist.

$$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}, \quad x \in \mathbb{R},$$
$$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, \quad x > 1,$$
$$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}, \quad |x| < 1.$$



Inverse cosh is defined at x=1, with value zero. Its finite derivative formula applies only for x greater than 1.

Keep the inner derivative and the domain

The **chain rule** 链式法则 gives $\frac{d}{dx}f(u(x)) = f'(u(x))u'(x)$. A correct outer formula is incomplete without the inner derivative. For $\cosh^{-1}(2x)$ the function needs $x \geq 1/2$, but the finite derivative below needs $x > 1/2$. For $\sin^{-1}(2x)$ the finite derivative needs $|x| < 1/2$.

$$\frac{d}{dx}\cosh^{-1}(2x) = \frac{2}{\sqrt{4x^2-1}},$$
$$\frac{d}{dx}\sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}},$$
$$\frac{d}{dx}\tanh^{-1}(x^2) = \frac{2x}{1-x^4}, \qquad |x| < 1.$$

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Differentiate an implicit relation once

An **implicit** 隐式 relation mixes x and y. Treat y as a function of x: $(y^2)' = 2yy'$ and $(xy)' = y + xy'$ by the product rule. For $x^2 + xy + y^2 = 7$, collect all terms containing y' before dividing. At $(1, 2)$, the denominator is $5 \neq 0$ and $y' = -4/5$.

$$\begin{aligned} 2x + y + xy' + 2yy' &= 0, \\ (x + 2y)y' &= -(2x + y), \\ y' &= -\frac{2x + y}{x + 2y}, \qquad x + 2y \neq 0. \end{aligned}$$

Differentiate the implicit relation again

A **second derivative** 二阶导数 is $y'' = d^2y/dx^2$. Differentiate the unsimplified first relation again; this often avoids a long quotient. Differentiating $2x + y + xy' + 2yy' = 0$ gives the expression below. At $(1, 2)$ substitute $y' = -4/5$ only after forming the second-derivative equation.

$$\begin{aligned} 2 + 2y' + 2(y')^2 + (x + 2y)y'' &= 0, \\ y'' &= -\frac{2 + 2y' + 2(y')^2}{x + 2y}, \\ y''\big|_{(1,2)} &= -\frac{2 - 8/5 + 32/25}{5} = -\frac{42}{125}. \end{aligned}$$

The derivative of yy' is $(y')^2 + yy''$, not just yy''. A denominator zero in a slope formula needs separate analysis; it does not automatically prove that the original curve has no point there.

Combine an inverse with an implicit product

For $y^2 - \sinh^{-1}(xy) = 1 - \ln 2$, the point $(3/4, 1)$ lies on the curve because $\sinh^{-1}(3/4) = \ln 2$. Set $p = y'$ and $S = \sqrt{1 + x^2y^2}$ to organise the calculation. The first relation below gives $p = 4/7$ at the point. For the second derivative, differentiate before substituting: $(y + xp)' = 2p + xy''$ and $S' = xy(y + xp)/S$.

$$2yp - \frac{y + xp}{S} = 0.$$

$$2p^2 + 2yy'' - \frac{2p + xy''}{S} + \frac{xy(y + xp)^2}{S^3} = 0.$$

At $(3/4, 1)$, $S = 5/4$ and $p = 4/7$. The second relation becomes

$$\frac{7}{5}y'' + \frac{128}{245} = 0, \qquad y'' = -\frac{128}{343}.$$

The square-root factor contains the variable y too. Treating it as a function of x alone loses a chain-rule term.

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Change the parameter derivative into a slope

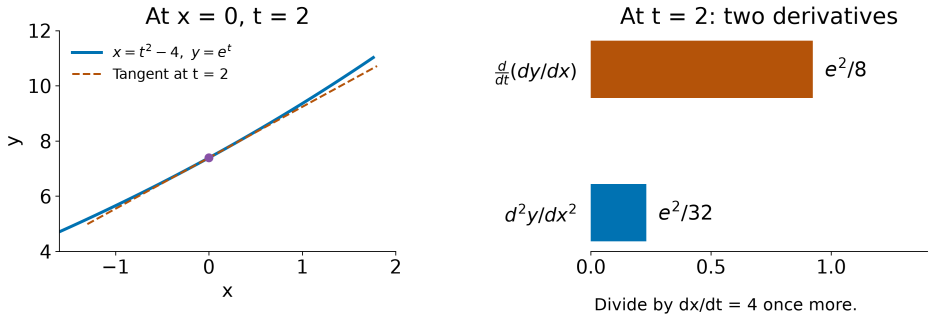
A **parametric** 参数 description uses $x = x(t)$ and $y = y(t)$. Where $x'(t) \neq 0$, $dy/dx = y'(t)/x'(t)$. The prime here must refer to the variable you state. For $x = t^2 - 4$, $y = e^t$, $t > 0$, the slope is $e^t/(2t)$. At $x = 0$ the chosen branch has $t = 2$, so the point is $(0, e^2)$ and its slope is $e^2/4$.

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = e^t, \qquad \frac{dy}{dx} = \frac{e^t}{2t}.$$

Divide by dx over dt for the second derivative too

Differentiating dy/dx with respect to t is not yet a second derivative with respect to x. Divide that result by dx/dt once more. In the worked curve, the extra factor changes the denominator. At $t = 2$, the result is $e^2/32$, rather than the parameter derivative $e^2/8$.

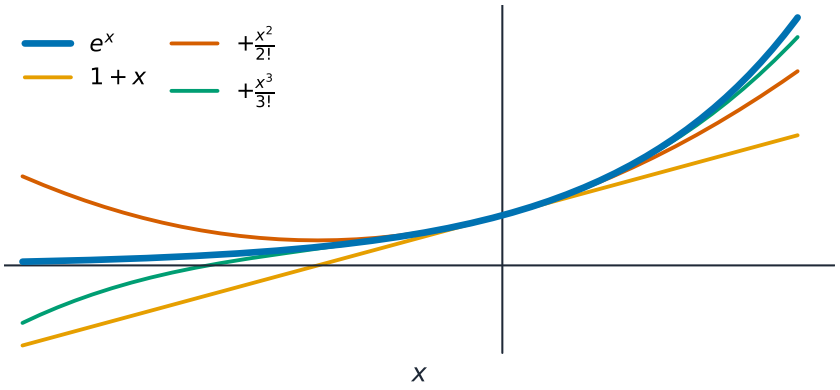
$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}, \\ \frac{d}{dt}\left(\frac{e^t}{2t}\right) &= \frac{e^t(t-1)}{2t^2}, \\ \frac{d^2y}{dx^2} &= \frac{e^t(t-1)}{4t^3}. \end{aligned}$$



Build a polynomial from derivatives at zero

A **Maclaurin's series** 麦克劳林级数 uses derivatives with respect to x, evaluated at $x = 0$. Its x^n **coefficient** 系数 is $f^{(n)}(0)/n!$; do not omit the factorial. To find terms through x^2, obtain $f(0), f'(0), f''(0)$ before substituting. For e^x, these values are all 1, so the first terms are $1 + x + x^2/2$.

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots .$$



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n \quad \cdot \quad e^x = 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots$$

Matching derivatives describes local behaviour. It does not by itself prove that an infinite series equals the function everywhere.

Obtain a composite expansion by differentiation

Let $f(x) = e^{1/(x+3)}$ and write $E = e^{1/3}$. The chain rule gives $f' = -(x+3)^{-2}f$; the product rule is then needed for f''. At zero, $f(0) = E$, $f'(0) = -E/9$ and $f''(0) = 7E/81$. Divide the last value by 2! to obtain the quadratic coefficient.

$$f''(x) = \left(2(x+3)^{-3} + (x+3)^{-4}\right) f(x),$$
$$f(x) = E - \frac{E}{9}x + \frac{7E}{162}x^2 + \cdots .$$

For $f(x) = \ln(1+x)$, the values at zero are $0, 1, -1, 2$ through the third derivative. Thus the first three non-zero terms are $x - x^2/2 + x^3/3$. Keep the factorial: $2/3! = 1/3$.

Use successive implicit derivatives for tan

For $y = \tan x$, start with $y' = \sec^2 x = 1 + y^2$. At $x = 0$, $y = 0$ and $y' = 1$; this relation keeps later derivatives in terms of y and its derivatives. The next two derivatives below give $y''(0) = 0$ and $y'''(0) = 2$. Therefore $\tan x = x + x^3/3 + \cdots$; there is no constant or quadratic term. A general coefficient formula is not required here.

$$y'' = 2yy', \qquad y''' = 2(y')^2 + 2yy''.$$

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Locate x zero before expanding a parametric curve

A Maclaurin expansion is centred on $x = 0$, which need not mean $t = 0$. For $x = t^2 - 4$, $y = e^t$, $t > 0$, use $t = 2$ and the derivatives already obtained. Here $y(0) = e^2$, $y'_x(0) = e^2/4$ and $y''_{xx}(0) = e^2/32$. The quadratic coefficient is $e^2/64$, not $e^2/32$.

$$y = e^2\left(1 + \frac{x}{4} + \frac{x^2}{64}\right) + \cdots .$$

Without the branch $t > 0$, $t = \pm 2$ gives two different points with $x = 0$. Identify the local branch before assigning a single value $y(0)$.

Join hyperbolic parameters to a local expansion

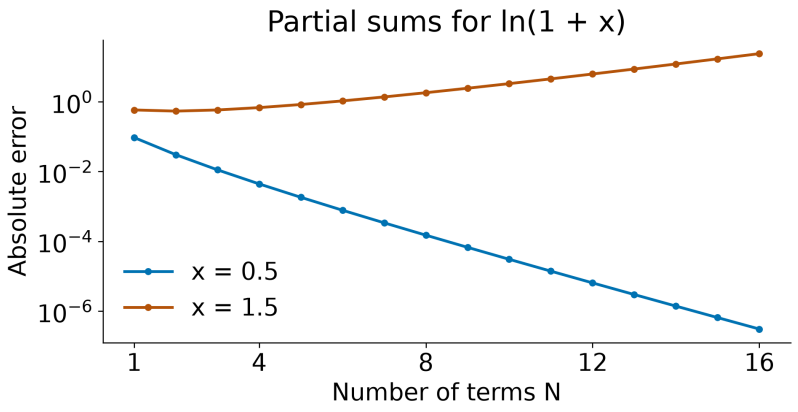
For $x = \sinh t$, $y = 2t + \cosh t$, the first derivative is $(2 + \sinh t)/\cosh t$. Since $\cosh t > 0$, division by dx/dt is valid for all real t. Apply the quotient rule, then divide by $\cosh t$ again. At $x = 0$, $t = 0$, so $y = 1$, $dy/dx = 2$ and $d^2y/dx^2 = 1$.

$$\frac{d^2y}{dx^2} = \frac{\cosh^2 t - (2 + \sinh t)\sinh t}{\cosh^3 t} = \frac{1 - 2\sinh t}{\cosh^3 t}.$$

$$y = 1 + 2x + \frac{x^2}{2} + \cdots .$$

Keep approximation and convergence separate

A finite Taylor polynomial is a local approximation; an infinite series needs an **interval of convergence** 收敛区间. For $\ln(1+x)$ that interval is $-1 < x \le 1$; for e^x, sine and cosine it is all real x. Outside its interval a series may **diverge** 发散, so adding terms need not improve the approximation. Even within the interval, convergence alone is not a promise of improvement at every finite step.



The plotted error is the absolute difference between a finite sum and ln(1+x). Both function values exist; only x=0.5 lies inside the series interval.

Choose the variable before differentiating

For an inverse function, state its branch and finite-derivative domain, then apply the inner derivative. For an implicit relation, carry every y' factor and use the product rule again for y''. For a parametric curve, divide by dx/dt for each derivative with respect to x. For a Maclaurin expansion, locate $x = 0$, evaluate the needed derivatives, and divide by factorials.

An inverse value can exist where its finite derivative does not. A parameter value need not equal the x-coordinate. A second parameter derivative need not equal d^2y/dx^2. Checking these distinctions prevents correct formulas being used at the wrong point.

Integration

Length and paint are different integrals

A curved strip needs material along its length; rotating it around an axis creates a surface to paint. Neither quantity is simply the area under its graph. First decide what is being measured. Then choose an **antiderivative** 原函数, a **standard form** 标准形式 or a **substitution** 代换, keeping the limits and any sign restrictions.



A potter shaping a vessel: a volume of revolution rotates a plane region about an axis

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Integrate the hyperbolic functions

Reverse the known derivatives to integrate **hyperbolic** functions. Include a constant C for an indefinite integral; use absolute values where a logarithm's argument can be negative. Differentiate a proposed answer to check it. Factors in an argument still matter: $\int \cosh(3x)\,dx = \sinh(3x)/3 + C$.

$$\int \sinh x\,dx = \cosh x + C,$$
$$\int \cosh x\,dx = \sinh x + C,$$
$$\int \tanh x\,dx = \ln(\cosh x) + C,$$
$$\int \coth x\,dx = \ln|\sinh x| + C,$$
$$\int \operatorname{sech} x\,dx = \tan^{-1}(\sinh x) + C,$$
$$\int \operatorname{cosech} x\,dx = \ln|\tanh(x/2)| + C.$$

The coth and cosech formulas apply on intervals that exclude $x = 0$. Useful product forms are $\int \operatorname{sech}^2 x\,dx = \tanh x + C$ and $\int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C$.

Match the three square-root forms

For $a > 0$, match the expression under the square root before choosing an inverse function. A **surd** 根式 with a plus sign uses inverse sinh; exchanging the sign changes both the inverse function and the real domain. The inverse-cosh answer below uses the positive branch $x > a$. On either real interval $|x| > a$, the logarithmic expression $\ln|x + \sqrt{x^2 - a^2}| + C$ gives an antiderivative instead.

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) + C, \quad |x| < a,$$
$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}(x/a) + C, \quad x \in \mathbb{R},$$
$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}(x/a) + C, \quad x > a.$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}(x/a) + C.$$

■ Complete the square before substituting

Completing the square 配方法 reveals the standard form. Here $x^2+4x+13=(x+2)^2+9$, so a shift makes the inverse-sinh form visible. If the coefficient of x^2 is not one, factor it carefully too. Check the outside multiplier by differentiating the final answer.

$$\int \frac{dx}{\sqrt{x^2+4x+13}} = \sinh^{-1}\left(\frac{x+2}{3}\right)+C,$$
$$\int \frac{dx}{4x^2+8x+13} = \frac{1}{6}\tan^{-1}\left(\frac{2(x+1)}{3}\right)+C.$$

■ Use a hyperbolic substitution for a surd

For $\sqrt{x^2+4}$, put $x=2\sinh u$. Then $dx=2\cosh u\,du$ and the non-negative square root is $2\cosh u$, since $\cosh u>0$. Use $\cosh^2 u=(1+\cosh 2u)/2$ and return to x. For a definite integral, either transform both limits or return to x before evaluating them.

$$\begin{aligned}\int \sqrt{x^2+4}\,dx &= 4\int \cosh^2 u\,du\\ &= 2u+\sinh 2u+C\\ &= 2\sinh^{-1}(x/2)+\frac{x}{2}\sqrt{x^2+4}+C.\end{aligned}$$

From $x=0$ to $x=3/2$, the new limits are $u=0$ and $u=\ln 2$. The exact integral is $2\ln 2+15/8$.

■ Choose a trigonometric branch

A **trigonometric** 三角 substitution $x=2\sin\theta$ gives $\sqrt{4-x^2}=2|\cos\theta|$. Choose $-\pi/2\le\theta\le\pi/2$ so $\cos\theta\ge 0$ before removing the absolute value. For the definite integral below, $x=0,1$ become $\theta=0,\pi/6$. The double-angle identity integrates $\cos^2\theta$.

$$\begin{aligned}\int_0^1 \sqrt{4-x^2}\,dx &= 4\int_0^{\pi/6}\cos^2\theta\,d\theta\\ &= [2\theta+\sin 2\theta]_0^{\pi/6}=\frac{\pi}{3}+\frac{\sqrt{3}}{2}.\end{aligned}$$

■ Reuse integration by parts and a half angle

For $I=\int e^x\sin x\,dx$, two integrations by parts return to I. Collect that repeated integral rather than starting the same calculation again. For a rational expression in sine and cosine, $t=\tan(x/2)$ gives $\sin x=2t/(1+t^2)$, $\cos x=(1-t^2)/(1+t^2)$ and $dx=2dt/(1+t^2)$. Work on a branch that does not cross an odd multiple of π.

$$I=e^x\sin x-(e^x\cos x+I)\quad\Longrightarrow\quad I=\frac{e^x}{2}(\sin x-\cos x)+C.$$

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$$\int_0^{\pi/2}\frac{dx}{2+\cos x}=2\int_0^1\frac{dt}{3+t^2}=\frac{\pi}{3\sqrt{3}}.$$

■ Derive a trigonometric reduction

A **reduction formula** 递推公式 links an integral to one with a smaller index. For $I_n=\int_0^{\pi/2}\sin^n x\,dx$, take $u=\sin^{n-1}x$, $dv=\sin x\,dx$ in integration by parts. For $n\ge 2$, the boundary term is zero. Replacing $\cos^2 x$ by $1-\sin^2 x$ introduces both I_{n-2} and I_n; collect the latter before dividing.

$$\begin{aligned}I_n &= [-\sin^{n-1}x\cos x]_0^{\pi/2}+(n-1)\int_0^{\pi/2}\sin^{n-2}x\cos^2 x\,dx\\ &= (n-1)(I_{n-2}-I_n),\\ I_n &= \frac{n-1}{n}I_{n-2}.\end{aligned}$$

The base cases are $I_0=\pi/2$ and $I_1=1$. Thus $I_4=(3/4)(1/2)I_0=3\pi/16$. Even and odd indices lead to different base cases.

■ Retain a non-zero boundary term

For $J_n=\int_0^1 x^n e^{2x}\,dx$, integrate e^{2x} and differentiate x^n. At $x=1$, the boundary term is $e^2/2$, not zero. The recurrence below needs $n\ge 1$. Evaluate J_0 directly: the lower endpoint contributes $-1/2$. Applying the recurrence at $n=0$ would lose it.

$$J_n=\left[\frac{x^n e^{2x}}{2}\right]_0^1-\frac{n}{2}J_{n-1}=\frac{e^2}{2}-\frac{n}{2}J_{n-1},\quad n\ge 1.$$
$$J_0=\frac{e^2-1}{2},\qquad J_1=\frac{e^2+1}{4},\qquad J_2=\frac{e^2-1}{4}.$$

■ Two integrations by parts can reduce by two

Set $K_n=\int_0^2(2-x)^n\sinh x\,dx$. The first integration by parts gives $-2^n+n\int_0^2(2-x)^{n-1}\cosh x\,dx$. For $n\ge 2$, a second integration by parts has zero boundary term and gives $K_n=-2^n+n(n-1)K_{n-2}$. The base integrals must still be evaluated directly.

$$K_0=\cosh 2-1,\qquad K_1=\sinh 2-2,\qquad K_2=2\cosh 2-6.$$

A hint may suggest differentiating $\tan x\sec^{n-2}x$. Its derivative is $(n-1)\sec^n x-(n-2)\sec^{n-2}x$. Integrating from 0 to $\pi/4$ gives, for $n\ge 2$,

$$H_n=\frac{2^{(n-2)/2}+(n-2)H_{n-2}}{n-1},\qquad H_n=\int_0^{\pi/4}\sec^n x\,dx.$$

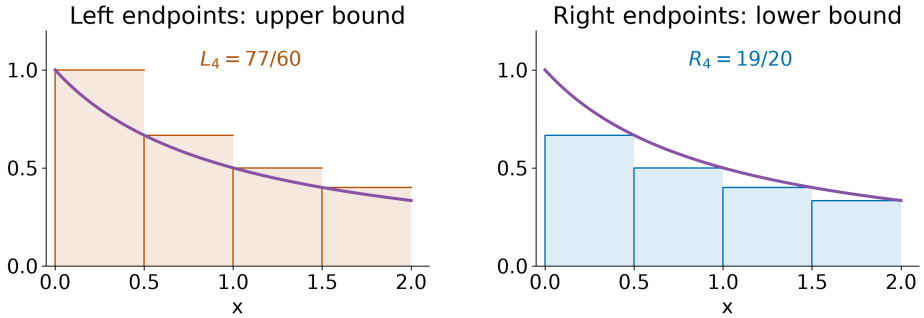
Here $H_0=\pi/4$, $H_2=1$ and $H_4=4/3$. The endpoint term is essential.

Draw rectangles before choosing an inequality

For a positive decreasing function, left-endpoint rectangles overestimate the integral and right-endpoint rectangles underestimate it. Each rectangle has area height times width; a height alone is not its area. For $f(x) = 1/(1+x)$ on $[0, 2]$ with four rectangles, $\Delta x = 1/2$. The sums below trap the exact integral $\ln 3$.

$$R_4 = \frac{1}{2}\left(\frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{3}\right) = \frac{19}{20},$$
$$L_4 = \frac{1}{2}\left(1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}\right) = \frac{77}{60}.$$

$$R_4 < \int_0^2 \frac{dx}{1+x} = \ln 3 < L_4.$$



y = 1/(1 + x), rectangle width = 1/2; integral = ln 3.

Keep unit-width and shrinking-width sums distinct

Unit-width rectangles under $1/x$ give, for integers $n \geq 2$, $\ln(n+1) < \sum_{r=1}^{n} 1/r < 1 + \ln n$. For the upper bound, keep the first term 1 separate and compare terms $r = 2, \ldots, n$ with strips from $r-1$ to r. With n equal strips on $[0, 1]$, the width is instead $1/n$. For a continuous function, a **Riemann sum** 黎曼和 tends to the integral; identify both the width and the sample point.

$$\frac{1}{n}\sum_{r=1}^{n} f(r/n) \longrightarrow \int_0^1 f(x)\,dx.$$

For an increasing function the left/right inequality directions reverse. Do not copy a decreasing-function diagram without checking the derivative or monotonicity on the actual interval.

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Rewrite a sum into height times width

Divide each denominator by n^2 to expose the sample point r/n. The sum below is a right-endpoint rectangle sum, not n times the limiting integral. Since $f(x) = 1/(4+x^2)$ decreases on $[0, 1]$, the finite sum is below the integral. The gap between the left and right sums is $(f(0) - f(1))/n = 1/(20n)$, which tends to zero.

$$\sum_{r=1}^{n} \frac{n}{4n^2 + r^2} = \frac{1}{n}\sum_{r=1}^{n} \frac{1}{4 + (r/n)^2} \longrightarrow \frac{1}{2}\tan^{-1}(1/2).$$

Find Cartesian arc length from ds

A small displacement has length $ds = \sqrt{dx^2 + dy^2}$. For a single traversal with increasing x, the **arc length** 弧长 is $L = \int_a^b \sqrt{1 + (dy/dx)^2}\,dx$; it is not $\int y\,dx$. For $y = \cosh x$, $0 \leq x \leq \ln 2$, $\sqrt{1 + \sinh^2 x} = \cosh x > 0$. Thus $L = [\sinh x]_0^{\ln 2} = 3/4$. For $y = x^2$, $0 \leq x \leq 1$, a hyperbolic substitution in $\int_0^1 \sqrt{1 + 4x^2}\,dx$ gives $L = \sqrt{5}/2 + \sinh^{-1}(2)/4$. The straight-line distance between the endpoints is $\sqrt{2}$, which is smaller.

Use speed for a parametric curve

For $x = x(t)$ and $y = y(t)$, $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$. Check that the chosen parameter interval traces the required arc once. For $x = 3(t - \sin t)$, $y = 3(1 - \cos t)$, $0 \leq t \leq \pi$, the speed is $3\sqrt{2 - 2\cos t} = 6\sin(t/2)$, since the half-angle sine is non-negative.

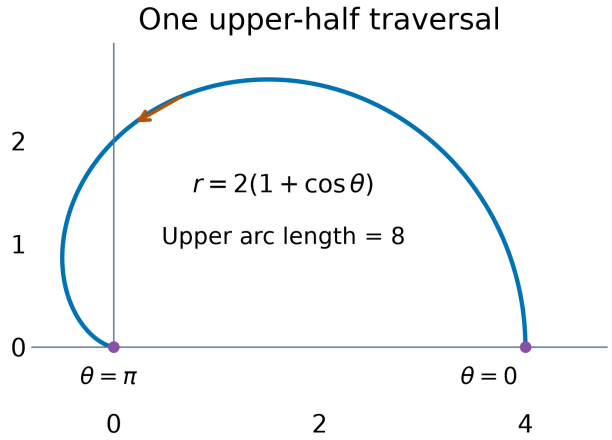
$$L = \int_0^{\pi} 6\sin(t/2)\,dt = 12.$$

For $x = \ln(\cosh 2t)$, $y = \tan^{-1}(\sinh 2t)$, $0 \leq t \leq \ln 2$, the derivatives are $2\tanh 2t$ and $2\operatorname{sech} 2t$. The identity $\tanh^2 + \operatorname{sech}^2 = 1$ makes the speed 2, so $L = 2\ln 2$.

Include both terms in polar arc length

For a polar curve $r = r(\theta)$, radial and sideways changes contribute separately: $ds^2 = dr^2 + r^2 d\theta^2$. Thus $L = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$, with angles in radians. For $r = 2(1 + \cos\theta)$ on $0 \leq \theta \leq \pi$, the square root is $4|\cos(\theta/2)| = 4\cos(\theta/2)$. This interval traces the upper half once; its length is 8.

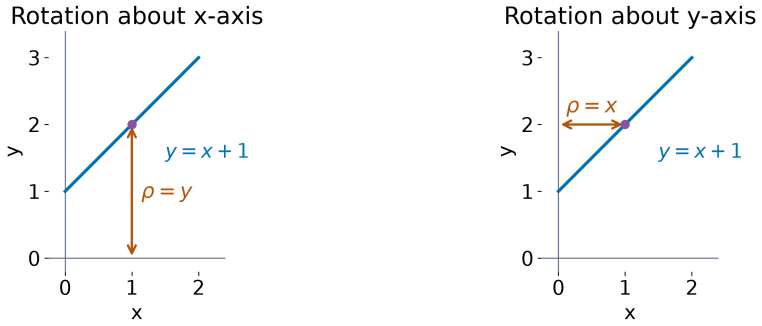
$$L = \int_0^{\pi} 4\cos(\theta/2)\,d\theta = 8.$$



Choose the radius before finding surface area

A narrow band of a **surface area of revolution** 旋转曲面面积 has area $2\pi\rho\,ds$, where $\rho \geq 0$ is distance to the rotation axis. About the x-axis use $|y|$; about the y-axis use $|x|$. For $y = x + 1$, $0 \leq x \leq 2$, $ds = \sqrt{2}\,dx$. Changing the axis changes the radius even though the arc is the same.

$$S_x = 2\pi\sqrt{2}\int_0^2 (x+1)\,dx = 8\pi\sqrt{2},$$
$$S_y = 2\pi\sqrt{2}\int_0^2 x\,dx = 4\pi\sqrt{2}.$$



$\pi\int y^2\,dx$ is a volume formula, not a surface-area formula. Negative coordinates do not give negative areas. The syllabus requires Cartesian and parametric surfaces, not surfaces specified directly in polar form.

Keep the parameter and speed in a surface integral

For a parametric arc, use $S_x = 2\pi\int |y(t)|\sqrt{x'(t)^2 + y'(t)^2}\,dt$ or replace $|y(t)|$ by $|x(t)|$ for the y-axis. The square-root speed handles decreasing coordinates without producing negative length. For $x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq \pi/2$, the speed is 2. Rotation about the x-axis gives a hemisphere of radius 2; the same is true about the y-axis.

$$S_x = 2\pi\int_0^{\pi/2} (2\sin t)(2)\,dt = 8\pi,$$
$$S_y = 2\pi\int_0^{\pi/2} (2\cos t)(2)\,dt = 8\pi.$$

Change the limits when the substitution reverses direction

A surface integral can need a further substitution after its geometric setup. For $y = \tanh x$, rotating $0 \leq x \leq \tanh^{-1}(1/3)$ about the x-axis gives $2\pi\int \tanh x\sqrt{1 + \operatorname{sech}^4 x}\,dx$. Put $u = \sqrt{1 + \operatorname{sech}^4 x}$. Then $du/dx = -2\operatorname{sech}^4 x \tanh x/u < 0$ for $x > 0$. Reversing the transformed limits removes the minus sign.

$$S = \pi\int_{\sqrt{145}/9}^{\sqrt{2}} \frac{u^2}{u^2 - 1}\,du = \pi\left[u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_{\sqrt{145}/9}^{\sqrt{2}}.$$

At the upper x endpoint, $\tanh x = 1/3$, so $\operatorname{sech}^2 x = 8/9$ and $u = \sqrt{145}/9$. Split $u^2/(u^2-1) = 1 + 1/(u^2-1)$ before integrating. The transformed integral is positive.

Set up, simplify, evaluate and check

For substitutions and reductions, show the transformed differential, endpoint terms and valid index range. For rectangles, draw the comparison and include the rectangle width before taking a limit. For arc length or surface area, write ds first, choose the parameter interval and a non-negative radius, then integrate. A plausible antiderivative cannot repair the wrong geometric setup.

Check an indefinite answer by differentiation. Check a reduction at a low allowed index. Check a sum against its diagram. Check a length against its chord and a surface against a simple geometric case where possible.

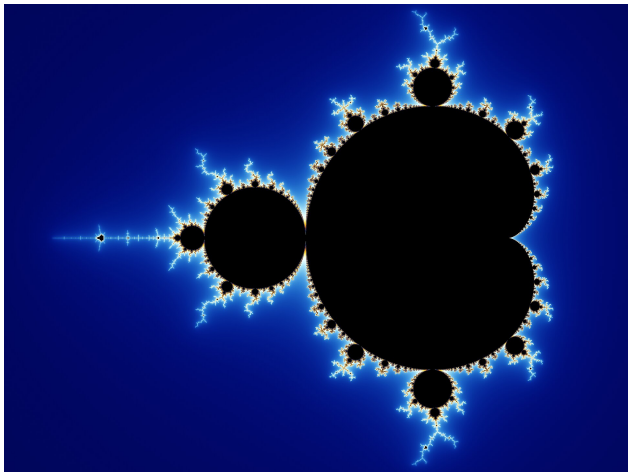
Complex numbers

Repeat a turn and a change of scale

Picture an arrow that turns and changes length after each step. Multiplication of non-zero **complex numbers** 复数 combines these two changes; a power repeats them. **De Moivre's theorem** 棣莫弗定理 links those powers to angles. It also helps turn

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a trigonometric identity, a polynomial equation or two real series into one complex calculation.



The Mandelbrot set: the complex numbers c for which repeating $z \to z^2 + c$, starting from $z = 0$, stays bounded

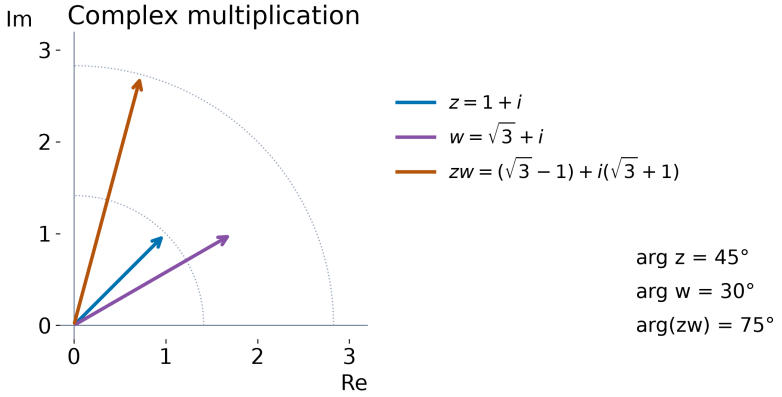
Multiplication by i preserves length. A general non-zero multiplier can change both length and angle.

Multiply lengths and add angles

Write a non-zero number in **polar form** 极坐标形式 as $z = r\operatorname{cis}\theta$, where $\operatorname{cis}\theta = \cos\theta + i\sin\theta$. Its **modulus** 模 is $r > 0$ and its **argument** 辐角 is defined modulo 2π. Multiplication multiplies moduli and adds arguments; division divides moduli and subtracts arguments. Adjust the final angle to the interval requested in the question.

$$(r\operatorname{cis}\alpha)(s\operatorname{cis}\beta) = rs\operatorname{cis}(\alpha+\beta),$$
$$\frac{r\operatorname{cis}\alpha}{s\operatorname{cis}\beta} = \frac{r}{s}\operatorname{cis}(\alpha-\beta), \quad s>0.$$

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Apply positive and negative integer powers

For an integer n, $z^n = r^n\operatorname{cis}(n\theta)$. A negative exponent needs $z \neq 0$; taking a reciprocal reverses the angle and replaces the modulus by its reciprocal. For $z = 1 + i = \sqrt{2}\operatorname{cis}(\pi/4)$, $z^6 = 8\operatorname{cis}(3\pi/2) = -8i$. Also $z^{-2} = (1/2)\operatorname{cis}(-\pi/2) = -i/2$.

The power multiplies the argument; it does not raise the real and imaginary parts separately. Zero has no argument: its positive integer powers can be handled directly, but negative powers of zero are undefined.

Prove the positive-integer theorem by induction

Use **mathematical induction** 数学归纳法. The claim is true at $n = 1$. Assume $(\operatorname{cis}\theta)^k = \operatorname{cis}(k\theta)$ for an arbitrary positive integer k. Multiply this assumed equality by $\operatorname{cis}\theta$. The angle-addition identities give the next case below, so the base case and conditional step prove every positive integer case.

$$\begin{aligned}(\operatorname{cis}\theta)^{k+1} &= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos k\theta\cos\theta - \sin k\theta\sin\theta)\\ &\quad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\\ &= \operatorname{cis}((k+1)\theta).\end{aligned}$$

For $m > 0$, $(\operatorname{cis}\theta)^{-m} = 1/\operatorname{cis}(m\theta) = \operatorname{cis}(-m\theta)$. This extends the proved result to negative integers without claiming that the induction itself covered them.

Compare real and imaginary parts

Expand $(c + is)^n$ with $c = \cos\theta$, $s = \sin\theta$, then compare real parts with $\cos n\theta$ and **imaginary parts** 虚部 with $\sin n\theta$. For the fifth power, replace s^2 by $1 - c^2$ in the real part, and c^2 by $1 - s^2$ in the imaginary part. Show that substitution rather than quoting the final identity.

$$\begin{aligned}\cos 5\theta &= c^5 - 10c^3s^2 + 5cs^4\\ &= 16c^5 - 20c^3 + 5c,\\ \sin 5\theta &= 5c^4s - 10c^2s^3 + s^5\\ &= 16s^5 - 20s^3 + 5s.\end{aligned}$$

The real part of the cube similarly gives $\cos 3\theta = c^3 - 3cs^2 = 4c^3 - 3c$.

■ Divide parts to obtain a tangent formula

If $t = \tan\theta$, divide the imaginary part of $(\cos\theta + i\sin\theta)^3$ by its real part, then divide numerator and denominator by $\cos^3\theta$. Both tangents must exist. Thus $\cos\theta \neq 0$ and the denominator $1 - 3t^2 \neq 0$; an equation obtained by cross-multiplication still needs those checks.

$$\tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\cos\theta\sin^2\theta} = \frac{3t - t^3}{1 - 3t^2}.$$

■ Express a power using multiple angles

Put $z = \operatorname{cis}\theta$, so $z + z^{-1} = 2\cos\theta$ and $z - z^{-1} = 2i\sin\theta$. Expand powers and pair z^m with z^{-m}. The fourth-power expansion below turns $\sin^4\theta$ into a constant plus multiple-angle cosines. Keep the sign of i^n when the power is odd.

$$\begin{aligned}(z - z^{-1})^4 &= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4},\\ 16\sin^4\theta &= 2\cos 4\theta - 8\cos 2\theta + 6,\\ \sin^4\theta &= \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}.\end{aligned}$$

The fifth power gives $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$. Only after deriving this may you take a reciprocal: for $\sin\theta \neq 0$,

$$\operatorname{cosec}^5\theta = \frac{16}{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}.$$

■ Turn a polynomial into an angle equation

For $32x^5 - 40x^3 + 10x - 1 = 0$, put $x = \cos\theta$. The fifth-angle identity gives $2\cos 5\theta - 1 = 0$. Choose $0 \leq \theta \leq \pi$, where cosine is one-to-one. Solve $5\theta = \pm\pi/3 + 2k\pi$ in $[0, 5\pi]$, giving the five distinct roots below. Five distinct verified roots exhaust a degree-five polynomial.

$$x = \cos\left(\frac{q\pi}{15}\right), \qquad q \in \{1, 5, 7, 11, 13\}.$$

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Listing five angles does not automatically give five distinct roots when using sine on a larger interval. Check the actual sine/cosine values and the polynomial degree; remove repetitions, not valid different roots.

■ Handle reciprocal trigonometric roots

To express roots using secant, put $x = \sec\theta$ and therefore $1/x = \cos\theta$. Divide by a power of x only after excluding $x = 0$ from the original polynomial. For $x^5 - 10x^4 + 40x^2 - 32 = 0$, zero is not a root. Dividing by x^5 gives $1 - 2\cos 5\theta = 0$, so the roots are the secants of the same five angles as in the previous step.

$$1 - \frac{10}{x} + \frac{40}{x^3} - \frac{32}{x^5} = 0,$$
$$x = \sec\left(\frac{q\pi}{15}\right), \quad q \in \{1, 5, 7, 11, 13\}.$$

■ Rational powers need a branch convention

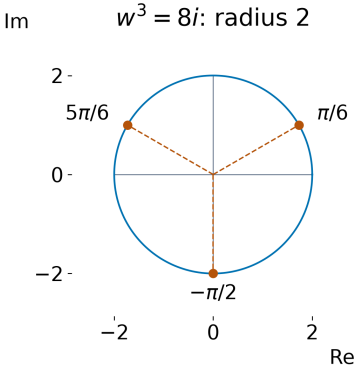
For $z = r\operatorname{cis}\theta \neq 0$ and coprime integers p, q with $q > 0$, the complex branches of the rational power use all equivalent arguments before dividing the angle by q. A specified principal value chooses a branch; solving an equation for all roots must retain all q values. Negative p also needs $z \neq 0$. Do not use an unreduced fraction to decide the branch count.

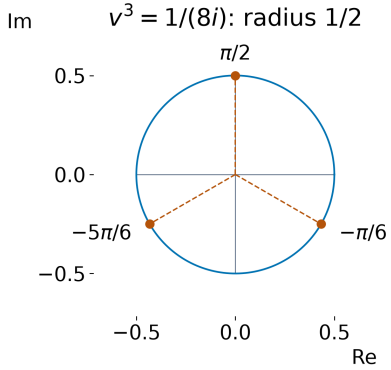
$$w_k = r^{p/q}\operatorname{cis}\left(\frac{p(\theta + 2k\pi)}{q}\right), \qquad k = 0, \ldots, q - 1.$$

For $z = 8$ and exponent $2/3$, the three values solve $w^3 = 64$: they are $4\operatorname{cis}(0)$, $4\operatorname{cis}(2\pi/3)$ and $4\operatorname{cis}(4\pi/3)$. The familiar real value 4 is one of them.

■ Find roots of a non-unit number

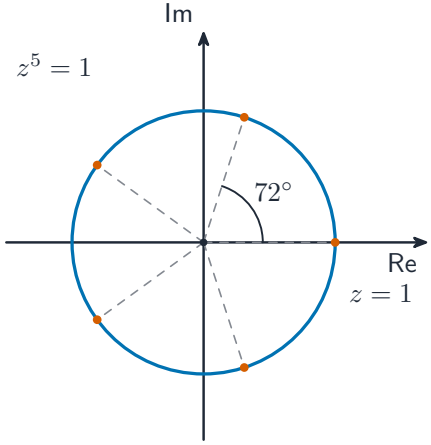
To solve $w^n = R\operatorname{cis}\Theta$ for $R > 0$, use modulus $R^{1/n}$ and arguments $(\Theta + 2k\pi)/n$, $k = 0, \ldots, n - 1$. Convert each argument to the requested interval after obtaining the complete set. For $w^3 = 8i$, the roots have modulus 2 and arguments $\pi/6, 5\pi/6, -\pi/2$. Reciprocating them gives the three values of $(8i)^{-1/3}$, with modulus $1/2$ and opposite arguments.





Use the roots of unity

The n **roots of unity** 单位根 solve $z^n = 1$: $z_k = \operatorname{cis}(2k\pi/n)$ for $k = 0, \dots, n-1$. They lie on the **unit circle** 单位圆 and are separated by $2\pi/n$ radians, or $360°/n$. For $n > 1$, their sum is zero. They give the linear factors $z - z_k$ of $z^n - 1$. If $\omega^3 = 1$ and $\omega \neq 1$, then $1 + \omega + \omega^2 = 0$; do not cancel $\omega - 1$ without its non-zero condition.

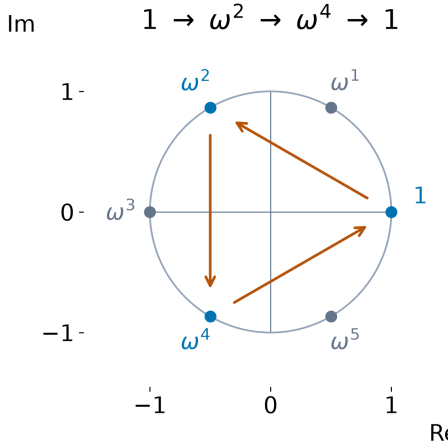


Check whether powers visit every root

A **primitive root of unity** 本原单位根 visits all n roots before its powers return to 1. For $\omega = \operatorname{cis}(2\pi/n)$ this is the full cycle; a power of ω can have a shorter cycle. For $n = 6$, powers of ω^2 visit only $1, \omega^2, \omega^4$ and then repeat. For any integer m, the complete sum of mth powers below is 6 if 6 divides m, and zero otherwise.

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$$\sum_{k=0}^{5} \omega^{mk} = \begin{cases} 6, & 6 \mid m, \\ 0, & 6 \nmid m. \end{cases}$$



$\omega = e^{i\pi/3}$ cycles through six values; ω^2 visits only three.

Add C and iS before summing separately

Let $C = \sum_{r=0}^{n} \cos(r\theta)$ and $S = \sum_{r=0}^{n} \sin(r\theta)$. Then $C + iS$ is a **geometric series** 等比级数 with ratio $z = \operatorname{cis}\theta$. For $z \neq 1$, sum the series first, then factor the complex exponentials into half-angle forms. When θ is a multiple of 2π, use the original sums: $C = n + 1$, $S = 0$.

$$C + iS = \frac{1 - z^{n+1}}{1 - z} = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \operatorname{cis}(n\theta/2).$$

$$C = \frac{\sin((n+1)\theta/2)\cos(n\theta/2)}{\sin(\theta/2)},$$

$$S = \frac{\sin((n+1)\theta/2)\sin(n\theta/2)}{\sin(\theta/2)}.$$

Include a geometric weight

For $C_n = \sum_{r=1}^{n} \rho^r \cos(r\theta)$ and $S_n = \sum_{r=1}^{n} \rho^r \sin(r\theta)$, take $q = \rho \operatorname{cis}\theta$. The finite sum is $q(1 - q^n)/(1 - q)$ for $q \neq 1$. Multiply by $1 - \overline{q}$ to make the denominator real, then compare parts. With $D = 1 - 2\rho\cos\theta + \rho^2$, the imaginary part is the expression below; obtain the real part from the same numerator.

$$C_n + iS_n = \frac{q - q^{n+1} - \rho^2 + \rho^2 q^n}{D}.$$

$$S_n = \frac{\rho \sin\theta - \rho^{n+1}\sin((n+1)\theta) + \rho^{n+2}\sin(n\theta)}{D}.$$

For $q = (1+i)/2$ and $n = 4$, $q/(1-q) = i$ and $q^4 = -1/4$. Hence $\sum_{r=1}^{4} q^r = 5i/4$: the cosine-weighted sum is 0 and the sine-weighted sum is $5/4$.

■ Use a binomial expansion for binomial weights

For $C = \sum_{r=0}^{n} \binom{n}{r}\cos(r\theta)$ and $S = \sum_{r=0}^{n} \binom{n}{r}\sin(r\theta)$, the **binomial expansion** 二项式展开 gives $C + iS = (1 + e^{i\theta})^n$. On $-\pi < \theta < \pi$, $1 + e^{i\theta} = 2\cos(\theta/2)e^{i\theta/2}$ has non-negative modulus. Apply de Moivre, then compare real and imaginary parts. Starting the sum at $r = 1$ removes 1 from C only.

$$C = 2^n \cos^n(\theta/2)\cos(n\theta/2),$$
$$S = 2^n \cos^n(\theta/2)\sin(n\theta/2).$$

For $n = 6$ and $\theta = \pi/3$, $1 + e^{i\theta} = \sqrt{3}e^{i\pi/6}$, so $C + iS = 27e^{i\pi} = -27$. Thus $C = -27$, $S = 0$ for $r = 0, \ldots, 6$.

■ Take a justified infinite-series limit

If $|q| = \rho < 1$, $q^n \to 0$ and the complex geometric sum from $r = 1$ tends to $q/(1-q)$. Take real and imaginary parts after using this convergence condition. With the real denominator $D = 1 - 2\rho\cos\theta + \rho^2$, the formulas below follow. For $q = (1+i)/2$, the infinite sum is i, so $C = 0$, $S = 1$.

$$C_\infty = \frac{\rho\cos\theta - \rho^2}{D}, \qquad S_\infty = \frac{\rho\sin\theta}{D}.$$

A finite geometric formula does not justify an infinite sum when $|q| \geq 1$. Check convergence before discarding its last term. The special case $q = 1$ also makes the finite quotient undefined and must be evaluated directly.

■ Bring down an index by differentiation

Differentiating the finite sine sum with respect to θ gives $\sum_{r=1}^{n} r\rho^r\cos(r\theta)$. Alternatively apply $q\,d/dq$ to the finite geometric polynomial. The result below has a justified limit for $|q| < 1$, using $n\rho^n \to 0$. This uses the finite identity before taking the limit, rather than assuming that an infinite series can always be differentiated term by term.

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$$\sum_{r=1}^{n} rq^r = \frac{q\{1 - (n+1)q^n + nq^{n+1}\}}{(1-q)^2},$$
$$\sum_{r=1}^{\infty} rq^r = \frac{q}{(1-q)^2}, \qquad |q| < 1.$$

At $q = i/2$, the limiting weighted sum is $-8/25 + 6i/25$. Therefore the weighted cosine sum is $-8/25$ and the weighted sine sum is $6/25$.

■ Choose the complex expression that matches the task

For a power, state the modulus, argument and branch conditions. For a trigonometric identity, expand before selecting parts. For polynomial roots, choose an angle interval, exclude invalid divisions and count distinct values. For sums, choose geometric or binomial weights correctly, combine $C + iS$, and retain exceptional cases and convergence conditions. For roots of unity, check whether powers give a full cycle or a shorter one.

Multiplication combines scale and rotation. De Moivre links powers and angles. Comparing parts produces identities and sums; retaining all allowed arguments produces the complete root set.

Differential equations

■ Balance what enters and leaves

- A tank has 50 litres of well-mixed liquid. Each minute, 10 litres enters carrying 2 grams of salt, and 10 litres leaves. If M is salt mass in grams and t is time in minutes, the outflow removes $10(M/50) = M/5$ grams per minute.
- The **differential equation** 微分方程 is $M' = 2 - M/5$. It links a **function** 函数 to its rate of change. We will find the mass, use the starting mass and check what happens after a long time.



A cooling cup of coffee: Newton's law of cooling, $dT/dt = -k(T - T_0)$, is a first-order differential equation

Put a first-order equation in linear form

- A **first-order linear equation** 一阶线性方程 has the form $y' + P(x)y = Q(x)$. First divide by the coefficient of y', on an interval where it is non-zero. Here a prime means differentiation with respect to the independent variable.
- For $2xy' - 4y = 2x^3$, division gives $y' - 2y/x = x^2$, so $P = -2/x$, not -4. Use an interval such as $x > 0$; do not extend the result across $x = 0$ without checking the original equation.

Build the integrating factor from the product rule

- We want $\mu y' + \mu P y = (\mu y)'$. The product rule requires $\mu' = P\mu$, giving an **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$. Any non-zero constant multiple works; choose the simplest one.
- Multiply every term by μ, integrate, then divide by μ. Keep one arbitrary constant in the **general solution** 通解:

$$(\mu y)' = \mu Q,$$
$$y = \frac{\int \mu Q\,dx + C}{\mu}.$$

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Finish an algebraic integrating-factor example

- For $y' - 2y/x = x^2$ on $x > 0$, choose $\mu = e^{-2\ln x} = x^{-2}$. Multiplying gives $(x^{-2}y)' = 1$, not x^2.
- Integrate to get $x^{-2}y = x + C$, so $y = x^3 + Cx^2$. If $y(1) = 2$, then $C = 1$ and $y = x^3 + x^2$. Substitution gives $2xy' - 4y = 2x^3$.

Let the integrating factor reveal a standard integral

- Consider $y' - \dfrac{x+2}{(x+2)^2+9}y = 1$, with $y(1) = 0$. The numerator is half the derivative of the denominator. Thus $\mu = ((x+2)^2+9)^{-1/2}$ and

$$\left(\frac{y}{\sqrt{(x+2)^2+9}}\right)' = \frac{1}{\sqrt{(x+2)^2+9}}.$$

- Use the inverse-hyperbolic integral from the previous topic. The **initial condition** 初始条件 fixes the constant:

$$y = \sqrt{(x+2)^2+9}\left[\sinh^{-1}\left(\frac{x+2}{3}\right) - \sinh^{-1}1\right].$$

Here $\sinh^{-1}$ *means inverse hyperbolic sine, not a reciprocal.*

Use hyperbolic factors on a stated interval

- For $y' + y\coth x = \cosh x$ on $x > 0$, $\int \coth x\,dx = \ln(\sinh x)$. Choose $\mu = \sinh x$; then $(y\sinh x)' = \sinh x\cosh x$.
- Hence $y\sinh x = \frac{1}{2}\sinh^2 x + C$ and $y = \frac{1}{2}\sinh x + C\operatorname{csch} x$. If $y(\ln 2) = 1$, use $\sinh(\ln 2) = 3/4$ to find $C = 15/32$. The equation itself is undefined at $x = 0$.

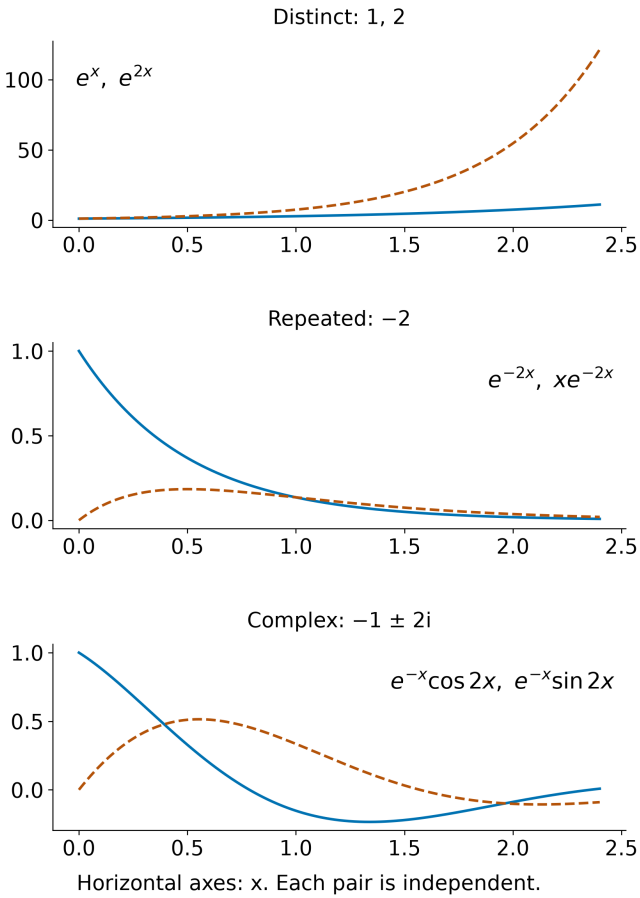
Separate the complementary and particular parts

- For a linear equation $L[y] = f(x)$, the **complementary function** 余函数 (CF) solves $L[y] = 0$. A **particular integral** 特积分 (PI) is any one solution of $L[y] = f(x)$. Linearity gives $y = \text{CF} + \text{PI}$.
- This also works at first order: for $y' + 2y = e^x$, CF $= Ce^{-2x}$ and PI $= e^x/3$. A **particular solution** 特解 is the complete solution after its arbitrary constants have been fixed; it is not another name for the PI.

Find all three complementary-function forms

- For $ay'' + by' + cy = 0$ with constant coefficients and $a \neq 0$, substituting $y = e^{mx}$ gives the **auxiliary equation** 辅助方程 $am^2 + bm + c = 0$. Use two independent terms, including the extra x for a repeated root.
- These examples cover distinct, repeated and conjugate-complex roots:

equation	roots	CF
$y'' - 3y' + 2y = 0$	$1, 2$	$Ae^{x} + Be^{2x}$
$y'' + 4y' + 4y = 0$	$-2, -2$	$(A + Bx)e^{-2x}$
$y'' + 2y' + 5y = 0$	$-1 \pm 2i$	$e^{-x}(A\cos 2x + B\sin 2x)$



Match every power in a polynomial particular integral

- For $y'' + y' - 2y = 2x^2 + 3x + 1$, try $y_p = ax^2 + bx + c$. Include all lower powers even if the right side omits one. Substitute $y_p' = 2ax + b$ and $y_p'' = 2a$.
- Match coefficients, starting with the highest power:

$$-2a = 2, \quad 2a - 2b = 3,$$
$$2a + b - 2c = 1.$$

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Thus $a = -1$, $b = -5/2$, $c = -11/4$ and $y_p = -x^2 - 5x/2 - 11/4$.

Try an exponential and substitute its derivatives

- For $y'' + 2y' + 5y = 4e^{-x}$, try $y_p = ke^{-x}$. Its derivatives are $-ke^{-x}$ and ke^{-x}, so $(1 - 2 + 5)ke^{-x} = 4e^{-x}$ and $k = 1$.
- Combine it with the complex-root CF. If $y(0) = y'(0) = 0$, the full solution is $y = e^{-x}(1 - \cos 2x)$. The derivative condition must include the PI derivative as well as the CF derivative.

Keep both sine and cosine in the trial

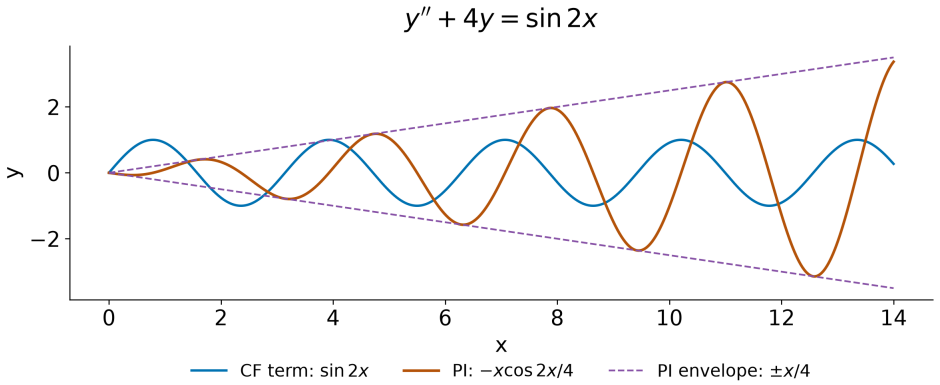
- For $y'' + 2y' + 2y = 2\sin 2x + 6\cos 2x$, try $y_p = a\sin 2x + b\cos 2x$. Differentiation mixes the two functions, even if the forcing has only one of them.
- Substitution and coefficient matching give

$$-2a - 4b = 2, \quad 4a - 2b = 6,$$
$$a = 1, \qquad b = -1.$$

So $y_p = \sin 2x - \cos 2x$ and the full solution adds $e^{-x}(A\cos x + B\sin x)$.

Repair a trial that is already in the CF

- If a trial is in the CF, applying L gives zero, so no coefficient can match a non-zero right side. Multiply the trial by x until it is independent of the CF. This is the **resonance** 共振 adjustment in the trial method.
- For $y'' + 4y = \sin 2x$, $\sin 2x$ is in the CF. Use $y_p = kx\cos 2x$; substitution gives $y_p'' + 4y_p = -4k\sin 2x$, so $k = -1/4$. Thus PI $= -x\cos 2x/4$.



Count repeated overlaps before choosing the power of x

- For $y''-4y'+4y=e^{2x}$, the auxiliary root 2 is repeated. Both e^{2x} and xe^{2x} already occur in the CF, so the trial is $y_p=kx^2e^{2x}$.
- Differentiating and substituting gives $y_p''-4y_p'+4y_p=2ke^{2x}$. Hence $k=1/2$ and

$$y=e^{2x}\left(A+Bx+\frac{x^2}{2}\right).$$

Use a supplied form even when it is unfamiliar

- A question may supply a PI form outside the standard polynomial/exponential/sine-cosine list. Differentiate that form, substitute into the equation and solve for its coefficients; do not guess an extra family.
- Given $y_p=k/x$ for $y''+y=2/x^3+1/x$ on $x>0$, $y_p''=2k/x^3$. Substitution gives $2k/x^3+k/x=2/x^3+1/x$, so $k=1$. The full solution is $A\cos x+B\sin x+1/x$.

Apply conditions to the complete solution

- Return to $y''+y'-2y=2x^2+3x+1$. Its complete solution is $Ae^x+Be^{-2x}-x^2-5x/2-11/4$. Differentiate before applying $y(0)=y'(0)=0$.
- The conditions give $A+B=11/4$ and $A-2B=5/2$, so $A=8/3$, $B=1/12$. Two independent conditions usually determine two constants; some **boundary conditions** 边界条件 can instead be inconsistent or leave a constant free.

Interpret the transient and the lasting response

- For $y''+2y'+2y=2\sin 2x+6\cos 2x$, the **transient** 暂态 term is $e^{-x}(A\cos x+B\sin x)$. It tends to zero as $x\to\infty$, whatever the finite starting constants.
- The lasting response is $y\approx\sin 2x-\cos 2x=\sqrt{2}\sin(2x-\pi/4)$. The **amplitude** 振幅 is $\sqrt{2}$, the angular frequency is 2 and the **phase lag** 相位滞后 relative to $\sin 2x$ is $\pi/4$. A positive auxiliary-root real part would prevent this decay argument.

Change independent variable using the chain rule

- If $x=e^t$, write $Y(t)=y(e^t)$ and $t=\ln x$, so $x>0$. The chain rule gives $Y_t=xy_x$; differentiating once more gives $Y_{tt}=xy_x+x^2y_{xx}$.
- Therefore the correct replacement is $x^2y_{xx}=Y_{tt}-Y_t$, not just Y_{tt}:

$$ax^2y_{xx}+bxy_x+cy=aY_{tt}+(b-a)Y_t+cY.$$

Solve the transformed equation and change back

- Use $x=e^t$ in $x^2y''-3xy'+4y=x^2$. The transformed equation is $Y_{tt}-4Y_t+4Y=e^{2t}$, with the repeated-resonance solution just found.
- Change back to the original variable before applying the stated conditions. Since $y(1)=0$ and $y'(1)=1$, $A=0$, $B=1$:

$$y=x^2\left(\ln x+\frac{(\ln x)^2}{2}\right),\qquad x>0.$$

Use y equals ux to separate variables

- Given $xy'=y+\sqrt{4x^2+y^2}$ and $y(1)=0$, work on $x>0$ and set $y=ux$. The product rule gives $y'=u+xu'$. Also $\sqrt{x^2(4+u^2)}=x\sqrt{4+u^2}$ on this interval.
- After cancelling xu, use **separable variables** 可分离变量 and integrate:

$$xu'=\sqrt{4+u^2},$$
$$\sinh^{-1}(u/2)=\ln x+C.$$

The condition gives $C=0$, so $u=2\sinh(\ln x)=x-1/x$ and $y=x^2-1$. Check the positive square-root sign in the original equation.

A sum substitution can expose a lost solution

- Given $y'=(1+2x+2y)/(2x+2y-1)$, put $z=x+y$. Then $z'=1+y'=4z/(2z-1)$. For $z\neq 0$, integrate $(1/2-1/(4z))\,dz=dx$.
- With $y(1)=0$, the local branch has $z>1/2$ and the result is

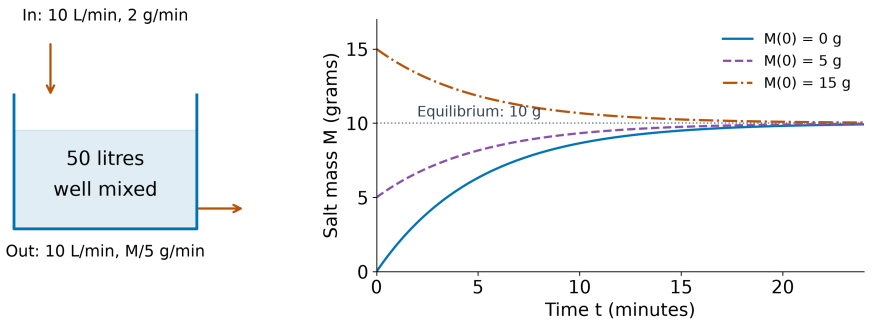
$$2(x+y)-\ln(x+y)-4x+2=0.$$

Dividing by z loses the separate solution $z=0$, or $y=-x$, which does not meet this initial condition. The original equation is undefined at $z=1/2$.

Answer the model question in its own quantities

- For the tank equation $M'+M/5=2$, an integrating factor is $e^{t/5}$, giving $M=10+Ce^{-t/5}$. Initially fresh liquid means $M(0)=0$, so $M=10(1-e^{-t/5})$ grams.
- Half the limiting mass is reached at $t=5\ln 2$ minutes. As $t\to\infty$, $M\to 10$ grams and inflow/outflow salt rates both approach 2 grams per minute. The model assumes constant volume, fixed input and immediate uniform mixing.

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■ Choose a method and verify the original equation

- Normalise a first-order linear equation before choosing its integrating factor. For constant coefficients, find CF, choose and check PI, add them, then apply all conditions. For a given substitution, transform derivatives as well as variables.
- Differentiate the final answer and substitute it into the original equation, check the conditions, retain the allowed interval and interpret any units, limits or model assumptions. A correct transformed equation is not yet a solution in the original variable.

Common mistakes.

- Giving only the positive solution of $\cosh x = k$: for $k > 1$ there are two, $\pm\ln\left(k+\sqrt{k^2-1}\right)$.
- Using Osborn's rule carelessly: change the sign of every product of two sines, including those hidden in $\tanh^2 x$.
- Putting eigenvectors into $\mathbf{P}$ in a different order from the eigenvalues in $\mathbf{D}$.
- Finding $\dfrac{d^2y}{dx^2}$ for parametric curves as $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ without dividing by $\dfrac{dx}{dt}$.
- Losing the general root in de Moivre questions: an nth root problem has n roots, spaced $\dfrac{2\pi}{n}$ apart.
- Forgetting to multiply the right-hand side by the integrating factor, or dropping the constant before dividing through.
- Choosing a particular integral that is already in the complementary function; then multiply the trial by x.
- Applying initial conditions to the complementary function alone instead of the general solution.

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Practice

Answer each before reading the answers below.

1. Solve $\cosh x = 2$, giving exact answers in logarithmic form.
2. Starting from the exponential definitions, prove that $\cosh^2 x - \sinh^2 x = 1$.
3. Find the eigenvalues and corresponding eigenvectors of $\mathbf{M} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$. Hence write $\mathbf{M} = \mathbf{PDP}^{-1}$ with $\mathbf{D}$ diagonal.
4. A curve is given by $x = t^2$, $y = t^3$ for $t > 0$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.
5. Find the Maclaurin series of $e^x \cos x$ up to and including the term in x^3.
6. Find (a) $\displaystyle\int \frac{1}{\sqrt{x^2+9}}\,dx$; (b) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$.
7. Let $I_n = \displaystyle\int_0^1 x^n e^x\,dx$. Show that $I_n = e - nI_{n-1}$ for $n \geq 1$, and hence find I_2 exactly.
8. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$.
9. Find the three cube roots of $8i$ in the form $a + bi$.
10. Solve $\dfrac{dy}{dx} + \dfrac{2y}{x} = x$ for $x > 0$.
11. Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4$, given that $y = 0$ and $\dfrac{dy}{dx} = 1$ when $x = 0$.

Answers

1. $\cosh x = 2$ gives $e^{2x} - 4e^x + 1 = 0$, so $e^x = 2 \pm \sqrt{3}$ and $x = \pm\ln(2+\sqrt{3})$ (since $\ln(2-\sqrt{3}) = -\ln(2+\sqrt{3})$).
2. $\cosh^2 x - \sinh^2 x = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \frac{4}{4} = 1$.
3. $\lambda^2 - 7\lambda + 10 = 0$, so $\lambda = 2$ or 5. For $\lambda = 5$: $-x + y = 0$, eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = 2$: $2x + y = 0$, eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$; the columns of $\mathbf{P}$ must be in the same order as the eigenvalues in $\mathbf{D}$.
4. $\dfrac{dy}{dx} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$. $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{3t}{2}\right) \div \dfrac{dx}{dt} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$.
5. $\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right) = 1 + x + \left(\frac{1}{2} - \frac{1}{2}\right)x^2 + \left(\frac{1}{6} - \frac{1}{2}\right)x^3 = 1 + x - \frac{1}{3}x^3$.
6. (a) $\sinh^{-1}\left(\dfrac{x}{3}\right) + c$, or $\ln\left(x + \sqrt{x^2+9}\right) + c$. (b) $\left[\tan^{-1} x\right]_0^1 = \dfrac{\pi}{4}$.
7. By parts with $u = x^n$, $dv = e^x\,dx$: $I_n = \left[x^n e^x\right]_0^1 - n\displaystyle\int_0^1 x^{n-1}e^x\,dx = e - nI_{n-1}$. $I_0 = e - 1$, $I_1 = e - (e-1) = 1$, $I_2 = e - 2(1) = e - 2$.

8. $\cos 3\theta = \mathrm{Re}(\cos\theta + i\sin\theta)^3 = c^3 - 3cs^2$ with $c = \cos\theta$, $s = \sin\theta$. Using $s^2 = 1 - c^2$: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
9. $8i = 8e^{i\pi/2}$. The roots are $2e^{i(\pi/6 + 2k\pi/3)}$ for $k = 0, 1, 2$: arguments $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{3\pi}{2}$, giving $\sqrt{3} + i$, $-\sqrt{3} + i$ and $-2i$.
10. Integrating factor $e^{\int 2/x\,dx} = x^2$. Then $\dfrac{d}{dx}(x^2 y) = x^3$, so $x^2 y = \frac{1}{4}x^4 + c$ and $y = \frac{1}{4}x^2 + \dfrac{c}{x^2}$.
11. Auxiliary equation $m^2 - 3m + 2 = 0$, so $m = 1, 2$: complementary function $Ae^x + Be^{2x}$. A constant particular integral $y = k$ gives $2k = 4$, so $k = 2$. $y = Ae^x + Be^{2x} + 2$. Then $A + B + 2 = 0$ and $A + 2B = 1$, so $B = 3$, $A = -5$: $y = -5e^x + 3e^{2x} + 2$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.

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Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics** 进阶力学. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10$ m s^{-2} unless stated otherwise.

Motion of a projectile

■ Predict where a thrown ball will go

A ball is thrown across a court. Its position, speed and direction change during the flight. We need a model that predicts all three, including a landing point below the thrower's hand. A **projectile** 抛射体 moves freely under **gravity** 重力. Model it as a **particle** 质点, ignore **air resistance** 空气阻力 and take gravity as constant. These assumptions become less accurate for a light shuttlecock, strong wind or a very long flight.



The water arcs suggest the model; drag and interactions between water drops limit its accuracy.

■ Set the origin and resolve the launch velocity

Put the origin at launch, with **horizontal** 水平 **displacement** 位移 x to the right and **vertical** 竖直 displacement y upwards. Let the **angle of projection** 抛射角 α be measured above the horizontal. The initial **velocity component** 速度分量 in each direction is $u_x = u\cos\alpha$, $u_y = u\sin\alpha$. For $u = 20$ $\dfrac{\text{m}}{\text{s}}$ and $\tan\alpha = 4/3$, a 3–4–5 triangle gives $u_x = 12$, $u_y = 16$, in metres per second.

Use the same time in both directions

There is no horizontal force. The **constant acceleration** 匀加速 is 0 horizontally and $-g$ vertically. With displacement measured from launch, the position and velocity equations are:

$$x = u_x t, \quad y = u_y t - \frac{1}{2} g t^2,$$
$$v_x = u_x, \quad v_y = u_y - g t.$$

Both equations use the same t. Unless a question states otherwise, take $g = 10\ \frac{\text{m}}{\text{s}^2}$ in the worked projectile examples.

Find position and speed at a stated time

For $u_x = 12$, $u_y = 16$, $g = 10$ and $t = 1.5$ s, the position is $x = 18$ m, $y = 24 - 11.25 = 12.75$ m. The velocity components are 12 and 1, in metres per second. Speed is the magnitude of velocity: $v = \sqrt{12^2 + 1^2} = \sqrt{145}\ \frac{\text{m}}{\text{s}}$. Its direction is $\tan^{-1}(1/12) \approx 4.76°$ above the horizontal. Position coordinates are not velocity components.

Translate a direction into signed components

Later the same ball travels at 45° to the **downward vertical**. It is descending, so $v_y < 0$ and $|v_y| = v_x = 12$. Thus $16 - 10t = -12$ and $t = 2.8$ s. Then $(x, y) = (33.6, 5.6)$ m and $v = 12\sqrt{2}\ \frac{\text{m}}{\text{s}}$. For a downward angle β to the horizontal use $v_y = -v_x \tan\beta$; for angle γ to the downward vertical use $v_y = -v_x \cot\gamma$.

A height may be reached twice

At height 10 m, $16t - 5t^2 = 10$ gives $t = (8 \pm \sqrt{14})/5$ s. Both times are positive: one on the way up, one on the way down. The corresponding x values are $(96 \pm 12\sqrt{14})/5$ m. The vertical velocities are $\pm 2\sqrt{14}\ \frac{\text{m}}{\text{s}}$, with the positive velocity at the earlier time. Both speeds are $\sqrt{144 + 56} = 10\sqrt{2}\ \frac{\text{m}}{\text{s}}$. Keep both roots until the event wording selects one.

Derive the greatest height

For an upward launch, the **greatest height** 最大高度 above launch occurs when $v_y = 0$. Therefore $t_{\text{top}} = u_y/g$ and $H = u_y^2/(2g) = u^2 \sin^2\alpha/(2g)$. For $u = 20\ \frac{\text{m}}{\text{s}}$, $\alpha = 30°$ and $g = 10$, $u_y = 10$, so $t_{\text{top}} = 1$ s and $H = 5$ m. The horizontal speed is still $10\sqrt{3}\ \frac{\text{m}}{\text{s}}$; the velocity is not zero.

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Derive the level-ground range

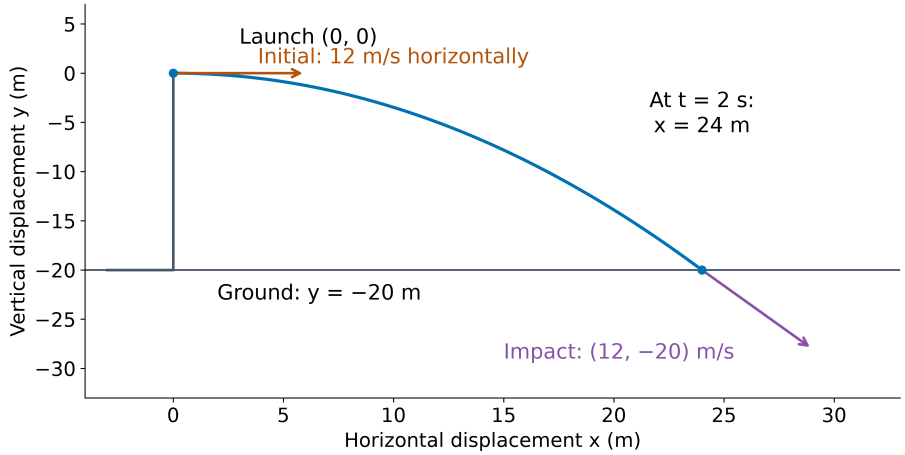
If landing is at the launch height, set $y = 0$: $t(u_y - gt/2) = 0$. The root $t = 0$ is launch; the positive **time of flight** 飞行时间 is $T = 2u_y/g$ for $u_y > 0$. The **range** 射程 on that horizontal plane is $R = u_x T = u^2 \sin 2\alpha/g$. For the $20\ \frac{\text{m}}{\text{s}}$, 30° example, $T = 2$ s and $R = 20\sqrt{3} \approx 34.6$ m. The formula does not apply unchanged when landing height differs.

State the conditions for a maximum range

For fixed u and g, no air resistance, level ground and an upward launch, $\sin 2\alpha \leq 1$ gives the largest range at $\alpha = 45°$. Complementary angles such as 30° and 60° give the same range, but different heights and flight times. A vertical throw has zero horizontal range. A horizontal or downward launch from a raised platform needs its actual landing height; the 45° conclusion cannot simply be carried over.

Land below the launch point

A ball leaves a 20 m platform horizontally at $12\ \frac{\text{m}}{\text{s}}$. With the origin at launch, the ground is $y = -20$, not $y = 0$. Thus $-5t^2 = -20$ gives the physical root $t = 2$ s and horizontal distance 24 m. At impact, $v_x = 12$, $v_y = -20$, so speed is $4\sqrt{34}\ \frac{\text{m}}{\text{s}}$ and direction is $\tan^{-1}(20/12) \approx 59.0°$ below horizontal. Reject $t = -2$ because it precedes launch.



Eliminate time to obtain the trajectory

The **trajectory** 轨迹 is the path. For $u_x \neq 0$, substitute $t = x/(u\cos\alpha)$ into the vertical equation. This gives its **Cartesian equation** 直角坐标方程:

$$\begin{aligned} y &= u\sin\alpha\frac{x}{u\cos\alpha} - \frac{g}{2}\left(\frac{x}{u\cos\alpha}\right)^2 \\ &= x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}. \end{aligned}$$

It is a **parabola** 抛物线 for a non-vertical launch. The equation describes the mathematical curve; keep only positions reached after launch and before the first impact. A vertical throw has $x = 0$ throughout, so this division is not allowed.

Use a known direction to find an unknown speed

Suppose $\tan\alpha = 3/2$ and the projectile passes through $(10, 10)$ m. Write $T = \tan\alpha$ and $1/\cos^2\alpha = 1 + T^2$. The trajectory equation gives

$$u^2 = \frac{gX^2(1+T^2)}{2(XT-Y)} = \frac{10(100)(13/4)}{2(15-10)} = 325.$$

Hence $u = 5\sqrt{13}\ \frac{\text{m}}{\text{s}}$, $u_x = 10$, $u_y = 15$ and $y = 3x/2 - x^2/20$. For a target to the right, $X > 0$ and a finite positive speed requires $XT > Y$: the initial straight direction must point above the target.

Find two launch angles for one target

Now fix $u = 20\sqrt{2}\ \frac{\text{m}}{\text{s}}$ and require the path to pass through the top $(40, 20)$ m of a wall. With $T = \tan\alpha$ and $g = 10$:

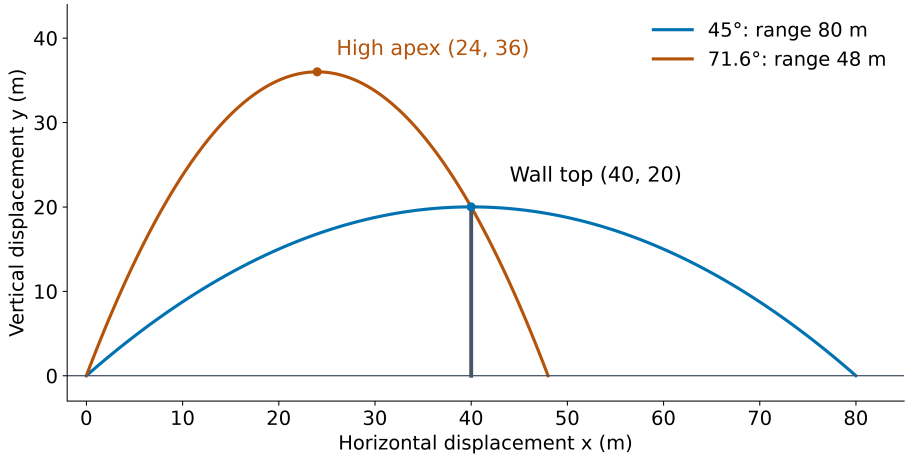
$$20 = 40T - 10(1+T^2) \quad \Rightarrow \quad (T-1)(T-3) = 0.$$

Both are positive launch angles: $\alpha = 45°$ or $\tan^{-1}3 \approx 71.6°$. The higher-angle path does not have the longer range. Substitute each root back into the target equation before accepting it.

Compare the two flights

The low-angle path is $y = x - x^2/80$: its vertex is $(40, 20)$ and range is 80 m. The high-angle path is $y = 3x - x^2/16$: its vertex is $(24, 36)$ and range is 48 m. Both pass through $(40, 20)$. The second is already descending at the wall. A larger angle raises the path but reduces the horizontal component; do not rank ranges by height alone.

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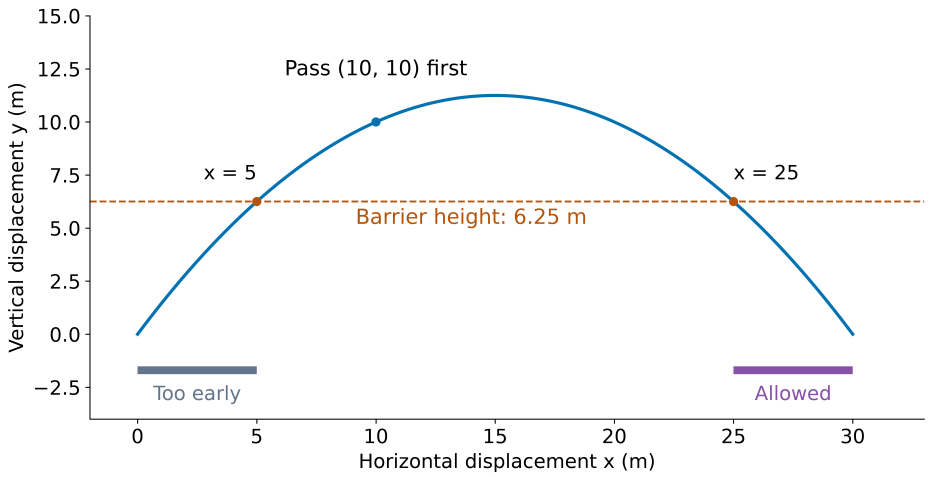


Turn a clearance condition into an interval

To reach or pass above that wall top, require $40T - 10(1+T^2) \geq 20$. Thus $(T-1)(T-3) \leq 0$ and $1 \leq T \leq 3$. The level-ground range is $R = 160T/(1+T^2)$. Its derivative is $160(1-T^2)/(1+T^2)^2$, so it decreases on $1 \leq T \leq 3$. The smallest landing distance is 48 m at $T = 3$. If touching the wall top is forbidden, the endpoints are excluded and 48 is an unattained lower bound.

Apply the event order after solving a height inequality

Return to $y = 3x/2 - x^2/20$. After passing $(10, 10)$, the ball hits a vertical barrier of height $25/4$ m at $x = D$. It is above ground for $0 \leq D \leq 30$. Hitting the barrier means $y(D) \leq 25/4$, giving $(D-5)(D-25) \geq 0$. On the ground interval this gives $D \leq 5$ or $D \geq 25$. But the barrier is reached **after** $x = 10$, so only $25 \leq D \leq 30$ remains, with boundary contact allowed. "After", "descending" and "before landing" are mathematical restrictions.



Recover both launch conditions from two points

Suppose an unknown launch passes through $(10, 15)$ and $(20, 20)$ in metres. Write $y = Ax - Bx^2$, with $A = \tan\alpha$ and $B = g/(2u_x^2) > 0$. The two points give $10A - 100B = 15$ and $20A - 400B = 20$. Subtract twice the first equation from the second: $-200B = -10$, so $B = 1/20$ and $A = 2$. Therefore $u_x = 10$, $u_y = 20$, $u = 10\sqrt{5}\ \frac{\text{m}}{\text{s}}$ and $\alpha = \tan^{-1} 2$. The range is 40 m and greatest height 20 m.

Choose the method from the event

Given a time, use the component equations. Given a direction, first relate the **signed velocity** components. Given a height, solve for time and check both roots. Given a target or unknown launch conditions, eliminate time and substitute coordinates. Check units, the coordinate origin, $t \geq 0$, the direction of travel and the first collision. For an upward launch, only the vertical component vanishes at the top; for a purely vertical throw the whole velocity does vanish there. The same-height range formula always needs its same-height assumption.

Equilibrium of a rigid body

Why a handle is far from the hinge

Push near a door hinge and you need a large force. Push at the handle and the same force has a larger **turning effect** 转动效应. To predict balance we must know both the forces and where they act. Model the object as a **rigid body** 刚体: its shape and the distances between its parts stay fixed. We consider **coplanar forces** 共面力, all in one plane. A particle model would lose the information needed to predict turning.

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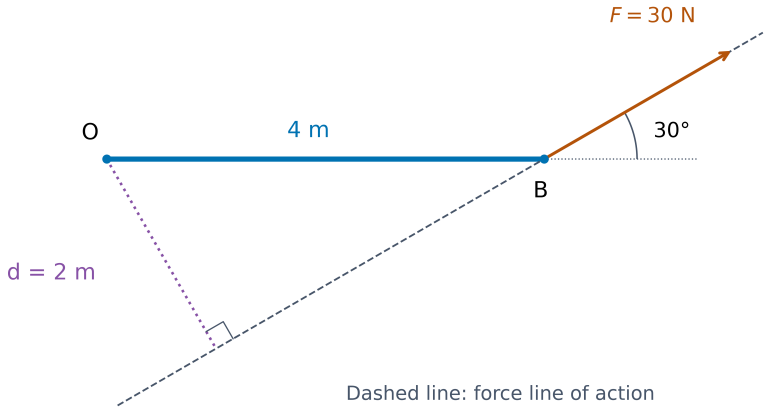
A tower crane: for a rigid body in equilibrium, both the resultant force and the total moment are zero

Measure a moment from the line of action

The **moment** 力矩 of a force about a point is force multiplied by the **perpendicular distance** 垂直距离 from that point to the force's **line of action** 作用线. The unit is N m. Choose a sign, such as anticlockwise positive. A 20 N upward force acting 0.6 m to the right of a **pivot** 支点 has moment +12 N m. If the line of action passes through the pivot, its moment is zero even when the force itself is not zero.

An angled force needs its perpendicular arm

A force of 30 N acts at 30° above a horizontal rod, 4 m from the pivot. Its **perpendicular** 垂直 component is $30\sin 30° = 15$ N, giving moment $15(4) = 60$ N m anticlockwise. Equivalently use the full force and arm $4\sin 30° = 2$ m. These are two ways to calculate the same moment; do not insert the sine factor twice.



Equilibrium needs two separate balances

For a body in **equilibrium** 平衡, the **resultant force** 合力 is zero and the total moment about any point is zero. For coplanar forces, resolve in two perpendicular directions and take moments about one chosen point. These conditions are also sufficient for equilibrium in this model. Equal opposite 10 N forces on different parallel lines 0.4 m apart have zero resultant but a moment of 4 N m. They do not balance rotationally. Zero resultant means no translational acceleration; it does not make an already moving body stop. For static equilibrium, the body starts at rest.

Replace gravity by one force at the centre of mass

In a uniform gravitational field, the weight $F_W = mg$ acts through the **centre of mass** 质心. A **uniform** 均匀 body has mass distributed evenly over its length, area or volume, as appropriate. Use **symmetry** 对称: a uniform rod has its centre of mass at its midpoint; a uniform rectangle has it where its diagonals meet. The centre of mass lies on every symmetry axis of the mass distribution. Shape symmetry alone is not enough if the **density** 密度 varies, and a centre of mass may lie in a hole outside the material.

Locate the centre of a triangular lamina

A **lamina** 薄片 is a thin plate. For a uniform triangular lamina, the centre of mass lies two-thirds of the way along each **median** 中线 from its vertex, or one-third from the opposite side. The MF19 formula is given; a proof is not required here. For vertices $(0,0)$, $(6,0)$ and $(0,3)$ in metres, the midpoint opposite the origin is $(3,1.5)$. Two-thirds of this displacement gives $(\bar{x},\bar{y}) = (2,1)$ m. Equivalently, average the three vertex coordinates for a uniform triangle.

Read the reference point in a solid-body formula

The formula list places the centre of mass of a uniform solid cone or pyramid $3h/4$ from its vertex, hence $h/4$ from its base. For a solid hemisphere of radius r, it is $3r/8$ from the base-circle centre along the axis towards the curved surface. A thin **hemispherical shell** 半球壳 instead has distance $r/2$. A shell is not a solid hemisphere. Convert every position to the same origin and signed axis before combining bodies.

Distinguish an arc from a sector

For a uniform **circular arc** 圆弧 of radius r and angle 2α, MF19 gives centre-of-mass distance $r\sin\alpha/\alpha$ from the circle centre on the symmetry axis. For a uniform **circular sector** 扇形 lamina, the distance is $2r\sin\alpha/(3\alpha)$. Here α is the half-angle in radians. A semicircle has $\alpha = \pi/2$: the wire arc gives $2r/\pi$, while the sector plate gives $4r/(3\pi)$. Use length for the mass of a uniform wire and area for a uniform plate of fixed thickness; these objects do not have the same mass model.

Replace each component by one particle

A **composite body** 复合物体 consists of simpler parts. Replace each part by a particle of mass m_i at its own centre of mass (x_i, y_i). Taking moments of the weights gives

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}, \qquad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}.$$

Mass is proportional to length for a uniform wire, area for a uniform lamina of fixed thickness, or volume for a uniform solid. Use an area or volume ratio only when the relevant densities are equal. A plain average of component positions works only for equal component masses.

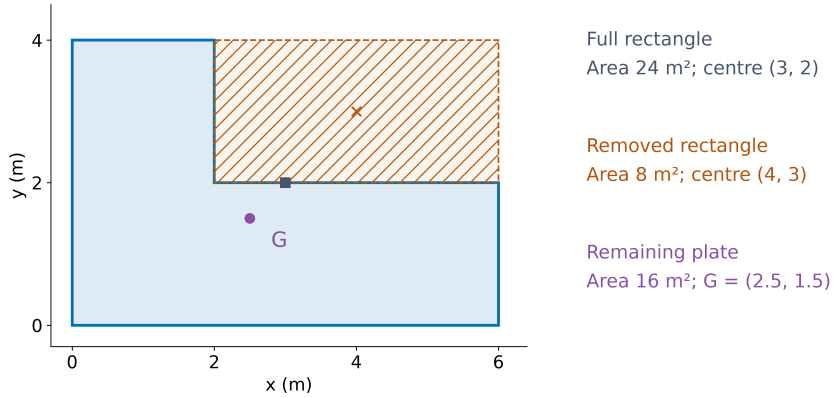
Add a rectangle and a triangle

Join a rectangle $0 \le x \le 4$, $0 \le y \le 2$ to a triangular plate with vertices $(4,0)$, $(4,2)$ and $(7,0)$, with lengths in metres. Both have the same thickness and density. Their areas are 8 and 3, and their centres are $(2,1)$ and $(5,2/3)$. Therefore $\bar{x} = (8\cdot 2 + 3\cdot 5)/11 = 31/11$ m and $\bar{y} = (8\cdot 1 + 3\cdot 2/3)/11 = 10/11$ m. The larger part contributes more to each moment.

Subtract a removed part from both mass and moments

Start with a 6 m by 4 m uniform rectangular plate. Remove the upper-right 4 m by 2 m rectangle, leaving an L-shape. Using the lower-left corner as origin, the original area is 24 at $(3,2)$ and the removed area is 8 at $(4,3)$. Subtract the hole's area and both its moments:

$$\bar{x} = \frac{24(3) - 8(4)}{24 - 8} = 2.5 \text{ m}, \qquad \bar{y} = \frac{24(2) - 8(3)}{24 - 8} = 1.5 \text{ m}.$$



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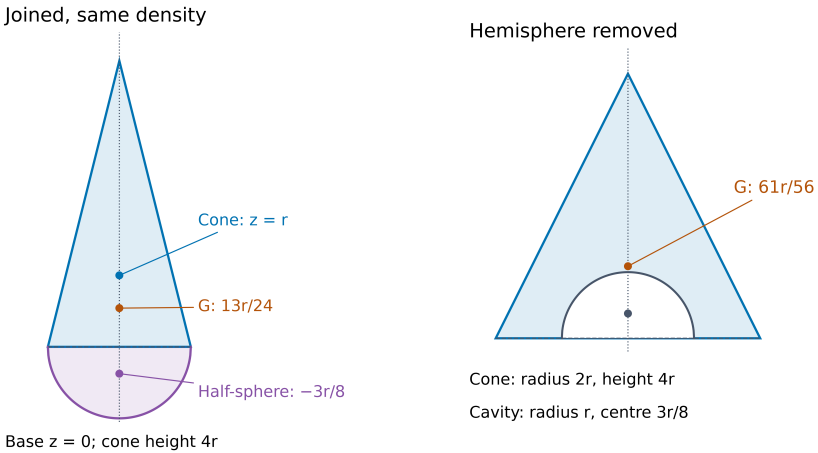
Change the mass weights when the material changes

Two uniform thin rods of the same material and cross-section form an L: a 6 m horizontal rod from the origin, and a 4 m vertical rod. Use lengths 6 and 4 as mass weights. Their centres are $(3, 0)$ and $(0, 2)$, giving $(\bar{x}, \bar{y}) = (1.8, 0.8)$ m. If the vertical rod has twice the mass per unit length, the weights become 6 and 8, giving $(9/7, 8/7)$ m. Do not use the area of the empty region between the rods.

Join solids with a signed common axis

Join a uniform solid cone of radius r, height $4r$, to a uniform solid hemisphere of radius r at their circular bases. They have the same density. Put the common base at $z = 0$ and take positive z towards the cone's vertex. The cone has volume $4\pi r^3/3$ and centre $z = r$. The hemisphere has volume $2\pi r^3/3$ and centre $z = -3r/8$. Thus

$$\bar{z} = \frac{2(r) + 1(-3r/8)}{2 + 1} = \frac{13r}{24}.$$



Axial sections of the solids; use their volumes as mass weights, rather than the drawn triangle and semicircle areas.

Remove a cavity without changing its original centre

A uniform cone has base radius $2r$ and height $4r$. Remove a solid hemisphere of radius r whose base lies in the cone's base plane, with the same axis and its curved surface inside the cone. The original cone's volume and centre are $16\pi r^3/3$ and $z = r$; the removed hemisphere's are $2\pi r^3/3$ and $z = 3r/8$. With z measured upwards from that base, the remaining centre is

$$\bar{z} = \frac{16(r) - 2(3r/8)}{16 - 2} = \frac{61r}{56}.$$

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Removing material near the base moves the centre upwards. A hole has a negative mass contribution in the calculation; it does not have an unknown new centre.

A freely suspended body puts its centre below the support

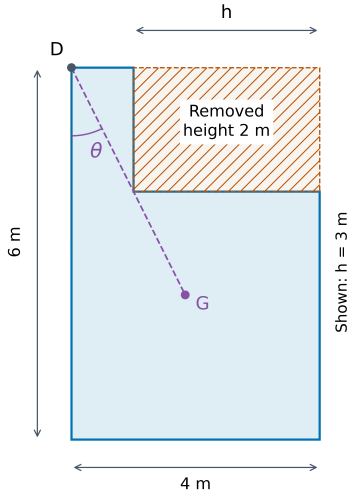
When a body hangs freely from a smooth pivot, the support force and weight must share a vertical line. For stable rest, the centre of mass is vertically below the support. A vertical line with the centre above the support is the unstable orientation. Suspend our L-shaped plate from its upper-left corner $D = (0, 4)$. In the plate's original coordinates, $G - D = (2.5, -2.5)$. The angle between the downward edge direction and DG is $\tan^{-1}(2.5/2.5) = 45°$. The plate rotates until DG points vertically down.

Check whether a parameter root leaves a real support point

A 4 by 6 uniform plate has an upper-right rectangle of width h and height 2 removed, where $0 < h < 4$. Distances are in metres; the support is its surviving upper-left corner $D = (0, 6)$. The area method gives

$$\bar{x} = \frac{h^2 - 8h + 48}{24 - 2h}, \qquad \bar{y} = \frac{72 - 10h}{24 - 2h}.$$

If the original downward edge makes angle θ with the hanging vertical and $\tan\theta = 1/2$, use $\bar{x}/(6 - \bar{y}) = 1/2$. This gives $(h - 3)(h - 4) = 0$. Only $h = 3$ is allowed: at $h = 4$ the entire top strip, including D, has been removed. Algebraic roots must obey the body's geometry.

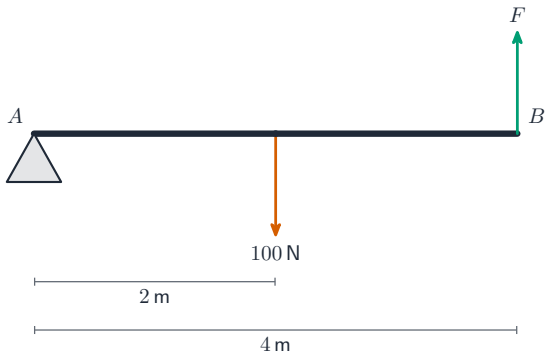


Draw every external force before choosing equations

A **free-body diagram** 受力图 contains the body and the forces acting on it. Weight acts at the centre of mass; a light string pulls along itself; a **normal reaction** 法向反力 is perpendicular to a contact surface. A smooth contact has no friction. A hinge can exert horizontal and vertical reaction components. At a rough contact, **friction** 摩擦力 opposes the direction of impending relative **sliding** 滑动. Static friction obeys $|F_f| \le \mu F_N$, where μ is the **coefficient of friction** 摩擦系数. The **limiting friction** 最大静摩擦 is μF_N. Equality holds at **limiting equilibrium** 极限平衡, not at every resting contact. A support that can only push must also have $F_N \ge 0$.

Balance a beam and then find its reaction

A uniform horizontal beam AB is 4 m long and weighs 100 N. It is pivoted at A, with an upward vertical force F at B. The weight acts 2 m from A. Moments about A give $4F = 100(2)$, so $F = 50$ N. Vertical force balance then gives the upward pivot reaction $R_A = 100 - 50 = 50$ N. Taking moments removed that reaction from one equation; it did not make the reaction zero.



A loaded beam needs the load positions

A uniform horizontal beam AB of length 6 m and weight 120 N rests on supports at its ends. A 60 N load is 5 m from A. The upward reactions satisfy $R_A + R_B = 180$. Moments about A: $6R_B = 120(3) + 60(5) = 660$, giving $R_B = 110$ N and $R_A = 70$ N. Both reactions are positive. If a calculated reaction at a simple support is negative, the assumed contact cannot supply it; check whether the body loses contact or **topples** 倾倒.

Resolve an angled rod and its hinge reaction

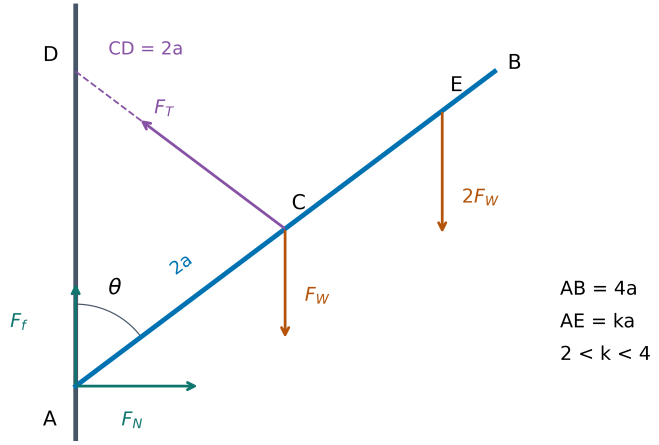
A uniform rod of length 4 m and weight 100 N is hinged at A and held at 30° above horizontal by a horizontal light string attached at its other end. The string pulls towards the wall through A. Moments about A use the vertical arm for the horizontal tension and horizontal arm for the weight:

$$F_T(4\sin 30^\circ) = 100(2\cos 30^\circ), \qquad F_T = 50\sqrt{3}\ \text{N}.$$

The hinge reaction components are $50\sqrt{3}$ N rightwards and 100 N upwards. Its magnitude is $50\sqrt{7}$ N, not their sum.

Set up the geometry of a rough-wall rod

A uniform rod AB has length $4a$ and weight F_W, with its lower end A against a rough vertical wall. Its midpoint is C. A light string of length $2a$ joins C to a point D higher on the wall, so $AC = CD$. The rod makes angle θ with the wall, where $\tan\theta = 4/3$. A particle of weight $2F_W$ is attached at $AE = ka$, between C and B. The coefficient of friction at A is $1/4$, and the rod is about to slip down. The unknowns are the string tension, contact forces and position parameter k.



Resolve the rough-wall forces first

The isosceles triangle ACD makes the string's angle with the vertical equal to θ. Friction acts upwards against impending downward slipping. Using $\sin\theta = 4/5$, $\cos\theta = 3/5$ and the limiting relation $F_f = F_N/4$ gives

$$F_N = \frac{4}{5}F_T, \qquad \frac{3}{5}F_T + \frac{1}{4}F_N = 3F_W.$$

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Substitute the first equation into the second: $4F_T/5 = 3F_W$. Hence $F_T = 15F_W/4$, $F_N = 3F_W$ and $F_f = 3F_W/4$. The extra particle doubles its own weight contribution; the total downward weight is $3F_W$.

Finish the rough-wall problem with one moment equation

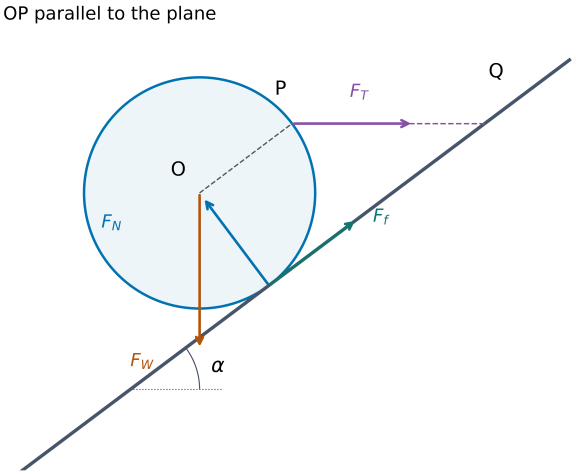
Taking moments about A removes both contact forces. Resolve the tension at C, or use its perpendicular arm. The moment equation is

$$F_T(2a\sin 2\theta) = F_W(2a\sin\theta) + 2F_W(ka\sin\theta).$$

Substitute $F_T = 15F_W/4$, $\sin 2\theta = 24/25$ and $\sin\theta = 4/5$. This gives $36/5 = (8/5)(1+k)$, hence $k = 7/2$. Check $2 < k < 4$: the particle is between C and B, as the model requires.

A ring can need friction even when its centre is stationary

A ring of radius r and weight F_W rests on a rough plane inclined at α, with $\tan\alpha = 3/4$. A horizontal string pulls rightwards at a point P whose radius OP points uphill, parallel to the plane. Let F_f act uphill and F_N normal to the plane. Moments about the centre give $F_f r = F_T r\sin\alpha$. Resolving along the plane gives $F_f + F_T\cos\alpha = F_W\sin\alpha$. Using $\sin\alpha = 3/5$, $\cos\alpha = 4/5$, obtain $F_T = 3F_W/7$ and $F_f = 9F_W/35$.



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Use the normal force before applying a friction limit

In the ring example, normal balance gives $F_N = F_W\cos\alpha + F_T\sin\alpha = 37F_W/35$. The horizontal string pushes into the plane as well as pulling uphill; the normal reaction is not simply $F_W\cos\alpha$. At limiting equilibrium, $\mu = F_f/F_N = 9/37$. If the plane has a larger coefficient, the ring can rest with the same required friction; friction does not automatically increase to its maximum value.



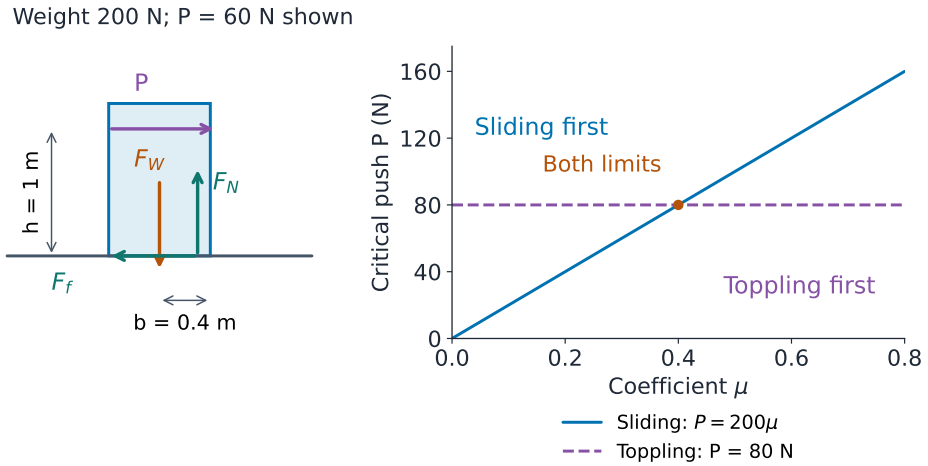
A ladder against a wall: weight, two normal reactions and friction at the foot, with moments taken about the base

Find the support interval before testing toppling

A body resting under its weight and contact forces alone can remain upright while the vertical through its centre of mass meets its **base of support** 支撑底面. At the point of **toppling** 翻倒, the contact resultant reaches an edge; the weight's line passes through that edge. Beyond it there is an unbalanced turning moment. The L-shaped plate has $\bar{x} = 2.5$ m. On its full 6 m lower edge it can balance. If turned over to stand on its short 2 m top edge, its weight line lies outside the support interval, so it topples. An extra applied force changes the moment balance: weight inside the base alone is then not a sufficient test.

A horizontal push can move the contact resultant to an edge

A uniform rectangular block of weight F_W rests on a horizontal rough floor. Its base half-width is b, and a horizontal push P acts at height h above the floor. Before motion, $F_N = F_W$ and the required friction is $F_f = P$. Moments show that the normal reaction shifts a distance $\delta = Ph/F_W$ from the base centre towards the leading edge. Toppling begins at $\delta = b$, so $P_{\text{topple}} = F_W b/h$. Sliding begins when $P_{\text{slide}} = \mu F_W$. Compare these thresholds; do not assume both limits occur together.



Decide which loss of equilibrium happens first

For $F_W = 200$ N, $b = 0.4$ m and $h = 1$ m, the toppling threshold is 80 N. With $\mu = 0.3$, the sliding threshold is 60 N, so sliding begins first. With $\mu = 0.5$, sliding would require 100 N, so toppling begins first. The thresholds coincide at $\mu = b/h = 0.4$. Increasing the push height makes toppling easier, but does not change the horizontal sliding threshold in this model. These predictions concern the first motion; the subsequent motion needs a different model.

On an incline, compare two critical angles

For a body with a symmetric base of half-width b and centre of mass a perpendicular distance c above its centre, the weight line reaches the downhill edge when $\tan\alpha = b/c$. With no other applied forces, impending sliding gives $\tan\alpha = \mu$. The smaller critical angle occurs first. For $b = 0.4$ m, $c = 0.6$ m and $\mu = 0.5$, sliding occurs at $\tan^{-1}(0.5) \approx 26.6°$, before toppling at $\tan^{-1}(2/3) \approx 33.7°$. For the cone with a hemispherical cavity, $c = 61r/56$, $b = 2r$, so the toppling angle satisfies $\tan\alpha = 112/61$, provided the plane is sufficiently rough to prevent earlier sliding.

Keep the geometry and the force model together

Find each component's mass and centre from the stated material and given formula. Use one origin, signed coordinates and weighted moments; subtract removed material from both the total mass and every moment. Check the geometric domain of an unknown length before accepting a root. Draw every external force. Resolve twice, take moments once and then check contact and friction restrictions. A freely hanging body's centre is below its support. For standing or pushed bodies, identify the actual support edge and compare toppling and sliding limits. Neither zero resultant alone nor zero moment alone proves equilibrium.

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Circular motion

A turn needs a change in velocity

A ball on a string changes direction even when its speed stays the same. The string pulls the ball inwards. Your hand feels the ball's pull outwards: those two forces act on different objects. In a frame fixed to the ground, draw the real forces on the ball. Its **inertia** 惯性 does not supply an extra outward force. The name **centrifugal** 离心 force belongs to a rotating-frame description, which we do not need here.



The wire pulls the hammer inwards during the turn. At release, its velocity is tangent; gravity then curves its flight.

Link angular speed to distance along the circle

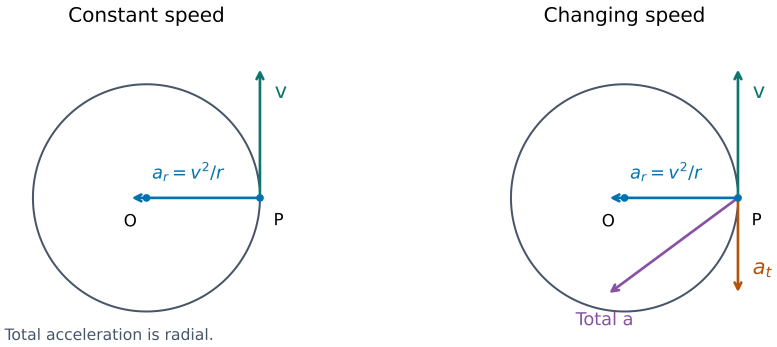
An angle of θ **radians** 弧度 sweeps arc length $s = r\theta$. For fixed radius, **angular speed** 角速度 is the rate of turning; **linear speed** 线速度 is distance travelled per second. Therefore $v = r\omega$. A point farther from the axis has greater speed at the same ω. Use radians per second in this formula; degrees per second must first be multiplied by $\pi/180$.

Convert a period or a revolution rate

The **period** 周期 τ is the time for one full turn. The **frequency** 频率 f is turns per second. Hence $\omega = 2\pi/\tau = 2\pi f$. At 30 revolutions per minute, $f = 1/2$ turn per second and $\omega = \pi \ \frac{\text{rad}}{\text{s}}$. At radius 0.4 m, $v = 0.4\pi \ \frac{\text{m}}{\text{s}}$. Do not put 30 straight into $v = r\omega$.

Separate inward acceleration from changing speed

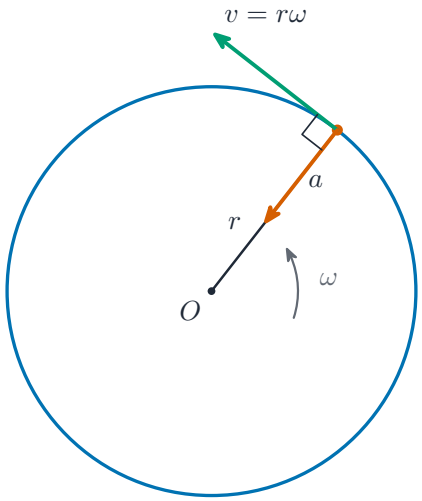
Velocity is along the **tangent** 切线. The **radial** 径向 component of acceleration points towards the centre and has magnitude $a_r = v^2/r = r\omega^2$. This inward component is the **centripetal acceleration** 向心加速度. A proof of the acceleration formulae is not required here. At constant speed this is the whole acceleration. If speed changes, there is also a **tangential** 切向 component. In a vertical circle under gravity, the full acceleration need not point exactly towards the centre.



Use both forms of the acceleration formula

For $r = 2$ m and constant $\omega = 3\ \frac{\text{rad}}{\text{s}}$, the speed is $v = 2(3) = 6\ \frac{\text{m}}{\text{s}}$ and $a_r = 2(3^2) = 18\ \frac{\text{m}}{\text{s}^2}$. Checking with $v^2/r = 6^2/2 = 18$ gives the same result. At fixed speed, doubling the radius halves a_r; at fixed angular speed, doubling the radius doubles it. State which quantity is fixed.

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This diagram shows constant-speed circular motion; at changing speed, acceleration also has a tangential component.

Resolve real forces towards the centre

Draw a **free-body diagram** and choose inwards as positive. Newton's law gives $\sum F_{\text{inward}} = mv^2/r$. A **centripetal force** 向心力 is the inward resultant needed for the motion, not another arrow to add beside weight, tension or contact forces. A **horizontal circle** 水平圆 need not have constant speed. In this syllabus's steady horizontal-circle problems, vertical acceleration and tangential acceleration are zero; the inward acceleration is not zero. Resolve vertically and radially as separate equations.

Friction can provide the inward resultant

A particle on a rough horizontal rotating disc has **normal reaction** $N = mg$. If friction is the only horizontal force, the required friction is $F_f = mr\omega^2$. It must satisfy $F_f \le \mu N$. Therefore $\omega^2 \le \mu g/r$. For $r = 1.2$ m, $\mu = 0.3$ and $g = 10\ \frac{\text{m}}{\text{s}^2}$, the greatest angular speed is $\sqrt{2.5}\ \frac{\text{rad}}{\text{s}}$, about 15.1 revolutions per minute. Below the limit, friction is not automatically μN.

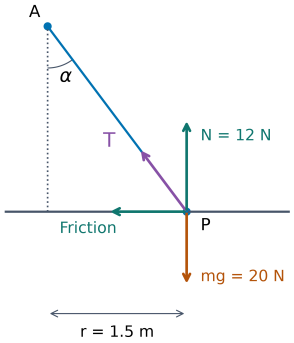
An inclined string changes the disc reaction

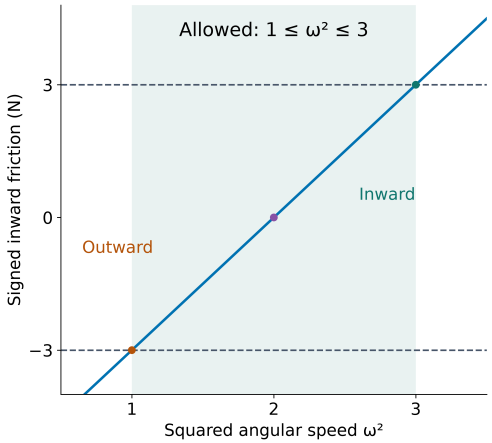
A 2 kg particle moves at radius 1.5 m on a rough disc. A string pulls towards a point above the axis, with **tension** 张力 $T = 10$ N, $\sin\alpha = 3/5$ and $\cos\alpha = 4/5$, where α is measured from vertical. Take $g = 10\ \frac{\text{m}}{\text{s}^2}$. Vertical balance gives $N = 20 - 10(4/5) =$

12 N. With $\mu = 1/4$ and impending outward slipping, **limiting friction** is 3 N inwards. Thus $10(3/5) + 3 = 2(1.5)\omega^2$, giving $\omega = \sqrt{3}\ \dfrac{\text{rad}}{\text{s}}$.

Friction may reverse when the speed changes

Keep that particle, tension and geometry fixed. The string already supplies 6 N inwards. Required signed inward friction is $F_f = 3\omega^2 - 6$, while $|F_f| \le 3$. Hence $1 \le \omega^2 \le 3$. At the lower limit friction acts outwards to reduce the inward resultant; at the upper limit it acts inwards. An outward real friction force is possible even though the net radial force is inwards.





A conical pendulum has two useful directions

A **conical pendulum** 圆锥摆 has a light inextensible string of length L making angle θ with the vertical. Its orbit radius is $r = L\sin\theta$, not L. The bob stays at one height. Resolve tension: $T\cos\theta = mg$ and $T\sin\theta = mr\omega^2$. For a non-zero orbit, substitution gives $\omega^2 = g/(L\cos\theta)$. The tension has both vertical and inward components.

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A swing ride: each seat is a conical pendulum, with the chain's tension giving both the vertical support and the inward force

Find the orbit before finding its speed

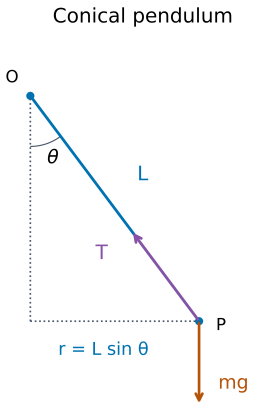
Let $L = 2.5$ m, $m = 0.4$ kg, $\cos\theta = 4/5$, $\sin\theta = 3/5$ and $g = 10\ \dfrac{\text{m}}{\text{s}^2}$. Then $r = 1.5$ m and $T = mg/\cos\theta = 5$ N. Now $\omega^2 = 10/[2.5(4/5)] = 5$, so $v = 1.5\sqrt{5}\ \dfrac{\text{m}}{\text{s}}$ and the period is $2\pi/\sqrt{5}$ s. Using the string length as the orbit radius would give the wrong speed.

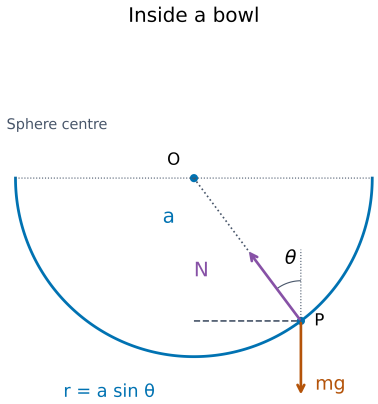
A second particle can fix the tension

A string of total length $9a$ passes through a fixed smooth ring R. Particle A of mass $2m$ hangs below it. Particle B of mass m moves in a horizontal circle whose centre is A. Let $RB = L$ and let RB make angle θ with the vertical. For A, $T = 2mg$. For B, $T\cos\theta = mg$, so $\cos\theta = 1/2$. Geometry gives $RA = L\cos\theta = L/2$ and $RA + RB = 9a$, so $L = 6a$. Then $\omega^2 = g/(L\cos\theta) = g/(3a)$.

A bowl radius is not the orbit radius

On the smooth inner surface of a hemispherical bowl of radius a, let the normal make angle θ with upward vertical. The orbit radius is $a\sin\theta$. Vertical and inward equations are $N\cos\theta = mg$ and $N\sin\theta = m(a\sin\theta)\omega^2$. Therefore $\omega^2 = g/(a\cos\theta)$. If $\cos\theta$ changes from $4/5$ to $5/13$, the angular-speed ratio is $\omega_1/\omega_2 = \sqrt{(5/13)/(4/5)} = 5/(2\sqrt{13})$. Compare squared speeds first, then take the square root.





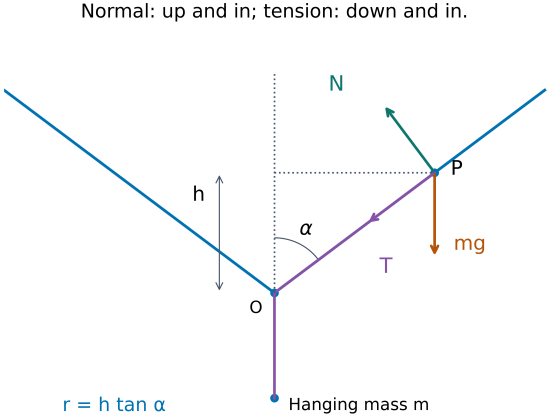
A cone needs its own geometry

A smooth hollow cone has its vertex downwards and semi-vertical angle α. At height h above the vertex the orbit radius is $r = h\tan\alpha$. The normal points upwards and inwards, making angle α above horizontal. With only gravity and contact, $N\sin\alpha = mg$ and $N\cos\alpha = mr\omega^2$. These give $\omega^2 = g\cot^2\alpha/h$. A cone's radius is set by height; do not borrow the bowl's radius formula.

Add a string through the cone vertex

Now attach the particle of mass m to another mass m hanging below the vertex through a smooth hole. The string pulls down the sloping surface towards the vertex. Take $\tan\alpha = 4/3$, so $\sin\alpha = 4/5$, $\cos\alpha = 3/5$ and $r = 4h/3$. The hanging mass gives $T = mg$. Then $N(4/5) = mg + T(3/5)$ gives $N = 2mg$. Radially, $N(3/5) + T(4/5) = mr\omega^2$, hence $\omega^2 = 3g/(2h)$. The string's downward component increases the reaction.

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Measure height in a vertical circle

For a **vertical circle** 竖直圆 of radius r, take angle θ from the downward vertical and measure height from the bottom. At the right-hand point of the circle, $x = r\sin\theta$ and height $h = r(1 - \cos\theta)$. At the bottom $\theta = 0$ and $h = 0$; at the side $\theta = \pi/2$ and $h = r$; at the top $\theta = \pi$ and $h = 2r$. Mark the angle's reference direction before writing an energy equation.



A vertical loop: speed changes with height, so energy conservation gives the speed and the reaction must stay positive at the top

Energy gives the speed at each height

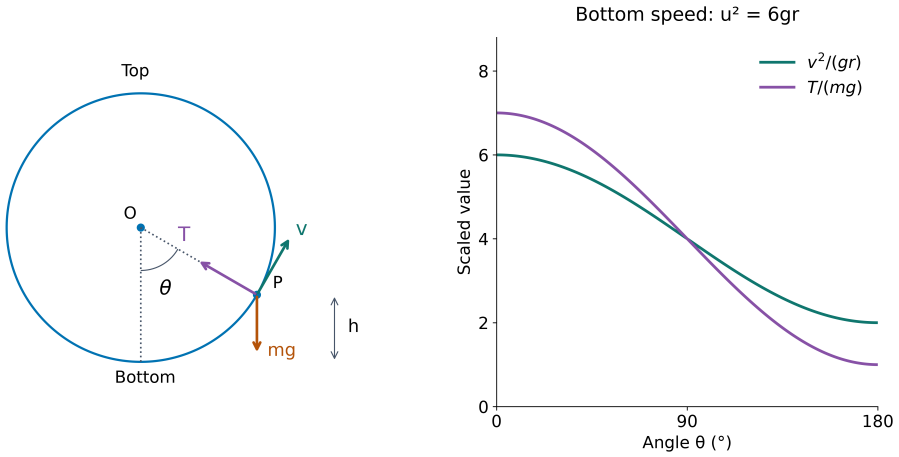
For a light inextensible string or a fixed smooth circular surface, the constraint force is perpendicular to velocity and does no work. With no resistance, **energy conservation** 能量守恒 gives $\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgr(1-\cos\theta)$, where u is bottom speed. Therefore $v^2 = u^2 - 2gr(1-\cos\theta)$. This formula applies while the particle follows the circle. It finds speed, not tension; and $v^2 \geq 0$ alone does not prove the circular path is possible.

Take inward components of weight and tension

For a string pulling towards the centre, the inward equation is $T - mg\cos\theta = mv^2/r$. At the bottom, weight acts outwards; at the top, weight acts inwards. The changing sign is already in $\cos\theta$. Combining with energy gives $T/m = u^2/r - 2g + 3g\cos\theta$. At the bottom $T = mu^2/r + mg$; at the top $T = m(u^2/r - 5g)$. These tensions use different speeds at the two heights.

Work from a specified bottom speed

Suppose $u^2 = 6gr$. At $\cos\theta = 1/2$, energy gives $v^2 = 5gr$, and the inward equation gives $T = 5mg + mg/2 = 11mg/2$. At the bottom the tension is $7mg$; at the top it is mg. The particle moves fastest and needs the greatest tension at the bottom. In between, gravity also changes its speed, so the full acceleration has a tangential component.



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A complete loop needs a feasible top force

A string can pull, not push: a **taut** 绷紧 string needs $T \geq 0$. The top has the least tension. There $T + mg = mv_{\text{top}}^2/r$, so $v_{\text{top}}^2 \geq gr$. Energy then gives the complete-loop condition $u^2 \geq 5gr$. Equality is the limiting complete loop, with zero tension at the top. Merely reaching top height by energy would require $u^2 \geq 4gr$; that weaker test allows a negative calculated tension before the top. The string would already have gone **slack** 松弛.

Start at a point other than the bottom

If the starting radius makes angle ϕ with the upward vertical, its height above the bottom is $r(1+\cos\phi)$. The climb to the top is $r(1-\cos\phi)$. A complete loop requires $v^2 \geq gr + 2gr(1-\cos\phi) = gr(3-2\cos\phi)$. At $\cos\phi = 1/4$, the minimum starting squared speed is $5gr/2$. At equality the bottom squared speed is $5gr$ and the greatest tension is $6mg$.

Use a tension ratio with two energy states

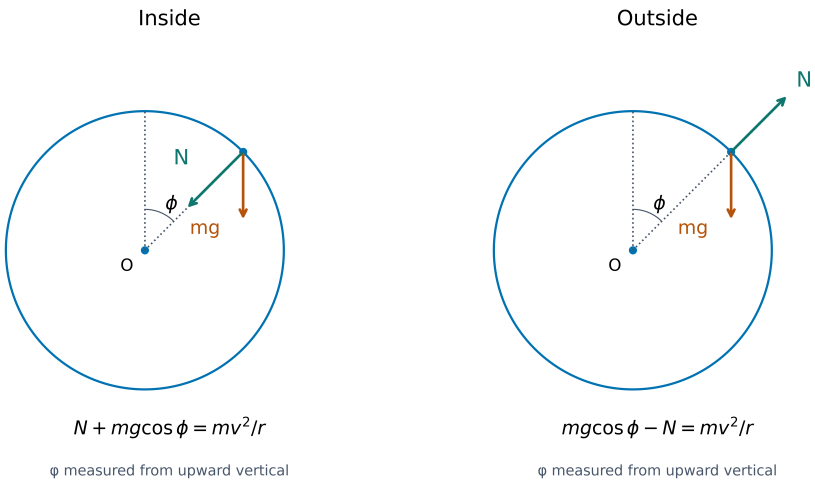
In a complete loop, bottom tension exceeds top tension by $6mg$. If their ratio is $9:1$, write $T_{\text{bottom}} = 9T_{\text{top}}$. Then $T_{\text{top}} = 3mg/4$, so $v_{\text{top}}^2 = 7gr/4$ and bottom squared speed is $23gr/4$. If the speed at angle θ from downward vertical is $\sqrt{4gr}$, energy gives $4gr = 23gr/4 - 2gr + 2gr\cos\theta$. Thus $\cos\theta = 1/8$. Equal top and bottom speeds would violate energy conservation.

A turning point and a slack point are different

If bottom squared speed is gr, the equation $v = 0$ gives $\cos\theta = 1/2$: the particle turns back at $60°$, with tension $mg/2 > 0$. It never reaches the horizontal level. If bottom squared speed is $3gr$, the equation $T = 0$ instead gives $\cos\theta = -1/3$, about $109.5°$. Here $v^2 = gr/3 > 0$: the string slackens while the particle is still moving. Stop using the string's circular-motion equation beyond this point.

An inner surface can push only inwards

On the smooth inner surface of a spherical shell, the normal reaction, or **normal contact force** 法向接触力 N, points towards the sphere centre. Its radial equation is $N - mg\cos\theta = mv^2/r$ with θ measured from downward vertical. The same energy and complete-loop results apply, replacing T by N. Require $N \geq 0$; a negative answer means the surface would have to pull. For **loss of contact** 脱离接触, test the boundary $N = 0$: the particle leaves if continuing the circle would require $N < 0$. A limiting complete loop can have $N = 0$ at the top without separating.



A reaction ratio can locate a rim

A particle starts vertically downwards from the side of a smooth inner shell with squared speed $u^2 = 2gr$. A rim is at height kr above the centre. At the start, $N_A = mu^2/r = 2mg$; at the rim, energy gives $v_B^2 = (2-2k)gr$. The inward weight component there is mgk, so $N_B/m = v_B^2/r - gk = (2-3k)g$. If $N_B = N_A/4$, then $k = 1/2$. Keeping contact as far as this rim requires $u^2 \geq 3gr/2$, not just the energy test $u^2 \geq gr$.

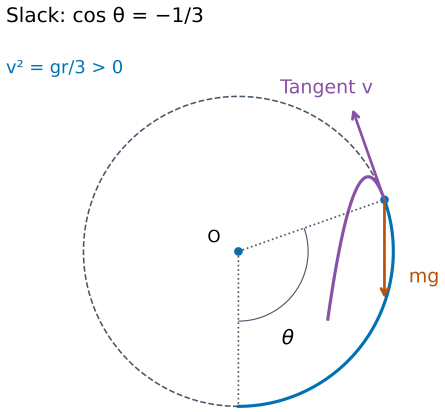
An outer surface has the opposite normal direction

A particle starts at the top of a smooth fixed sphere with tangential squared speed $gr/2$. Let ϕ be its angle from upward vertical. Energy gives $v^2 = gr/2 + 2gr(1-\cos\phi)$. On the outside, the normal points outwards. The inward equation is $mg\cos\phi - N = mv^2/r$, giving $N = mg(3\cos\phi - 5/2)$. It loses contact at $\cos\phi = 5/6$, with $v^2 = 5gr/6 > 0$. Do not copy the inner-surface sign.

Release gives a tangent velocity, then a curved path

At release, position and velocity remain continuous. The initial free-flight velocity is tangent to the circle. Gravity still acts, so the later path is a projectile parabola, not a straight tangent line. For the string that slackens at $\cos\theta = -1/3$ with bottom squared speed $3gr$, take centre O as origin. At release, $(x, y) = (2\sqrt{2}r/3, r/3)$ and speed is $\sqrt{gr/3}$. On the upward trip along the right side, velocity components are $(-v/3, 2\sqrt{2}v/3)$.

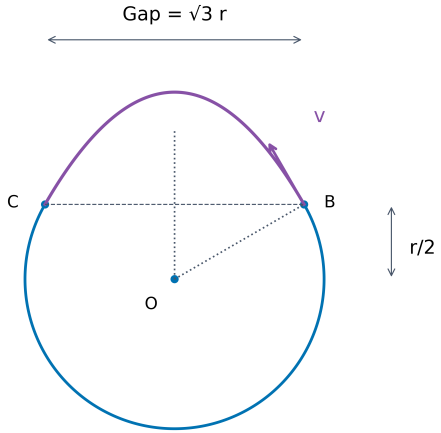
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Blue: circular path; purple: initial free flight.

Join circular motion to a flight across a rim

Cut an inner shell at height $r/2$ above its centre. To fly from the right rim to the left rim at the same height, let speed at release be v. The horizontal gap is $\sqrt{3}r$; velocity components are $(-v/2, \sqrt{3}v/2)$. The non-zero return time is $t = \sqrt{3}v/g$. Horizontal motion gives $\sqrt{3}r = (v/2)t$, so $v^2 = 2gr$. If the launch began at the side at centre height, energy gives side squared speed $u^2 = 3gr$. The rim reaction just before exit is $3mg/2 > 0$: the surface ends before contact is lost.



A positive reaction may remain at an open rim.

Check the model before accepting a root

Draw forces and identify the orbit radius. For steady horizontal motion, resolve vertically and inwards. For vertical motion, combine height-based energy with one inward force equation. Check $v^2 \geq 0$ and the sign of tension or reaction along the route. State which event happens first: a turning point, slack string, loss of contact, or the end of the surface. Then use the next model. A collision changes the motion through an impulse; do not carry the no-loss circular energy equation through an impact without a separate collision argument.

Keep the direction and constraint together

$v = r\omega$ links speed and angular speed; $a_r = v^2/r$ is the inward component of acceleration. The inward resultant comes from real forces. A horizontal path alone does not guarantee constant speed. For a light inextensible string in a lossless vertical loop, bottom squared speed must be at least $5gr$. At a zero-tension or zero-reaction boundary, test whether the constraint can continue. If release occurs, its velocity is initially tangent; gravity controls the following flight.

Hooke's law

Why a jump needs a model

A bungee rope stores energy as it stretches. Its pull changes during the jump, so constant-acceleration formulae are usually unsuitable. **Hooke's law** 胡克定律 models force proportional to extension within a stated range. Real ropes need not follow it over a whole jump. Here use ideal light **elastic** 弹性 strings and springs with no energy loss unless stated.



A bungee jump: the cord is slack until it reaches its natural length, then Hooke's law gives a tension proportional to the extension

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Measure extension from natural length

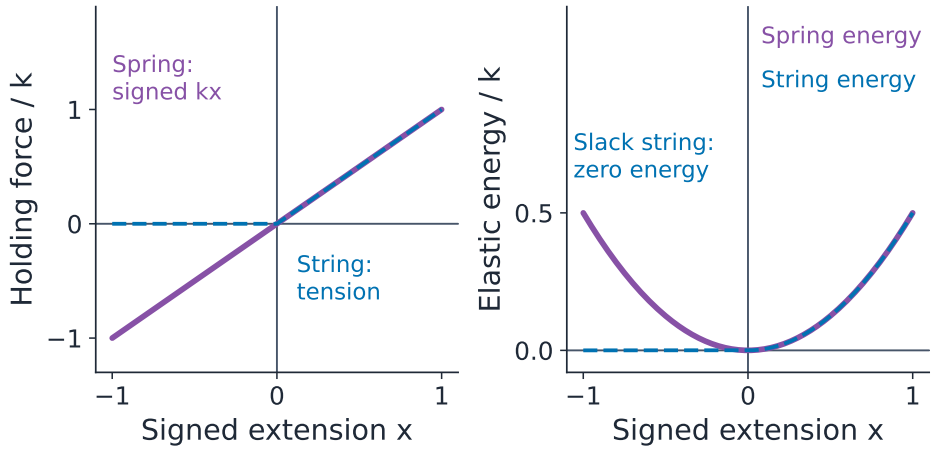
The **natural length** 自然长度 L is the unloaded length. If the actual length is ℓ, the **extension** 伸长量 is $x = \ell - L$, not ℓ. A taut elastic string has tension $T = kx$. With **modulus of elasticity** 弹性模量 λ, write $k = \lambda/L$, so $T = \lambda x/L$. For a spring the signed restoring force opposes displacement from its natural length.

Modulus and stiffness have different units

Stiffness 劲度系数 k is measured in $\frac{\mathrm{N}}{\mathrm{m}}$; this string model's λ is measured in newtons. It includes material and cross-section effects; it is not Young modulus, whose unit is pascals. Cut a uniform string into two equal pieces: each has the same λ but half the natural length, so each piece has twice the stiffness. Do not keep k fixed when changing L in this way.

A string and a spring have different branches

A string pulls only when **taut**. For $\ell \leq L$ it is **slack**: $T = 0$ and its stored elastic energy is zero. A negative value of $\ell - L$ is not a negative string tension. An ideal spring can pull in extension and push in **compression** 压缩. Its signed x may be negative, while the force magnitude is $k|x|$. Continue to use the spring law only within the stated model range.

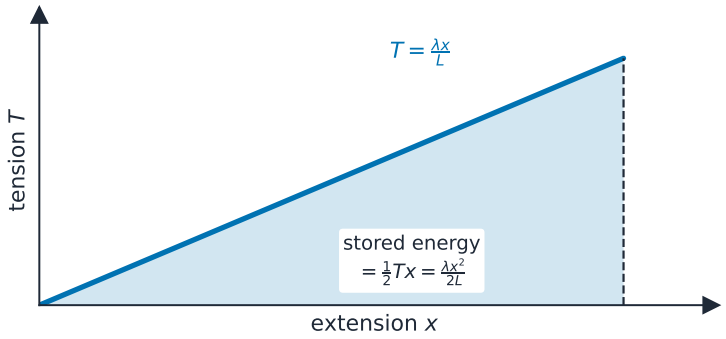


Store the area under the graph

Stretching slowly from natural length does work equal to the triangular area under the force–extension line. The stored **elastic potential energy** 弹性势能 is

$$E_e = \frac{1}{2}kx^2 = \frac{\lambda x^2}{2L}.$$

For a spring this also applies in compression. For a string use it only when $x \geq 0$; the slack branch has zero energy. Doubling extension doubles tension but multiplies energy by four. A proof of the energy formula is not required.



$T = \frac{\lambda}{L}x$ · $E = \frac{1}{2}Tx = \frac{\lambda x^2}{2L}$ (area under graph)

Use force and energy on the same string

A string has $L = 2$ m, $\lambda = 50$ N and extension 0.5 m. Its actual length is 2.5 m. First use $T = \lambda x/L$, giving $T = 50(0.5)/2 = 12.5$ N. For stored energy use $E_e = \lambda x^2/(2L)$, giving $E_e = 50(0.5)^2/4 = 3.125$ J. Force and energy have different units; do not substitute actual length for extension.

Infer a natural length from measurements

A string has tension 6 N at length 1.3 m and 10 N at length 1.5 m. Since $T = k(\ell - L)$, the gradient is $k = (10-6)/(1.5-1.3) = 20\ \dfrac{\text{N}}{\text{m}}$. Then $L = \ell - T/k = 1.3 - 6/20 = 1$ m and $\lambda = kL = 20$ N. Check the other measurement: $20(1.5 - 1) = 10$ N. The line's horizontal intercept gives natural length, not zero length.

Find a hanging equilibrium

A mass m hangs at rest from one vertical string attached above it. **Equilibrium** requires $T = mg$, so its equilibrium extension is $e = mg/k = mgL/\lambda$. A 0.6 kg mass on $k = 30\ \dfrac{\text{N}}{\text{m}}$ has $e = 0.2$ m when $g = 10\ \dfrac{\text{m}}{\text{s}^2}$. If the mass is moving through that position, acceleration is momentarily zero; speed need not be zero.

Release at natural length is not slow loading

Attach a mass to the lower end at natural length and release it from rest, with no resistance. After a downward extension x, energy gives $mgx = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$. At maximum stretch $v = 0$, so the non-zero root is $x = 2mg/k = 2e$. Maximum speed occurs earlier at $x = e$, where $mg - kx = 0$; $v_{\max}^2 = g^2 m/k$. Slow loading instead ends at e and removes energy from the system.

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A horizontal spring can pass natural length

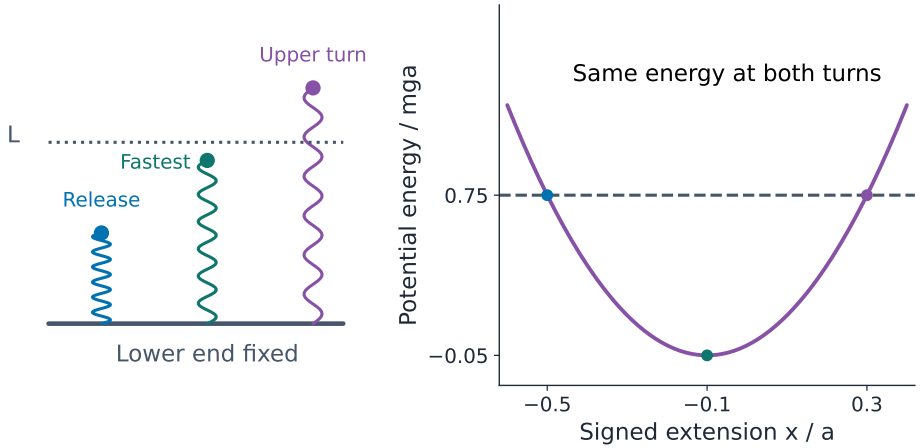
A 2 kg particle on a smooth horizontal line is attached to a spring with $k = 50\ \dfrac{\text{N}}{\text{m}}$. Release it from rest at extension 0.4 m. Its initial energy is $\frac{1}{2}k(0.4)^2 = 4$ J. At natural length the energy is all kinetic: $\frac{1}{2}mv^2 = 4$, so $v = 2\ \dfrac{\text{m}}{\text{s}}$. The particle continues into compression and first stops at $x = -0.4$ m, provided no other physical constraint intervenes.

A horizontal string changes model at natural length

Replace that spring by an elastic string with the same k and initial extension. It has the same speed at natural length, but beyond that point it has no tension and no elastic energy. On the smooth horizontal surface the particle then moves at constant velocity until another constraint acts. Do not predict spring-like compression and reversal. Specify geometry before following a particle past its anchor.

Release a compressed vertical spring

A spring has lower end fixed, natural length a, and $\lambda = 10mg$. A mass m is attached to its top and released from rest with compression $a/2$. Let x be signed extension upwards. Relative to natural length, total potential energy is $U = 5mgx^2/a + mgx$. Initially $x = -a/2$ and $U = 3mga/4$. At the next rest point, put $z = x/a$: $5z^2 + z = 3/4$. The roots are $-1/2$ and $3/10$. The first is the initial state; the maximum extension is $3a/10$.



Maximum speed is not a turning point

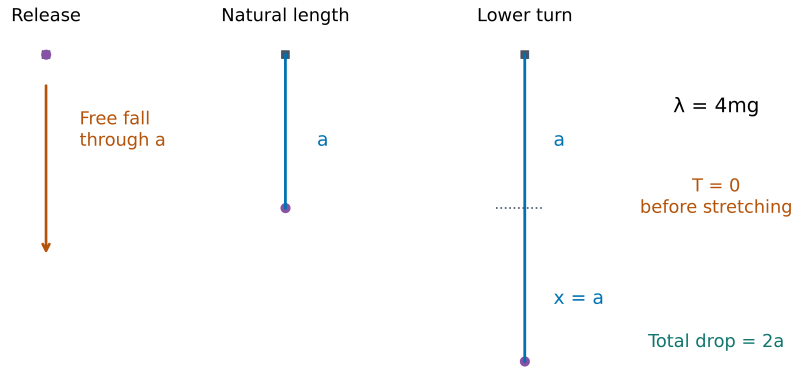
In that spring example, $-kx - mg = 0$ at $x = -a/10$. Potential energy there is $-mga/20$, so the maximum kinetic energy is $3mga/4 + mga/20 = 4mga/5$. Hence $v_{\max}^2 = 8ag/5$. At maximum extension $3a/10$, the mass is instantaneously at rest but the acceleration is $-4g$, downwards. A **turning point** 转折点 has $v = 0$; it need not have zero resultant force.

Record both end states before writing energy

Choose one height reference. Write kinetic energy, gravitational potential energy mgh, and every active elastic energy at the initial and final positions. With no resistance their total is conserved; with friction include its negative work. Use actual lengths to find each extension independently. A string at or below natural length contributes zero energy, not $\lambda(\ell - L)^2/(2L)$. If a path crosses a slack boundary, use the right force law in each interval.

Free fall can come before stretching

A mass m is released from rest at its anchor, on a vertical string with natural length a and $\lambda = 4mg$. It first falls freely through a; the string then begins stretching. At later extension x, energy gives $mg(a + x) = \frac{1}{2}mv^2 + 2mgx^2/a$. At the lowest point let $z = x/a$. Then $2z^2 - z - 1 = 0$, so $z = 1$; the negative root is outside the stretched branch. The total drop is $2a$. Maximum speed is at $x = a/4$, giving $v_{\max}^2 = 9ag/4$.



A slack string stores no compressive energy.

Removing a mass changes the equilibrium

A vertical string has natural length a and $\lambda = 4mg$. Masses m and $3m$ first hang at equilibrium, so the extension is a. Remove $3m$ without an impulse. The remaining mass starts at rest at length $2a$. On its taut branch put $z = x/a$ and take height zero at natural length. Energy becomes $\frac{1}{2}mv^2 + mga(2z^2 - z) = mga$. To find the first position where $v^2 = ag$, solve $2z^2 - z = 1/2$. The valid root gives $\ell = (5 + \sqrt{5})a/4$; the other root has negative extension.

A rejected root does not describe the slack flight

At natural length in that example, $v^2 = 2ag$ upwards. After that the string is slack; gravity alone acts until the next event. The taut-string quadratic cannot describe this phase. After rising a further distance s, use $v^2 = 2ag - 2gs$. The later position with $v^2 = ag$ is $s = a/2$, so the length is $a/2$. The negative-extension root from the old formula gives a different value because it invents compressive string energy.

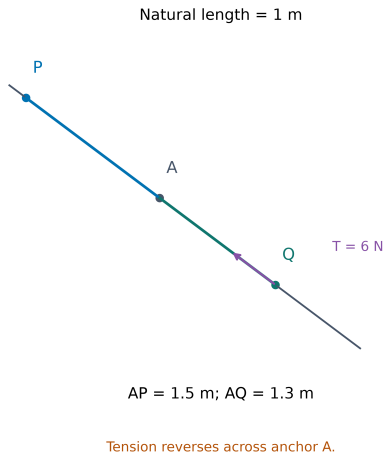
Resolve along an inclined plane

A mass m rests on a smooth plane inclined at β to horizontal, attached to an anchor upslope by a string along the plane. Normal reaction is $mg\cos\beta$; downslope weight is $mg\sin\beta$. Equilibrium requires $kx = mg\sin\beta$. With $m = 1$ kg, $g = 10\ \frac{\text{m}}{\text{s}^2}$, $\sin\beta = 3/5$ and $k = 20\ \frac{\text{N}}{\text{m}}$, $x = 0.3$ m. Friction, if present, must be included with its actual direction.

Follow a string past its anchor

Use that smooth plane and particle with natural length 1 m. Release from rest 1.5 m above the anchor. Initially the string pulls downslope; it then goes slack around the anchor and becomes taut again below it. Maximum speed is below the anchor at extension 0.3 m, where tension balances the 6 N downslope weight. Total travel to that point is $1.5 + 1 + 0.3 = 2.8$ m. Work–energy gives $\frac{1}{2}mv^2 = \frac{1}{2}k(0.5)^2 + mg\sin\beta(2.8) - \frac{1}{2}k(0.3)^2 = 18.4$ J. Thus $v_{\max}^2 = 36.8\ \frac{\text{m}^2}{\text{s}^2}$. Before this point the downslope resultant stays positive.

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Friction removes mechanical energy

On the same incline let $\mu = 1/4$, and release the particle from natural length below the anchor. For the first downslope motion, normal reaction is 8 N and friction is 2 N upslope. Net driving force before elasticity is $6 - 2 = 4$ N. Work–energy gives $\frac{1}{2}mv^2 = 4x - 10x^2$. The first maximum extension is 0.4 m, and maximum speed occurs at $x = 0.2$ m with $v_{\max}^2 = 0.8\ \dfrac{\mathrm{m}^2}{\mathrm{s}^2}$. Recheck friction's direction on any return journey; this equation describes the first descent.

Two opposing strings need two extensions

Fix anchors 6 m apart on a smooth horizontal line. Both strings have natural length 2 m; the left has $\lambda = 20$ N and the right 40 N. At coordinate q from the left, both are taut only when $2 < q < 4$. Balance $10(q-2) = 20(4-q)$ to find equilibrium $q = 10/3$ m. Total elastic energy is $U = 5(q-2)^2 + 10(4-q)^2$. A 2 kg mass released from $q = 3$ m has $U = 15$ J; at equilibrium $U = 40/3$ J, giving $v_{\max}^2 = 5/3\ \dfrac{\mathrm{m}^2}{\mathrm{s}^2}$. The other turn is $q = 11/3$ m, still in the same branch.

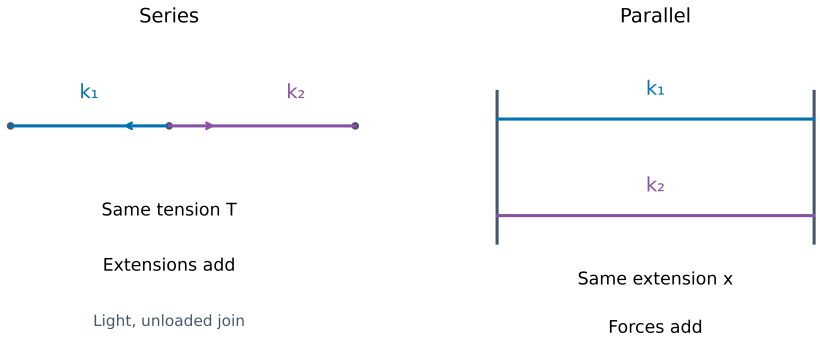
Series strings carry a common tension

For two strings in **series** 串联 with a light unloaded joining point, tensions are equal and extensions add. Use $x_i = T/k_i$ for each. The equivalent stiffness satisfies $1/k_{\mathrm{eq}} = 1/k_1 + 1/k_2$. Natural lengths $1, 2$ m and moduli $20, 30$ N give $k_1 = 20$, $k_2 = 15\ \dfrac{\mathrm{N}}{\mathrm{m}}$. Under 6 N tension their extensions are $0.3, 0.4$ m. Total extension is 0.7 m; energy is $\frac{1}{2}T(x_1 + x_2) = 2.1$ J.

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Parallel springs share an extension

Two springs in **parallel** 并联 between the same moving and fixed supports have the same extension if their natural lengths and geometry match. Their forces add: $T = (k_1 + k_2)x$, so $k_{\mathrm{eq}} = k_1 + k_2$. For $k_1 = 20$ and $k_2 = 30\ \dfrac{\mathrm{N}}{\mathrm{m}}$ at $x = 0.2$ m, the forces are 4 and 6 N; total force is 10 N and total energy 1 J. Equal extensions do not require equal tensions.



A lower string pulls downwards

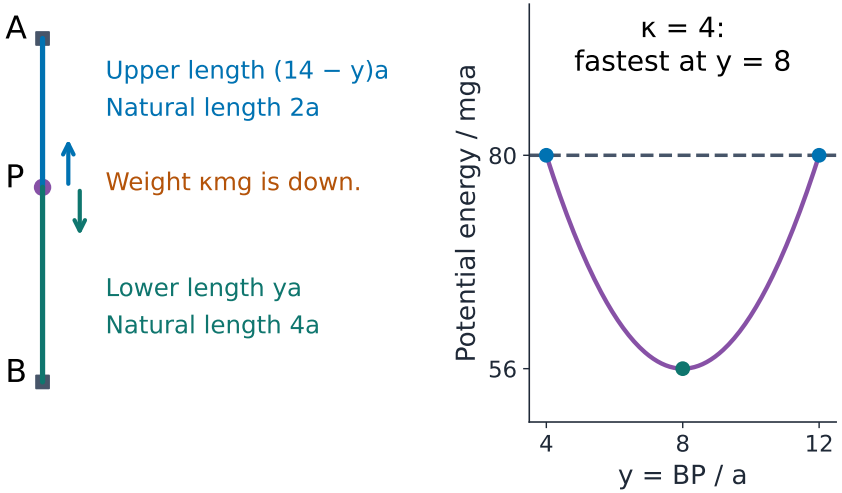
Two vertical anchors A above B are $14a$ apart. A particle of mass κm is joined to A by a string of natural length $2a$, and to B by one of natural length $4a$. Both moduli are $4mg$. Let $BP = ya$; then upper and lower extensions are $(12 - y)a$ and $(y - 4)a$. Both are taut for $4 < y < 12$. Their tensions are $2mg(12 - y)$ upwards and $mg(y - 4)$ downwards. Equilibrium gives $2(12 - y) - (y - 4) - \kappa = 0$, so $y = (28 - \kappa)/3$. Do not add both tensions in the same direction.

Two rest positions can determine the mass

Release the particle from rest at $y = 12$, and suppose its first lower turning point is $y = 4$. Take gravitational potential zero at B. Total potential energy divided by mga is $V = (12 - y)^2 + \frac{1}{2}(y - 4)^2 + \kappa y$. The endpoints give $32 + 12\kappa = 64 + 4\kappa$, hence $\kappa = 4$. Between them both extensions are nonnegative. With $\kappa = 4$, $V - 80 = \frac{3}{2}(y - 4)(y - 12) \le 0$, so the intermediate kinetic energy is feasible.

Use both strings to find maximum speed

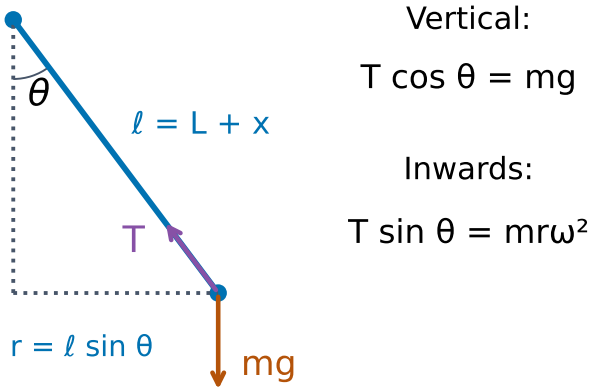
With $\kappa = 4$, equilibrium in that example is $y = 8$. At release the total energy is $80mga$; at $y = 8$ the potential energy is $(16 + 8 + 32)mga = 56mga$. Therefore $\frac{1}{2}(4m)v_{\max}^2 = 24mga$, giving $v_{\max}^2 = 12ag$. Use the actual moving mass $4m$ in kinetic energy; the symbol m in the moduli is only the reference mass.



An elastic conical pendulum stretches

A mass 0.6 kg rotates steadily with a string of natural length 1.5 m and modulus 30 N. Let its angle to vertical have $\cos\theta = 4/5$, $\sin\theta = 3/5$, and take $g = 10\ \frac{\text{m}}{\text{s}^2}$. Vertical balance gives $T\cos\theta = mg$, hence $T = 7.5$ N. Hooke's law gives $x = TL/\lambda = 0.375$ m, so actual length $\ell = 1.875$ m and orbit radius $r = \ell\sin\theta = 1.125$ m. The radial equation $T\sin\theta = mr\omega^2$ gives $\omega^2 = 20/3\ \frac{\text{rad}^2}{\text{s}^2}$. Use stretched length in the geometry.

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A positive length need not give a possible orbit

For a non-zero elastic conical orbit, $T = m\ell\omega^2$ and $T = \lambda(\ell - L)/L$. Therefore $\ell = \lambda L/(\lambda - mL\omega^2)$. Its denominator must be positive. Vertical balance also needs $\ell\cos\theta = g/\omega^2$, with $0 < \cos\theta < 1$. Take $L = 1$ m, $\lambda = 20$ N, $m = 1$ kg and $g = 10\ \frac{\text{m}}{\text{s}^2}$. If $\omega^2 = 5$, $\ell = 4/3$ m but the required vertical height is 2 m: impossible. If $\omega^2 = 10$, $\ell = 2$ m and $r = \sqrt{3}$ m: possible. At $\omega^2 = 20$ no finite length solves the equations.

Select a physically valid event

Mark the initial state, natural-length boundaries, equilibrium positions and candidate turning points. State the active model in each interval: a taut string pulls, a slack string has no force, and a spring may compress. Find roots, then check lengths, non-negative string tensions, nonnegative kinetic energy and the order of events. Maximum speed requires zero tangential resultant within a smooth branch, or comparison with a boundary; instantaneous rest requires $v = 0$.

Connect force, energy and geometry

First find $x = \ell - L$ for every element. Hooke's law gives force; $\lambda x^2/(2L)$ gives energy on its valid branch. Resolve directions before adding forces, and keep one height reference throughout energy calculations. Use original data to practise both one-element and connected systems. In circular motion, stretching determines actual length before orbit geometry. Check each final answer against its branch and units.

Linear motion under a variable force

Why a fast object slows differently

A bicycle usually feels more air resistance when it moves faster. Its **drag** 阻力 may depend on **speed** 速率. A different force might depend on **position** 位置 or time; these are different models. A **variable force** 变力 can produce changing acceleration. Start with Newton's law and set up a **differential equation** 微分方程 for the motion; constant-acceleration formulae need not apply.



A rocket's thrust and mass change during the burn, so the force is a function of time and $F = ma$ becomes a differential equation

Choose a positive direction first

Take displacement x positive to the right, and write **velocity** 速度 $v = dx/dt$. It can be negative. Speed is $|v|$, so a negative velocity does not mean a negative speed. Add signed forces on the particle: $m\,dv/dt = \sum F$. For motion to the right, resistance is negative. After reversal, ordinary resistance reverses too; a prescribed force towards a fixed point follows its own direction rule.

Change the variable using the chain rule

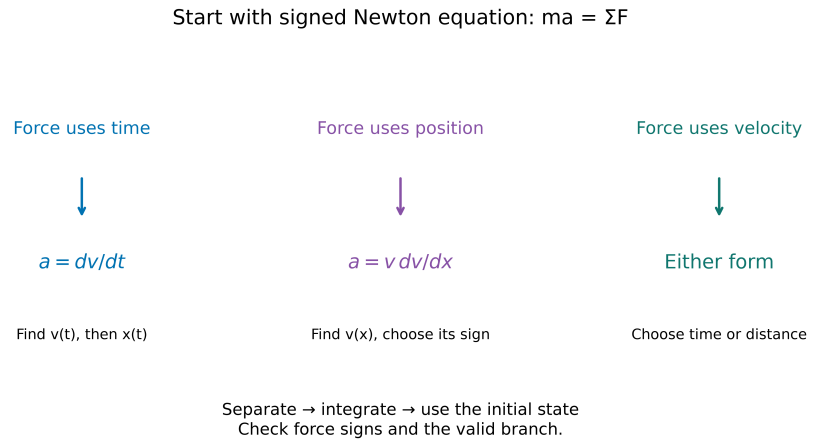
If velocity is written as a function of position along a branch of the motion, the **chain rule** 链式法则 gives

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

This is useful for finding v at a position without first finding time. At a **turning point** $v(x)$ may have a vertical tangent; do not assume $a = 0$ merely because $v = 0$. Use the original force law there.

Choose the variables for the requested answer

For a time-dependent force, $m\,dv/dt = F(t)$ often gives $v(t)$ directly. For a position-dependent force, $mv\,dv/dx = F(x)$ often gives $v(x)$ directly. If force depends on velocity, either form may help: use dv/dt for a time question and $v\,dv/dx$ for a distance question. These are choices of variables for the same Newton equation, not different laws.



Use each stated condition

A **separable differential equation** 可分离微分方程 can be rearranged so each side contains one variable and its differential. **Separate** 分离 the variables, **integrate** 积分, include a constant, then use an **initial condition** 初始条件 to determine it. A condition at a specified position is often called a **boundary condition** 边界条件. Here use only **separable** 可分离 equations and Pure Mathematics 3 calculus. Check any value divided out: division by a factor that can be zero may remove a constant solution.

A force depending on time can reverse the velocity

A 2 kg particle has resultant force $6t$ N to the right. At $t = 0$, $v = -1\ \frac{\text{m}}{\text{s}}$ and $x = 0$. Newton's law gives $2\,dv/dt = 6t$, so $v = \frac{3}{2}t^2 + C$; the initial velocity gives $C = -1$. Now integrate $dx/dt = -1 + \frac{3}{2}t^2$. The initial position gives $x = -t + \frac{1}{2}t^3$. There are two integrations and two initial conditions. The particle initially moves left even though acceleration is to the right for $t > 0$.

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Displacement and distance differ across a turn

In that example, $v = 0$ first at $t_* = \sqrt{2/3}$ s. Its position is $x_* = -2t_*/3$ m. At $t = 2$ s, displacement is $x = 2$ m. Total distance is the leftward distance $-x_*$ plus the rightward distance $2 - x_*$. Thus distance is $2 + 4\sqrt{2/3}/3$ m, greater than the displacement. Integrating signed velocity gives displacement; split at each reversal to find distance.

A position-dependent force links work to speed

With $F = mv\,dv/dx$, separation gives $mv\,dv = F(x)\,dx$. Integrating between two known states gives

$$\frac{1}{2}m(v^2 - u^2) = \int_{x_0}^{x} F(s)\,ds.$$

The signed area under a force–position graph is **work** 功. It equals the change in kinetic energy for the resultant force. This relation determines squared speed; use the direction of motion to choose the sign of velocity.

which form of acceleration?

force depends on **time** t	force depends on **position** x
$a = \dfrac{dv}{dt}$	$a = v\dfrac{dv}{dx}$
gives v as a function of t	gives v as a function of x

both come from $a = \frac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

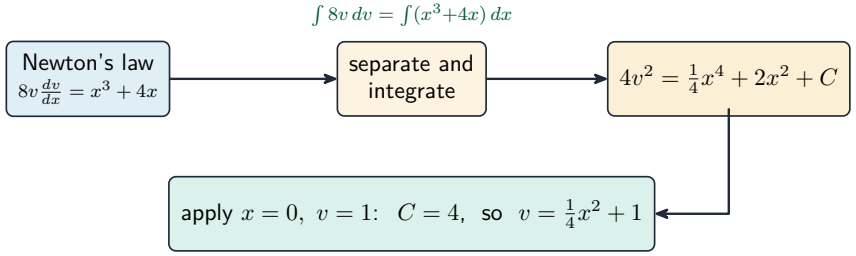
Work through a polynomial force

A particle of mass 8 kg starts at $x = 0$ with $v = 1$ $\frac{\text{m}}{\text{s}}$. For its outward motion the force is $(x^3 + 4x)$ N. Newton's law gives $8v\,dv/dx = x^3 + 4x$. Separate and integrate: $4v^2 = x^4/4 + 2x^2 + C$. Initially $C = 4$, so $v^2 = x^4/16 + x^2/2 + 1 = (x^2/4 + 1)^2$. The outward velocity is $v = x^2/4 + 1$; the negative square root would describe a different direction.

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A second separation can give position in time

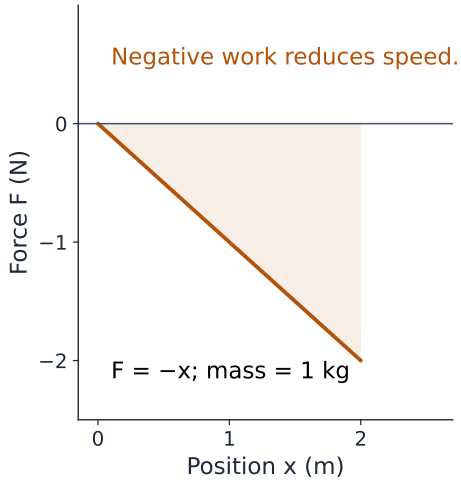
For that outward branch, $dx/dt = 1 + x^2/4$. Separate as $dt = 4\,dx/(x^2 + 4)$ and integrate: $t = 2\tan^{-1}(x/2) + C$. The initial position gives $C = 0$. Hence $x = 2\tan(t/2)$ on the branch $0 \le t < \pi$. The mathematical model predicts unbounded position as $t \to \pi$ from below. Do not continue the periodic tangent formula across that boundary or assume a real system can sustain the force model indefinitely.

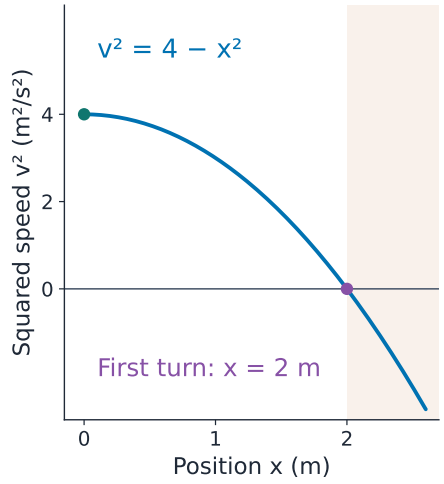


a position-dependent force becomes a **differential equation** in v and x

A turning root limits a position formula

A 1 kg particle starts at $x = 0$ with velocity 2 $\frac{\text{m}}{\text{s}}$, under force $-x$ N. Newton's law gives $v\,dv/dx = -x$, hence $v^2 = 4 - x^2$ from the initial state. It first stops at $x = 2$ m, where acceleration is -2 $\frac{\text{m}}{\text{s}^2}$, so it turns back. Use $v = +\sqrt{4 - x^2}$ before that turn and $v = -\sqrt{4 - x^2}$ immediately afterwards. The region $|x| > 2$ m would require negative squared speed and is inaccessible at this energy.





Linear resistance gives an exponential speed

A 2 kg particle moves horizontally with initial velocity 3 $\frac{\text{m}}{\text{s}}$, opposed by resistance of magnitude $4v$ N while $v > 0$. Newton's law gives $2\,dv/dt = -4v$, so $dv/v = -2\,dt$. Integration gives $\ln v = -2t + C$. The initial value gives $v = 3e^{-2t}$. It remains positive for every finite $t \geq 0$ and tends to zero as time grows. An exponential is not zero merely because a calculator rounds it to zero.

A limiting distance is not a finite stopping point

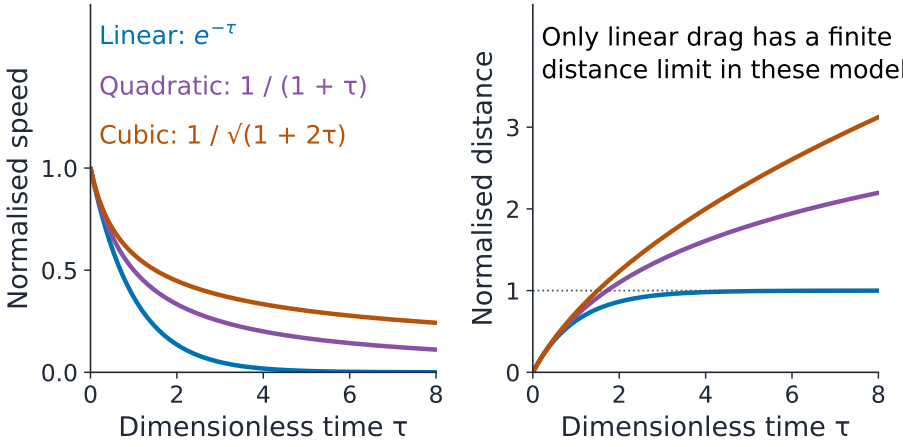
Integrating $v = 3e^{-2t}$ with $x(0) = 0$ gives $x = \frac{3}{2}(1 - e^{-2t})$. Thus the particle travels towards a limiting displacement of 1.5 m. It never reaches that limit in finite time under this ideal model. Equivalently $v = 3 - 2x$ on the moving branch, but putting $v = 0$ in that relation gives a limit, not a finite arrival time. Check the time law before describing a stop.

Quadratic resistance has a different distance limit

Suppose horizontal motion satisfies $dv/dt = -v^2/2$ with $v(0) = 4$. Separation gives $-1/v = -t/2 + C$, so $v = 4/(1 + 2t)$ for $t \geq 0$. With $x(0) = 0$, integration gives $x = 2\ln(1 + 2t)$. Speed tends to zero, but displacement has no finite upper limit. A decreasing speed does not by itself imply a finite total distance.

A cubic resistance gives another power law

Suppose $dv/dt = -3v^3$ with $v(0) = 2$. Separate as $v^{-3}\,dv = -3\,dt$. Integration and the initial value give $1/v^2 = 6t + 1/4$. Choose the positive branch: $v = 2/\sqrt{1 + 24t}$. With $x(0) = 0$, integration gives $x = (\sqrt{1 + 24t} - 1)/6$. Both this model and the quadratic model slow forever while covering an unbounded distance; their rates of slowing differ.



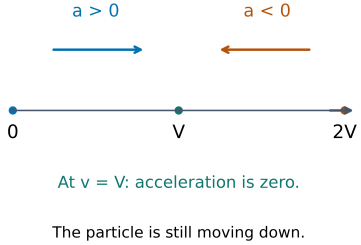
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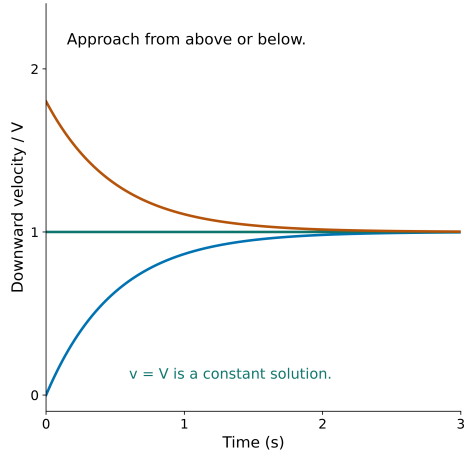
Falling with linear drag approaches a balance

Take downward as positive. A mass m falling under gravity and linear resistance kv obeys $m\,dv/dt = mg - kv$. Its **terminal velocity** 终端速度 is $V = mg/k$, where the resultant is zero. For $m = 2$ kg, $k = 4$ $\frac{\text{N s}}{\text{m}}$ and $g = 10$ $\frac{\text{m}}{\text{s}^2}$, $V = 5$ $\frac{\text{m}}{\text{s}}$. Released from rest, separation gives $v = 5(1 - e^{-2t})$ and downward distance $y = 5t - \frac{5}{2}(1 - e^{-2t})$.

Check a constant solution before dividing

In $dv/dt = (k/m)(V - v)$, dividing by $V - v$ excludes $v = V$. Test it in the original equation: it is a valid constant-velocity solution if that is the initial velocity. An initial downward velocity below V increases towards V; one above V decreases towards it. At terminal velocity the particle is moving with zero acceleration. It is not at rest, and the model need not reach the limit in finite time.





Separate the falling quadratic-drag equation

Take downward speed $v \geq 0$ with resistance per unit mass gv^2/V^2. Then $dv/dt = g(1 - v^2/V^2)$. From rest, $0 \leq v < V$, so separation gives $dv/(V^2 - v^2) = (g/V^2)\,dt$.

$$\frac{1}{V^2 - v^2} = \frac{1}{2V}\left(\frac{1}{V + v} + \frac{1}{V - v}\right).$$

Partial fractions 部分分式 give $\ln[(V + v)/(V - v)] = 2gt/V$ after using $v(0) = 0$. Rearranging gives $v = V(e^{2gt/V} - 1)/(e^{2gt/V} + 1)$. The formula approaches V from below; keep logarithm arguments and the initial branch consistent.

Falling speed can be found from distance instead

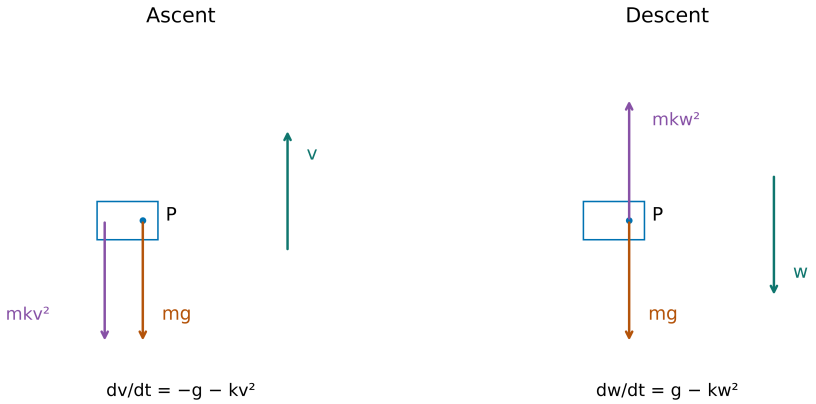
For the same quadratic-drag fall from rest, choose $v\,dv/dy = g(1 - v^2/V^2)$ with downward distance y. Separation and the initial state give

$$v^2 = V^2\left(1 - e^{-2gy/V^2}\right).$$

With $g = 10\ \frac{\text{m}}{\text{s}^2}$ and $V = 10\ \frac{\text{m}}{\text{s}}$, this is $v^2 = 100(1 - e^{-y/5})$. Gravity's lost potential energy does not all become kinetic energy: resistance does negative work.

Upward motion puts gravity and drag together

Launch a particle vertically upwards with speed u. Take upwards as positive on the ascent, so $v > 0$. Gravity and resistance both act downwards. For quadratic resistance per unit mass kv^2, the equation is $dv/dt = -g - kv^2$. Define $V^2 = g/k$ to simplify the algebra. Here V is the terminal speed for a later downward fall, not a possible steady upward speed. Using $g - kv^2$ during ascent would reverse the resistance incorrectly.



Find maximum height from the position equation

On ascent use $v\,dv/dx = -g - kv^2$. Separate and use the launch state in the integration:

$$\int_u^v \frac{w}{g + kw^2}\,dw = -\int_0^x ds.$$

$$\ln\left(\frac{g + kv^2}{g + ku^2}\right) = -2kx.$$

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Rearranging gives $v^2 = (u^2 + V^2)e^{-2kx} - V^2$ on ascent until its first zero. For $g = 10$, $k = 1/10$ and $u = 10$ in consistent SI units, $V = 10$ and $v^2 = 200e^{-x/5} - 100$. Maximum height is $H = 5\ln 2$ m. It is less than the no-drag height $u^2/(2g) = 5$ m.

Find ascent time by an inverse tangent

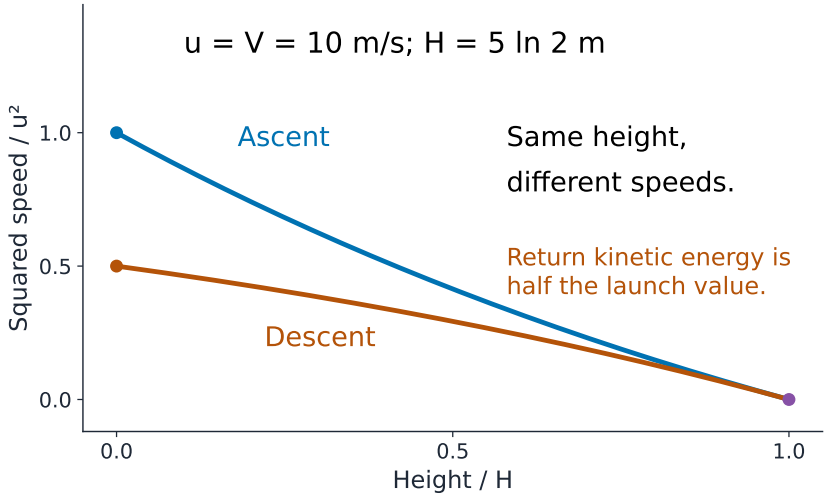
Separate $dv/dt = -k(V^2 + v^2)$. Integration gives $\tan^{-1}(v/V) = \tan^{-1}(u/V) - gt/V$. Use radians for the inverse tangent in this calculus calculation. In the example $u = V = 10$ and $g = 10$, $v = 10\tan(\pi/4 - t)$. The first zero gives ascent time $t = \pi/4$ s. Stop using this ascent equation at the top; resistance reverses during the descent.

Join the descent to the top state

At the top, velocity is continuous and equals zero. Restart a downward coordinate y from the top, with downward speed $w \geq 0$. The new equation is $dw/dt = g - kw^2$, because resistance now acts upwards. The falling-distance result gives $w^2 = V^2(1 - e^{-2ky})$. The ascent and descent each have a consistent sign convention; record the new initial state rather than carrying a negative ascent speed into an unchanged force equation.

Compare the return speed with the launch speed

In the example $u = V = 10$, $k = 1/10$ and $H = 5\ln 2$, the particle returns to launch height after falling distance H. Its downward squared speed is $w^2 = 100(1 - e^{-H/5}) = 50$. The return speed is $5\sqrt{2}\ \frac{\text{m}}{\text{s}}$, less than the launch speed $10\ \frac{\text{m}}{\text{s}}$. Gravity has the same potential energy at both endpoints, but resistance has removed mechanical energy during both parts.

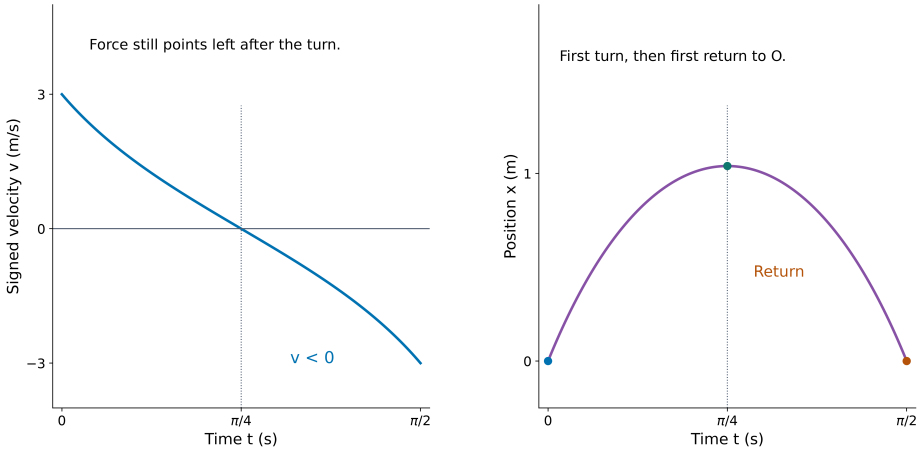


A force towards a point is not always drag

Suppose acceleration is $-(v^2+9)/3$ while a particle is to the right of O. It is prescribed to point towards O, even after the particle reverses and moves left. It then speeds up leftwards rather than resisting that motion. With $x(0) = 0$ and $v(0) = 3$, separation gives $\tan^{-1}(v/3) = \pi/4 - t$, so $v = 3\tan(\pi/4 - t)$. The particle first stops at $t = \pi/4$. A negative velocity after this time correctly describes the return towards O.

Integrate the return branch with its domain

Integrating that velocity and using $x(0) = 0$ gives $x = 3\ln[\cos(t - \pi/4)] + \frac{3}{2}\ln 2$ before the next return to O. At the turn the maximum displacement is $\frac{3}{2}\ln 2$. The next zero of x is $t = \pi/2$, found from the first return branch, not a later periodic tangent value. At O the direction rule changes; a formula derived for $x > 0$ is not automatically valid beyond that event.



Infer a force from a velocity relation

If a 2 kg particle has velocity $v = 1 + x^2$ on its outward branch, then $a = v\,dv/dx = (1 + x^2)(2x)$. Its resultant force is $F = ma = 4x(1 + x^2)$ in consistent SI units. To infer unknown parameters, substitute measured states into the integrated model and solve for the constants. Then check another state when available. A force coefficient has units chosen so the whole expression is a force, not necessarily newtons alone.

Check a proposed solution in the original equation

Differentiate the proposed $v(t)$ or $v(x)$, substitute into Newton's law, and check the initial value. For $v = 4/(1 + 2t)$, differentiation gives $dv/dt = -8/(1 + 2t)^2 = -v^2/2$, as required. Check signs, units and domains as well as algebra. A positive drag magnitude must oppose velocity; a squared-speed expression must stay nonnegative;

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logarithms need valid arguments; a new branch needs its own physical assumptions.

Decide which event ends the current model

Mark any turning point, crossing of a force-direction boundary, change of resistance law, or endpoint of the allowed motion. Decide whether a zero-speed state occurs at a finite time or only as a limit. A terminal solution can disappear when dividing by a zero factor, and a square root can hide a return branch. Restore such possibilities by checking the original equation and the order of events.

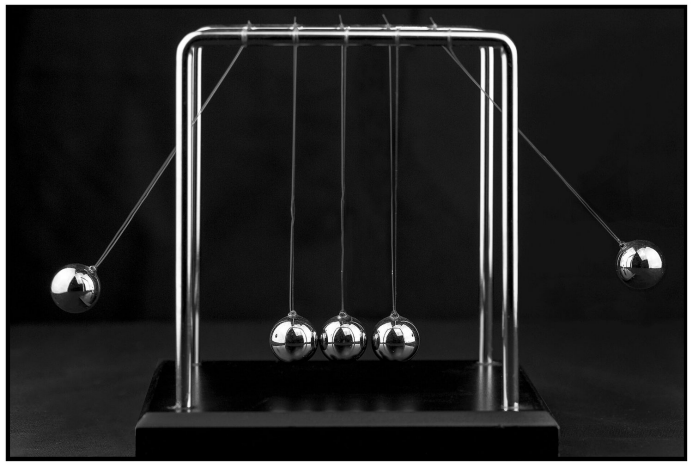
Keep the force law tied to the motion

Choose a signed direction, add the real forces, then select $a = dv/dt$ or $a = v\,dv/dx$ to suit the variables. Separate, integrate and apply the initial conditions at every stage. Distinguish force, acceleration, velocity and speed. Check each answer against the original equation, its branch and its limits; after a physical event, reconsider the model before continuing.

Momentum

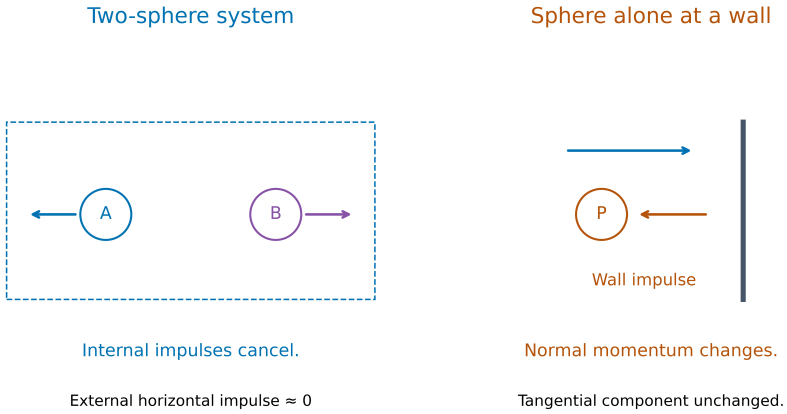
A bounce needs a system and a direction

A **collision** 碰撞 brings two low-friction carts into contact. Their velocities change, but their combined **momentum** 动量 stays constant if the external impulse during contact is negligible. A ball hitting a fixed wall is different: the wall gives the ball an impulse. Define your system before writing a conservation equation.



Decide which momentum is conserved

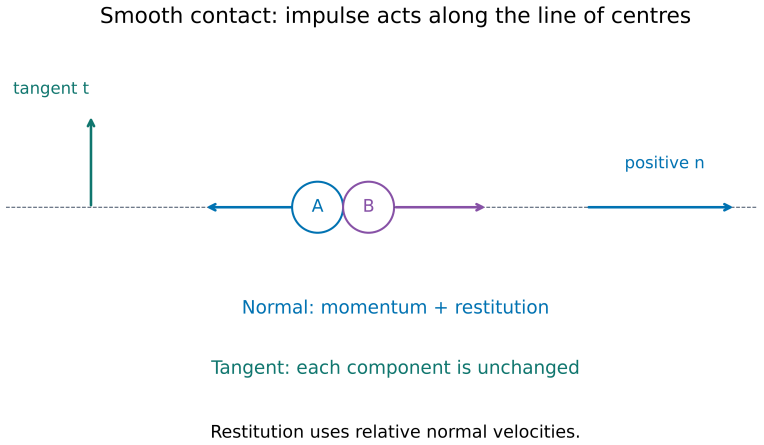
Linear momentum 线动量 is a vector: $\mathbf{p} = m\mathbf{v}$. The law is **conservation of linear momentum** 动量守恒, often shortened to **conservation of momentum**. It applies to a system with zero resultant external impulse, or to a component with zero external impulse in that direction. For two smooth spheres on a smooth horizontal plane, their mutual impulses cancel. For the sphere alone at a fixed wall, normal momentum changes; the wall belongs outside that one-sphere system.



Draw the line of impact

For two smooth spheres, the **line of impact** 碰撞线 is their **line of centres** 球心连线 at contact. The collision impulse acts along it. Call that direction normal, n, and a perpendicular direction tangent, t. A **direct impact** 正碰撞 has motion along n; an **oblique impact** 斜碰撞 also has tangential motion.

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Write restitution with signed components

Choose positive n from A towards B, with $u_{An} > u_{Bn}$ before contact. **Newton's experimental law** 牛顿实验定律 is

$$v_{Bn} - v_{An} = e(u_{An} - u_{Bn}).$$

The **coefficient of restitution** 恢复系数 e is normal separation speed divided by normal approach speed. In this syllabus model, $0 \le e \le 1$; use relative components, not the two objects' separate speed magnitudes.

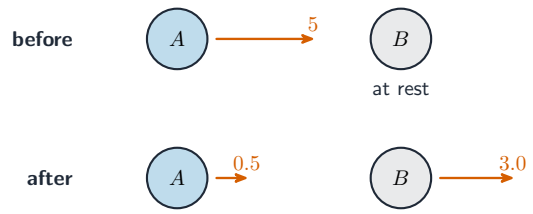
Interpret the two limiting cases

A **perfectly elastic** 完全弹性 impact has $e = 1$: **kinetic energy** 动能 is conserved for the ideal smooth collision. An **inelastic** 非弹性 impact with $e = 0$, also called **perfectly inelastic** 完全非弹性, gives equal final normal velocities. In a direct collision, $e = 0$ gives a common velocity. In an oblique collision, tangential velocities can differ, so equal normal velocities do not mean that the spheres become glued together.

A particle glancing along a smooth wall with $e = 0$ may keep moving along it. Apply the law to the normal component first.

Solve the original direct-collision example

A sphere A of mass 2 kg moves at 5 $\frac{\text{m}}{\text{s}}$ into stationary B of mass 3 kg on a smooth horizontal plane. Take rightwards positive and $e = 0.5$. Momentum gives $2v_A + 3v_B = 10$; restitution gives $v_B - v_A = 2.5$. Thus $v_A = 0.5$ $\frac{\text{m}}{\text{s}}$ and $v_B = 3$ $\frac{\text{m}}{\text{s}}$, both rightwards.



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

Use signed velocities in both equations.

A head-on collision needs negative velocities

Let A have mass 2 kg and velocity $+3\ \frac{\text{m}}{\text{s}}$, while B has mass 1 kg and velocity $-3\ \frac{\text{m}}{\text{s}}$. They approach at $6\ \frac{\text{m}}{\text{s}}$, not zero. With $e = 0.5$, $2v_A + v_B = 3$ and $v_B - v_A = 3$. Hence $v_A = 0$ and $v_B = 3\ \frac{\text{m}}{\text{s}}$; A is instantaneously at rest after impact.

Check whether contact can occur

If A is left of B and both move right, a first collision requires $u_A > u_B$. A negative answer for v_A means rebound, not a negative speed. After contact, check $v_B - v_A \geq 0$. With $e > 0$ the bodies separate; with $e = 0$ their normal separation speed is zero at that instant.

Derive a reusable direct-impact result

Solving normal momentum and restitution gives

$$v_A = \frac{(m_A - em_B)u_A + (1+e)m_B u_B}{m_A + m_B}.$$

Then $v_B = v_A + e(u_A - u_B)$. If B starts at rest, $v_B = \dfrac{(1+e)m_A u_A}{m_A + m_B}$; use the equations to check any memorised formula.

Infer a mass ratio from rebound

A sphere of mass m hits a stationary sphere of mass km in a perfectly elastic direct impact. If its final velocity is $-u/3$, restitution gives $v_B = 2u/3$. Momentum gives $mu = -mu/3 + km(2u/3)$, so $k = 2$. The sign of the rebound matters in both equations.

A momentum ratio is not a speed ratio

A sphere of mass $3m$ moving at u hits a stationary sphere of mass m. Afterwards $p_A = 2p_B$, both rightwards. Since total momentum is $3mu$, $p_A = 2mu$ and $p_B = mu$. Thus $v_A = 2u/3$, $v_B = u$ and $e = 1/3$. Dividing momenta without accounting for the masses would give the wrong velocity ratio.

Keep energy loss separate from momentum

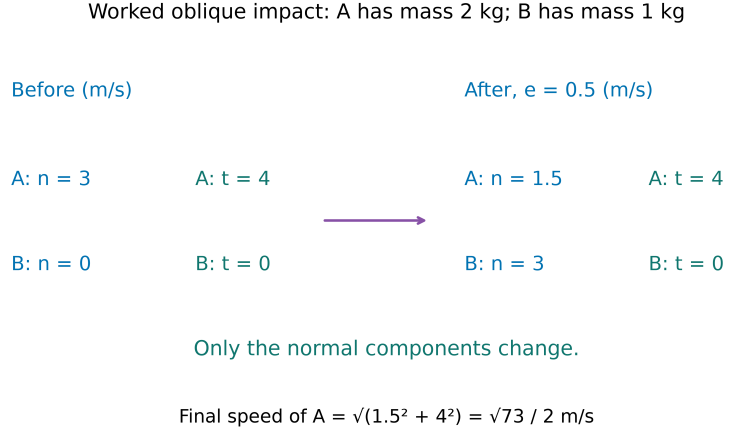
In the original 2 kg and 3 kg example, initial kinetic energy is 25 J and final kinetic energy is 13.75 J. The loss is 11.25 J, although momentum is conserved. For an isolated smooth two-sphere collision, the loss is $\frac{1}{2}\mu(1-e^2)(u_{An} - u_{Bn})^2$, where the **reduced mass** 约化质量 is $\mu = m_A m_B/(m_A + m_B)$. This checks a calculation; it does not replace the signed impact equations.

Resolve an oblique sphere impact

Smooth spheres exert no tangential impulse on each other. Therefore $v_{At} = u_{At}$ and $v_{Bt} = u_{Bt}$ separately; normal momentum and restitution determine v_{An}, v_{Bn}. Draw a component table before solving. Recombine each sphere's own components using $v = \sqrt{v_n^2 + v_t^2}$; the final normal component alone is usually not its speed.

Work through an oblique example

Sphere A has mass 2 kg and initial components $(3, 4)\ \frac{\text{m}}{\text{s}}$ in (n, t). Stationary B has mass 1 kg. For $e = 0.5$, $2v_{An} + v_{Bn} = 6$ and $v_{Bn} - v_{An} = 1.5$. Hence $\mathbf{v}_A = (1.5, 4)$ and $\mathbf{v}_B = (3, 0)$ in the same units. Their speeds are $\sqrt{73}/2$ and $3\ \frac{\text{m}}{\text{s}}$; A keeps its tangential component.



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Zero restitution does not remove tangential motion

Two equal-mass smooth spheres have initial components $\mathbf{u}_A = (4, 3)$ and $\mathbf{u}_B = (0, -1)$, with positive normal from A to B. If $e = 0$, both final normal velocities are 2. Their final vectors are $(2, 3)$ and $(2, -1)$ in the stated SI units. They have equal normal velocities but different tangential velocities; the particles do not move as a single rigid object.

Recover direction and deflection

For final components (v_n, v_t) with $v_n > 0$, the angle to positive normal satisfies $\tan\phi = v_t/v_n$. If $v_n < 0$, draw the quadrant; a calculator's acute angle alone can hide a rebound. In the previous $A = (1.5, 4)$ example, $\tan\phi = 8/3$. The **deflection** 偏转 is the change from its incoming direction, whose angle satisfies $\tan\theta = 4/3$, not simply ϕ itself.

Infer restitution from final kinetic energies

Equal-mass spheres collide with $\mathbf{u}_A = (4, 3)$ and $\mathbf{u}_B = (0, 0)$. The final normal velocities are $2(1 - e)$ and $2(1 + e)$, while their tangential velocities remain 3 and 0. Equal final kinetic energies require $4(1 - e)^2 + 9 = 4(1 + e)^2$, so $e = 9/16$. Both final speeds are $25/8$ $\frac{\text{m}}{\text{s}}$; keep both components in the energy equation.

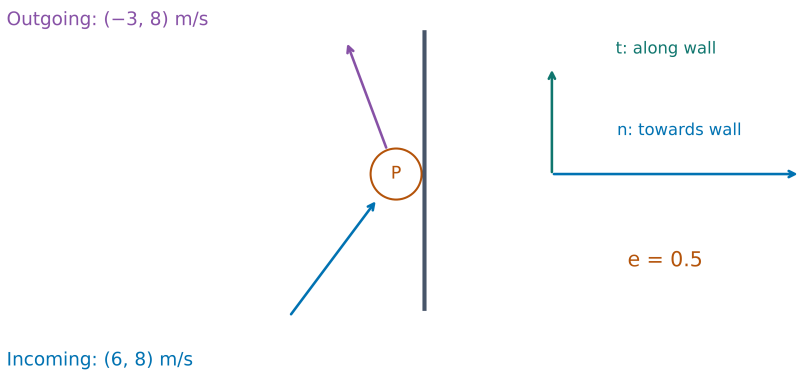
A fixed wall changes the sphere's momentum

At a smooth **fixed surface** 固定表面, take positive normal towards the wall. The surface has zero velocity, so the sphere's final component is $v_n = -eu_n$ and its tangential component is $v_t = u_t$. Do not conserve the sphere's normal momentum. The wall's impulse supplies the change: $\Delta p_n = -m(1 + e)u_n$. The wall–Earth system can take momentum while its speed is negligible.

Recombine components after a wall impact

A 2 kg sphere approaches a smooth fixed wall with components $(6, 8)$ $\frac{\text{m}}{\text{s}}$ in normal/tangent directions. For $e = 0.5$, the outgoing components are $(-3, 8)$. Speed changes from 10 to $\sqrt{73}$ $\frac{\text{m}}{\text{s}}$. Kinetic energy loss is $100 - 73 = 27$ J; the overall speed ratio is not e.

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Signed normal reverses and scales; tangent stays the same.

State which line an angle uses

If incoming and outgoing acute angles α, β are measured from the wall, $u_n = u\sin\alpha$ and $u_t = u\cos\alpha$. Thus $\tan\beta = e\tan\alpha$. If θ, ϕ are measured from the normal, $\tan\phi = \tan\theta/e$ for $e > 0$. At $e = 0$, use components directly; a nonzero tangent component leaves motion parallel to the wall.

Name the reference line before resolving

Angles from wall	Angles from normal
tan β = e tan α	tan φ = tan θ / e
Normal = u sin α	Normal = u cos θ
Tangent = u cos α	Tangent = u sin θ

For e = 0, use components; do not divide by e.

Infer a wall coefficient from angles

A sphere strikes a smooth fixed wall at $45°$ to the wall and leaves at angle β, with $\tan\beta = 1/3$. Then $e = \tan\beta/\tan 45° = 1/3$. Its final speed is $u\sqrt{\cos^2 45° + e^2 \sin^2 45°} = u\sqrt{5}/3$. A change in direction is why the speed ratio differs from $1/3$.

Check wall-impact energy and limits

At a smooth fixed wall, lost kinetic energy is $\frac{1}{2}m(1-e^2)u_n^2$. Only the normal contribution changes; at $e = 1$, speed is unchanged and the normal component reverses. At $e = 0$, normal speed becomes zero. If $u_t \neq 0$, the sphere continues along the wall; if both final components are zero, it is at rest immediately after contact.

A time ratio can reveal restitution

A particle travels between two parallel smooth fixed walls separated by d. If its first flight takes T, its normal speed is d/T. After a bounce with coefficient $e > 0$, the next flight takes T/e. Thus if the second time is kT, $e = 1/k$ and $k \geq 1$. The case $e = 0$ gives no finite second crossing; do not cancel a zero normal speed as though a return occurs.

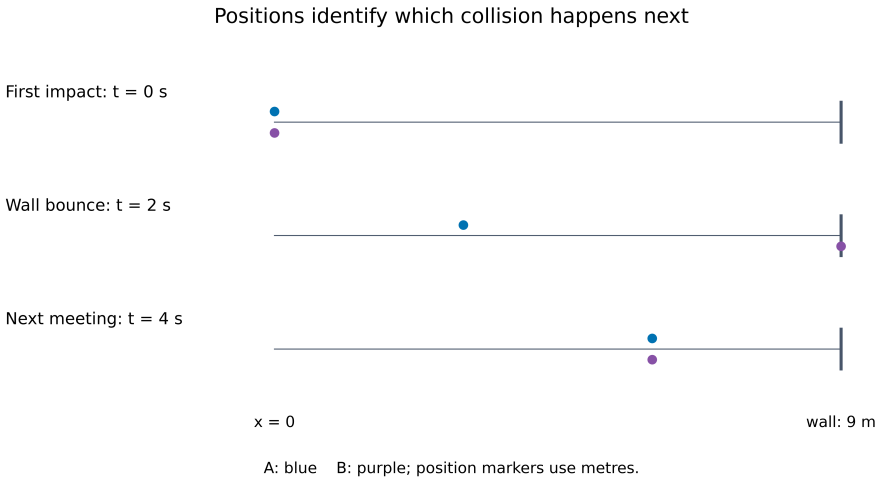
Treat successive impacts as separate events

Equal-mass particles collide directly: A initially has velocity $6\ \frac{\text{m}}{\text{s}}$, B is at rest and $e_{AB} = 0.5$. They leave with velocities 1.5 and $4.5\ \frac{\text{m}}{\text{s}}$. If B then hits a smooth wall with $e_{BW} = 1/3$, its velocity becomes $-1.5\ \frac{\text{m}}{\text{s}}$. Their next mutual impact uses incoming velocities $+1.5, -1.5$ and e_{AB}, giving $v_A = -0.75$, $v_B = +0.75\ \frac{\text{m}}{\text{s}}$.

Check the next event using distances

In that example, put the first mutual collision at $x = 0$ and the wall at $x = 9$ m. Particle B reaches the wall at $t = 2$ s; then A is at $x = 3$ m. The gap is 6 m and their closing speed after the wall bounce is $3\ \frac{\text{m}}{\text{s}}$. They meet 2 s later at $x = 6$ m, before A can reach the wall. Different pairs can have different restitution coefficients.

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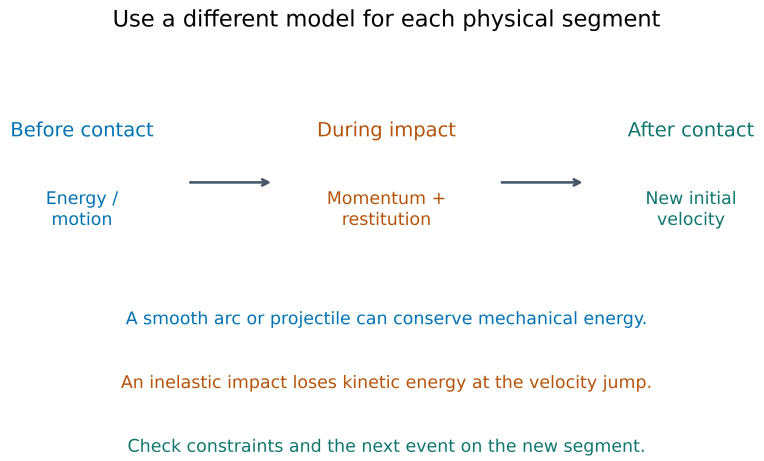


Use a fresh projectile segment after a barrier

A particle has incoming velocity $(4, -4)\ \frac{\text{m}}{\text{s}}$ at a smooth barrier inclined at $45°$ upwards to the right. Its velocity is purely normal to the barrier; $e = 0.5$ gives outgoing velocity $(-2, 2)\ \frac{\text{m}}{\text{s}}$. If impact is 4 m above horizontal ground and $g = 10\ \frac{\text{m}}{\text{s}^2}$, solve $0 = 4 + 2t - 5t^2$. The positive flight time is $(1+\sqrt{21})/5$ s and horizontal distance is $2(1+\sqrt{21})/5$ m to the left.

Link a collision to a vertical circle

At a circle's bottom, a particle of mass m and incoming speed $3\sqrt{5ag}$ hits a stationary particle of mass $2m$ in a perfectly elastic direct impact. Momentum and restitution give signed velocities $-\sqrt{5ag}$ and $2\sqrt{5ag}$. The first particle rebounds with squared speed $5ag$. Its later energy equation gives top squared speed ag, the limiting $T = 0$ condition for a taut string. Use energy on the smooth arcs and collision laws at contact; zero tension is a boundary test, not by itself proof of subsequent separation.



■ Reject a physically impossible model

An algebraic candidate must satisfy $0 \leq e \leq 1$, the stated incoming approach and outgoing directions. Reject inadmissible roots with a reason; do not turn a negative signed velocity into a positive one midway through the equations. Check momentum for the correct system, the restitution relation, unchanged smooth tangential components and nonnegative kinetic-energy loss. A fixed wall can violate one-sphere momentum conservation without violating the model.

■ Organise an independent impact solution

Draw the contact normal, define signed components and label the system. Write the normal momentum equation only where justified, then restitution and unchanged tangential components. Recombine velocities, state directions and check the next physical event. For a question combining topics, carry the outgoing velocity into a new motion segment with its own initial state.

■ Retrieve the model before calculating

Direct and oblique impacts use the same normal equations; tangential components survive smooth contact unchanged. For a fixed surface, $v_n = -eu_n$, not conservation of the sphere's normal momentum. Always define the system, the angle convention and the event order. Distinguish normal separation from total speed, equal normal velocity from sticking, and energy between impacts from energy across them.

Common mistakes.

- Using the same sign for g in both directions: horizontal acceleration is 0, vertical is $-g$ with up positive.

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- Treating speed and a velocity component as the same: speed needs both components.
- Taking moments with the full force instead of its perpendicular component, or about a point and then forgetting the forces through it.
- Subtracting a removed part's area but not its moment when finding a centre of mass.
- Adding a separate "centripetal force" to the force diagram; it is the resultant of the real forces.
- Using $v^2 = gr$ at the top of a vertical circle for a particle on a rod, which can have zero speed at the top.
- Confusing natural length with stretched length in Hooke's law: $T = \dfrac{\lambda x}{l}$ uses the extension x.
- Writing restitution as $v_A - v_B$: it is speed of separation over speed of approach, with signs fixed by one positive direction.

Practice

Answer each before reading the answers below.

1. A ball is projected from level ground at 25 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{3}{4}$. Find the time of flight, the range and the greatest height.
2. A stone is thrown horizontally at 12 m s^{-1} from the top of a cliff 45 m high. Find the time to reach the sea, the horizontal distance travelled and the speed on impact.
3. Find the equation of the trajectory of a particle projected from the origin at 20 m s^{-1} at 45° above the horizontal.
4. A uniform rod AB, of length 4 m and weight 60 N, is hinged to a wall at A. It is held horizontal by a light string attached at B, which makes an angle of 30° with the rod. Find the tension and the horizontal and vertical components of the force at the hinge.
5. A uniform lamina is a 6 by 4 rectangle with corners $(0, 0)$ and $(6, 4)$, from which the square with corners $(4, 2)$ and $(6, 4)$ is removed. Find its centre of mass.
6. A particle of mass 0.2 kg on a light string of length 0.5 m moves in a horizontal circle with the string at 60° to the vertical. Find the tension and the angular speed.
7. A particle on a light string of length 0.8 m moves in a vertical circle. Find the least speed at the top for the string to stay taut, and the speed at the bottom in that case.
8. A particle of mass 1.2 kg hangs in equilibrium on an elastic string of natural length 0.6 m and modulus 30 N. Find the extension and the elastic potential energy stored.

9. A particle of mass 2 kg starts from rest and is acted on by a force $6t$ N in a fixed direction. Find its speed and the distance travelled when $t = 2$.
10. Sphere A, of mass 2 kg and speed 6 m s^{-1}, hits sphere B, of mass 3 kg, at rest, directly. The coefficient of restitution is 0.5. Find the speeds after impact and the loss of kinetic energy.

Answers

1. $u_x = 20$, $u_y = 15$. Time of flight: $15t - 5t^2 = 0$, so $t = 3$ s. Range $= 20 \times 3 = 60$ m. Greatest height $= \dfrac{15^2}{2 \times 10} = 11.25$ m.
2. $5t^2 = 45$, so $t = 3$ s. Distance $= 12 \times 3 = 36$ m. At impact $v_y = 30$, so speed $= \sqrt{12^2 + 30^2} = \sqrt{1044} \approx 32.3$ m s^{-1}.
3. $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha} = x - \dfrac{10x^2}{2 \times 400 \times \frac{1}{2}} = x - \dfrac{x^2}{40}$.
4. Moments about A: $T\sin 30° \times 4 = 60 \times 2$, so $T = 60$ N. The string's horizontal pull on the rod, $T\cos 30°$, points towards the wall, so the hinge pushes the rod away from the wall with $30\sqrt{3} \approx 52.0$ N. Vertical component $= 60 - T\sin 30° = 30$ N upwards.
5. Areas: rectangle 24, square 4, lamina 20. $\bar{x} = \dfrac{24(3) - 4(5)}{20} = 2.6$ and $\bar{y} = \dfrac{24(2) - 4(3)}{20} = 1.8$, so the centre of mass is at $(2.6, 1.8)$.
6. Vertically: $T\cos 60° = 0.2 \times 10$, so $T = 4$ N. Radius $r = 0.5\sin 60°$. Horizontally: $T\sin 60° = 0.2\,\omega^2(0.5\sin 60°)$, so $4 = 0.1\,\omega^2$, $\omega^2 = 40$ and $\omega \approx 6.32$ rad s^{-1}.
7. At the top the tension can just be zero: $mg = \dfrac{mv^2}{r}$, so $v^2 = gr = 8$ and $v = 2\sqrt{2} \approx 2.83$ m s^{-1}. Energy: $v_{\text{bottom}}^2 = 8 + 2(10)(1.6) = 40$, so $v \approx 6.32$ m s^{-1}.
8. $T = mg = 12$ N and $T = \dfrac{\lambda x}{l} = \dfrac{30x}{0.6}$, so $x = 0.24$ m. EPE $= \dfrac{\lambda x^2}{2l} = \dfrac{30 \times 0.24^2}{1.2} = 1.44$ J.
9. $a = 3t$, so $v = \frac{3}{2}t^2$ (from rest) and $x = \frac{1}{2}t^3$. At $t = 2$: $v = 6$ m s^{-1} and $x = 4$ m.
10. Momentum: $12 = 2v_A + 3v_B$. Restitution: $v_B - v_A = 0.5 \times 6 = 3$. So $12 = 5v_A + 9$, $v_A = 0.6$ m s^{-1} and $v_B = 3.6$ m s^{-1}, both in the original direction. KE before $= 36$ J; after $= 0.36 + 19.44 = 19.8$ J; loss $= 16.2$ J.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.

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- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v\frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.

Further Probability & Statistics

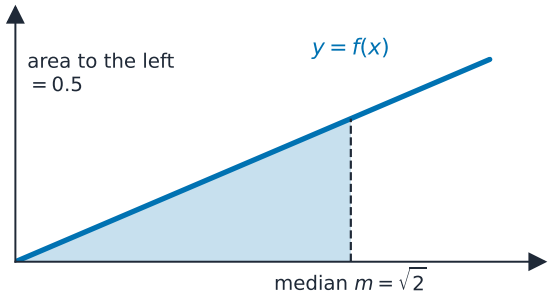
A-Level Further Mathematics

This handout covers Topic 4: **Further Probability & Statistics** 进阶概率统计. It adds continuous distributions, small-sample inference, the chi-squared and non-parametric tests, and probability generating functions.

Continuous random variables

A continuous variable X is described by a **probability density function** 概率密度函数 $f(x)$, which may be defined piecewise. The probability over a range is the area under f, and the mean of any function of X is

$$E(g(X)) = \int g(x)\, f(x)\, dx.$$

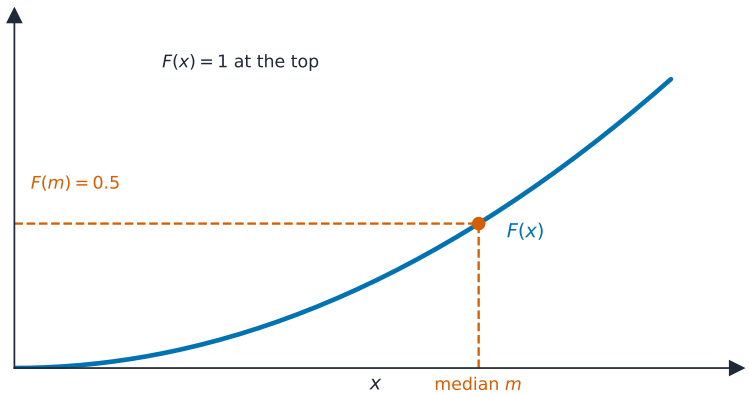


median m solves $F(m) = \int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$

The median sits where the area under $f(x)$ to its left is exactly 0.5.

The **cumulative distribution function** 累积分布函数 $F(x) = P(X \leqslant x)$ is the running total: $F(x) = \int_{-\infty}^{x} f(t)\,dt$, and $f(x) = F'(x)$ wherever the density is continuous. Use F to find probabilities and **percentiles** 百分位数 (for example, the **median** 中位数 solves $F(x) = 0.5$).

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$F(x) = P(X \le x) = \int_{-\infty}^{x} f$ · $0 \le F \le 1$ · median: $F(m) = \frac{1}{2}$

The cumulative graph $F(x)$ rises from 0 *to* 1*; the height* 0.5 *is reached at the median.*

Worked example. A variable has $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$. Find the median.

The cumulative distribution function is $F(x) = \int_0^x \frac{1}{2}t\,dt = \frac{1}{4}x^2$. Set $F(m) = 0.5$:

$$\tfrac{1}{4}m^2 = 0.5 \;\Rightarrow\; m^2 = 2 \;\Rightarrow\; m = \sqrt{2} = 1.41.$$

Worked example: mean, variance and $E(g(X))$. For the same $f(x) = \frac{1}{2}x$ on $[0, 2]$:

$$E(X) = \int_0^2 x \cdot \tfrac{1}{2}x\,dx = \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}, \qquad E(X^2) = \int_0^2 x^2 \cdot \tfrac{1}{2}x\,dx = \left[\frac{x^4}{8}\right]_0^2 = 2,$$

$$\operatorname{Var}(X) = E(X^2) - \big(E(X)\big)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

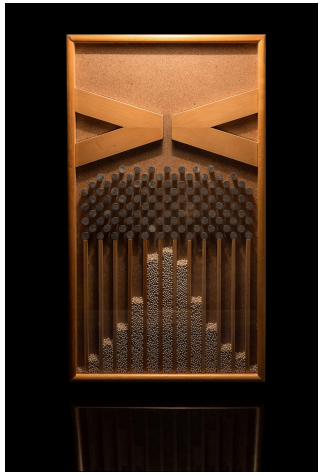
The same rule handles any g: $E\left(\frac{1}{X}\right) = \int_0^2 \frac{1}{x} \cdot \frac{1}{2}x\,dx = \int_0^2 \frac{1}{2}\,dx = 1$. Note that $E\left(\frac{1}{X}\right) = 1$ is **not** $\dfrac{1}{E(X)} = \frac{3}{4}$.

The distribution of a related variable. If $Y = g(X)$, find Y's distribution through its cumulative function. For $Y = X^2$: $F_Y(y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, then differentiate for $f_Y = F_Y'$. When g is monotonic increasing there is a shortcut, $F_Y(y) = F_X\big(g^{-1}(y)\big)$.

Worked example. With $f_X(x) = \frac{1}{2}x$ on $[0, 2]$ (so $F_X(x) = \frac{1}{4}x^2$), let $Y = X^2$. For $0 \leqslant y \leqslant 4$, $F_Y(y) = F_X(\sqrt{y}) = \frac{1}{4}y$, so $f_Y(y) = F_Y'(y) = \frac{1}{4}$ – that is, Y is uniform on $[0, 4]$.

For $Y = X^3$ (the syllabus example) the range becomes $0 \leqslant y \leqslant 8$, and $F_Y(y) = F_X\left(y^{1/3}\right) = \frac{1}{4}y^{2/3}$, so $f_Y(y) = \frac{1}{6}y^{-1/3}$. Always state the new range of Y first.

Inference using normal and t-distributions



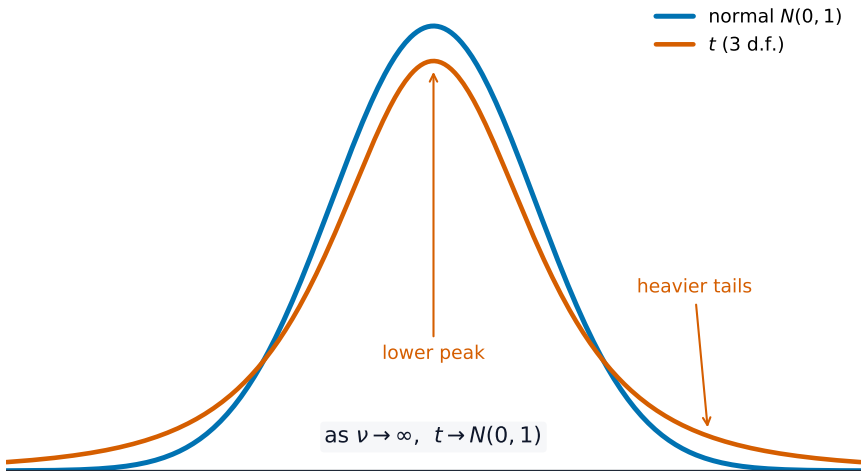
A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

For an independent random sample from a normal population with unknown variance, base your **hypothesis test** 假设检验 for the mean on the t-distribution. This assumption is especially important for a small sample. The same idea gives a **confidence interval** 置信区间 for the mean:

$$\bar{x} \pm t\,\frac{s}{\sqrt{n}},$$

where t comes from the t-tables with $n - 1$ degrees of freedom. To compare two populations, use a **two-sample** (2-sample) or **paired-sample** t-test, after finding a **pooled estimate** 合并估计 of the shared variance when appropriate.

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For a small sample the t-distribution is flatter with heavier tails, so its critical values are larger.

Worked example. An independent random sample of $n = 10$ from a normal population has mean $\bar{x} = 50$ and sample standard deviation $s = 4$ (using the $n - 1$ divisor). Find a 95% confidence interval for the mean (use $t = 2.262$ for 9 degrees of freedom).

$$50 \pm 2.262 \times \frac{4}{\sqrt{10}} = 50 \pm 2.86 \;\Rightarrow\; (47.1,\ 52.9).$$

Worked example: one-sample t-test. The same sample is used to test a claim that the population mean is 52, at the 5% level, assuming the population is normal. $H_0 : \mu = 52$, $H_1 : \mu \neq 52$.

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{50 - 52}{4/\sqrt{10}} = -1.58.$$

The critical values are ± 2.262 (9 degrees of freedom, two-tailed 5%). Since $|-1.58| < 2.262$, do not reject H_0: there is insufficient evidence that the mean differs from 52. This agrees with the interval, which contains 52.

Worked example: two-sample t-test with a pooled variance. Two independent samples from normal populations with a common variance: sample A has $n_A = 8$, $\bar{x}_A = 24.5$, $s_A^2 = 4.2$; sample B has $n_B = 10$, $\bar{x}_B = 22.1$, $s_B^2 = 5.0$ (unbiased estimates). Test $H_0 : \mu_A = \mu_B$ against $H_1 : \mu_A \neq \mu_B$ at the 5% level.

The pooled estimate weights each variance by its degrees of freedom:

$$s_p^2 = \frac{(n_A-1)s_A^2 + (n_B-1)s_B^2}{n_A+n_B-2} = \frac{7(4.2)+9(5.0)}{16} = \frac{74.4}{16} = 4.65.$$

$$t = \frac{\bar{x}_A - \bar{x}_B}{\sqrt{s_p^2\left(\frac{1}{n_A}+\frac{1}{n_B}\right)}} = \frac{24.5-22.1}{\sqrt{4.65\left(\frac{1}{8}+\frac{1}{10}\right)}} = \frac{2.4}{1.023} = 2.35.$$

With 16 degrees of freedom the two-tailed 5% critical value is 2.120. Since $2.35 > 2.120$, reject H_0: there is evidence that the population means differ. The matching 95% confidence interval for $\mu_A - \mu_B$ is $2.4 \pm 2.120(1.023)$, which is $(0.23,\ 4.57)$, and it excludes 0.

Worked example: paired t-test. Six runners are timed before and after a training plan. The improvements (before minus after, in seconds) are 1.2, 0.8, −0.3, 1.5, 0.9, 0.6. The same runners appear twice, so work with the **differences** as one sample: $\bar{d} = 0.783$ and $s_d = 0.618$. With $H_0 : \mu_d = 0$ and $H_1 : \mu_d > 0$,

$$t = \frac{0.783}{0.618/\sqrt{6}} = 3.11.$$

The one-tailed 5% critical value for 5 degrees of freedom is 2.015. Since $3.11 > 2.015$, reject H_0: there is evidence that the plan improves times, assuming the differences are normally distributed. A two-sample test here would be wrong, because the two sets of times are not independent.

Choosing the test. Same individuals measured twice → paired t. Two independent small samples from normal populations with a common variance → pooled two-sample t. Large samples (or known variances) → a test using the **normal distribution**, with $\sqrt{\frac{s_A^2}{n_A}+\frac{s_B^2}{n_B}}$ for independent large samples. If the population variances are known, use those variances in place of the sample estimates; exact small-sample normal inference also needs normal populations.

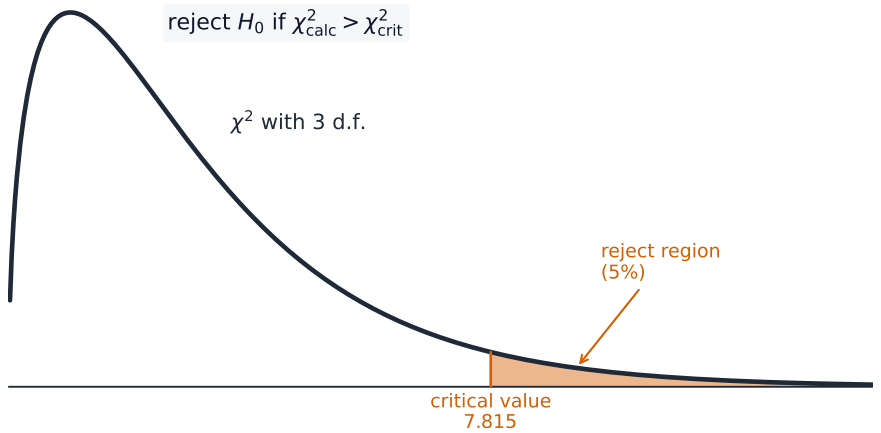
Chi-squared tests

A χ^2-test (**chi-squared test** 卡方检验) compares observed counts O with expected counts E from a **theoretical distribution** 理论分布:

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

Use independent observations, with positive expected counts large enough for the chi-squared approximation; combine suitable classes if needed. Compare with a table value for the right number of **degrees of freedom** 自由度. Two uses: a **goodness of fit** 拟合优度 test (does the data follow the proposed model?), and a test for **independence** 独立性 of two variables in a **contingency table** 列联表.

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The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5% tail.

Worked example. Four equally likely categories give observed counts 20, 30, 25, 25 (so each expected count is 25). Test the fit at the 5% level.

$$\chi^2 = \frac{(20-25)^2+(30-25)^2+0+0}{25} = \frac{25+25}{25} = 2.$$

With $4-1=3$ degrees of freedom the table value is 7.815. Since $2 < 7.815$, do not reject the model.

Worked example: fitting a prescribed distribution. 80 families with three children are counted by number of girls. Test, at the 5% level, whether $B(3,\ 0.5)$ fits.

Girls	0	1	2	3
O	6	28	34	12
$E = 80 \times P$	$80 \times \frac{1}{8} = 10$	$80 \times \frac{3}{8} = 30$	30	10

$$\chi^2 = \frac{(6-10)^2}{10} + \frac{(28-30)^2}{30} + \frac{(34-30)^2}{30} + \frac{(12-10)^2}{10} = 1.6 + 0.133 + 0.533 + 0.4 = 2.67.$$

All expected values are at least 5, so no classes are combined. The degrees of freedom are $4-1=3$ (critical value 7.815); since $2.67 < 7.815$, do not reject $B(3,\ 0.5)$. If p had been **estimated from the data**, one more degree of freedom is lost: $\nu = 4-1-1 = 2$.

Worked example: contingency table. 100 students are classified by course and by whether they passed.

	Pass	Fail	Total
Course A	30	20	50
Course B	20	30	50
Total	50	50	100

Each expected value is $\dfrac{\text{row total} \times \text{column total}}{\text{grand total}} = \dfrac{50 \times 50}{100} = 25$, so

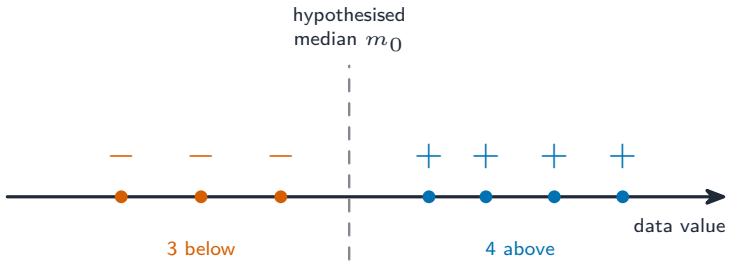
$$\chi^2 = \frac{(30-25)^2 + (20-25)^2 + (20-25)^2 + (30-25)^2}{25} = \frac{100}{25} = 4.$$

There is $(2-1)(2-1) = 1$ degree of freedom, and the 5% critical value is 3.841. Since $4 > 3.841$, reject H_0: there is evidence that the result is not independent of the course. (Yates' correction is not required in this syllabus.)

Non-parametric tests

A **non-parametric test** 非参数检验 does not require a normal population, but still has sampling assumptions; the signed-rank test also needs a symmetric distribution of differences to test its centre. The basic ones are:

- the **sign test** 符号检验: count how many values fall above and below a proposed median, and test those counts with a binomial model;
- the **Wilcoxon signed-rank test** 威尔科克森符号秩检验 (the **matched-pairs** test for paired data), which also uses the sizes of the differences, not just their signs;
- the **Wilcoxon rank-sum test** 威尔科克森秩和检验, for comparing two separate samples.



The sign test counts how many values fall above and below the proposed median, then tests those counts with a binomial model

Worked example. Test whether a median is 5. In a sample of 10 values (none equal to 5), 9 lie **above** 5 and 1 lies below. Test at the 5% level (two-tailed).

Under H_0 (median $= 5$) the number above follows $B(10, 0.5)$. The observed result (9 above) is extreme, so find $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + (0.5)^{10} = \dfrac{11}{1024} = 0.0107$. For a two-tailed test compare with $\frac{1}{2}(5\%) = 0.025$. Since $0.0107 < 0.025$, **reject** H_0: there is evidence the median is not 5.

Worked example: Wilcoxon signed-rank test. A machine is set so that the median mass of a bag is 20 g. Eight bags have masses 23.1, 18.4, 26.0, 21.7, 19.5, 27.3, 24.8, 22.2 g. Test at the 5% level whether the median is greater than 20 g, assuming the distribution is symmetric.

Subtract 20, then rank the differences by **size**, ignoring sign:

Difference	−0.5	−1.6	1.7	2.2	3.1	4.8	6.0	7.3
Rank	1	2	3	4	5	6	7	8

The sum of the negative ranks is $P_- = 1+2 = 3$; the positive ranks sum to $36-3 = 33$. Under H_1 (median > 20) negative differences should be rare, so the test statistic is $T = 3$. For $n = 8$ the one-tailed 5% critical value is 5, and $3 \leqslant 5$, so reject H_0: there is evidence that the median mass is above 20 g. A sign test on the same data (6 of 8 above) gives $P(X \geqslant 6) = 0.145$ and would **not** reject: using the sizes can give the Wilcoxon test greater power under its symmetry assumption; this one sample does not establish a general power comparison.

For n larger than the tables, a sum of positive ranks (or a sum of negative ranks) has the following approximate normal distribution under the null, when there are no zero differences or tied absolute differences. Use a continuity correction; ties need adjusted variance. The minimum of both rank sums is not itself distributed this way:

$$T \sim N\left(\frac{n(n+1)}{4},\ \frac{n(n+1)(2n+1)}{24}\right) \text{ approximately.}$$

Worked example: Wilcoxon rank-sum test. Two independent samples of reaction times (ms). A: 12, 15, 18, 21, 25. B: 14, 19, 22, 26, 28, 30. Test at the 5% level (two-tailed) whether the populations are identical.

Rank all 11 values together: 12(1), 14(2), 15(3), 18(4), 19(5), 21(6), 22(7), 25(8), 26(9), 28(10), 30(11). The smaller sample is A ($m = 5$, $n = 6$); its rank sum is $R_m = 1+3+4+6+8 = 22$. Also find $m(m+n+1) - R_m = 5(12) - 22 = 38$, and take the smaller: $W = 22$.

For $m = 5$, $n = 6$ the two-tailed 5% critical value is 18. Since $22 > 18$, do not reject H_0: there is insufficient evidence of a difference between the populations.

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Probability generating functions



Dice: a starting point for probability and discrete random variables

The **probability generating function** 概率母函数 of a non-negative integer-valued variable X is

$$G(t) = E(t^X) = \sum_x P(X = x)\, t^x.$$

It packs the whole distribution into one function. When the corresponding moments are finite, the mean and variance come from its derivatives at $t = 1$: $E(X) = G'(1)$ and $\mathrm{Var}(X) = G''(1) + G'(1) - \big(G'(1)\big)^2$. Also, the PGF of a sum of independent variables is the product of their PGFs.

> **Worked example.** X has $P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$. Find $E(X)$ using the PGF.

Here $G(t) = 0.5 + 0.3t + 0.2t^2$, so $G'(t) = 0.3 + 0.4t$ and

$$E(X) = G'(1) = 0.3 + 0.4 = 0.7.$$

Differentiate again for the variance: $G''(t) = 0.4$, so

$$\mathrm{Var}(X) = G''(1) + G'(1) - \big(G'(1)\big)^2 = 0.4 + 0.7 - 0.49 = 0.61.$$

(Check directly: $E(X^2) = 0.3 + 0.8 = 1.1$ and $1.1 - 0.7^2 = 0.61$.)

Standard PGFs (with $q = 1 - p$): discrete uniform on $1, \dots, n$: $\dfrac{t(1 - t^n)}{n(1 - t)}$; binomial $B(n, p)$: $(q + pt)^n$; geometric Geo(p) on $1, 2, \dots$: $\dfrac{pt}{1 - qt}$; Poisson Po(λ): $e^{\lambda(t-1)}$.

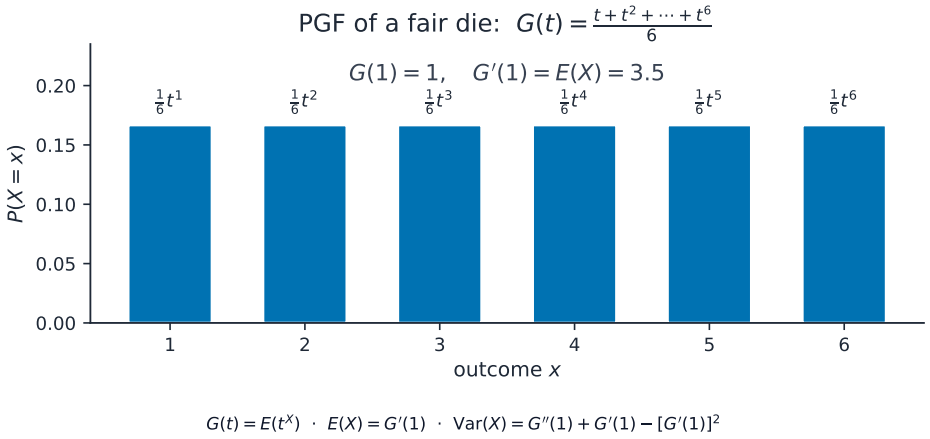
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> **Worked example: Poisson mean and variance.** For $G(t) = e^{\lambda(t-1)}$: $G'(t) = \lambda e^{\lambda(t-1)}$ and $G''(t) = \lambda^2 e^{\lambda(t-1)}$, so $G'(1) = \lambda$ and $G''(1) = \lambda^2$. Then $E(X) = \lambda$ and $\mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$.

> **Worked example: sum of independent variables.** X_1 and X_2 are independent copies of the variable above. The PGF of $Y = X_1 + X_2$ is the product:

$$G_Y(t) = \big(0.5 + 0.3t + 0.2t^2\big)^2 = 0.25 + 0.3t + 0.29t^2 + 0.12t^3 + 0.04t^4.$$

So, for example, $P(Y = 2) = 0.29$: the t^2 coefficient collects 0.5×0.2 twice and 0.3×0.3 once. The coefficients add to 1, a quick check.



PGF of a fair die packs equal masses at 1 through 6 into $G(t)$

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int x f(x)\, dx$.
- Use the **t-distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum (O - E)^2/E$, compare with the critical value at the right degrees of freedom, and **combine classes with** $E < 5$.
- State H_0 and H_1 and give the conclusion **in context** for every test.