Course Companion

A-Level Further Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

Contents 目录

Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics** 进阶纯数学 1. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

For a polynomial equation, the **roots** 根 are linked to the **coefficients** 系数. For $ax^2 + bx + c = 0$ with roots α, β:

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

For a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ:

$$\sum \alpha = -\frac{b}{a}, \qquad \sum \alpha\beta = \frac{c}{a}, \qquad \alpha\beta\gamma = -\frac{d}{a}.$$

Sums like $\sum \alpha$ and $\sum \alpha\beta$ are **symmetric functions** 对称函数 of the roots (unchanged if the roots are swapped). To find an equation whose roots are changed in a simple way, use a **substitution** 代换 (for example, put $w = \alpha + 1$).

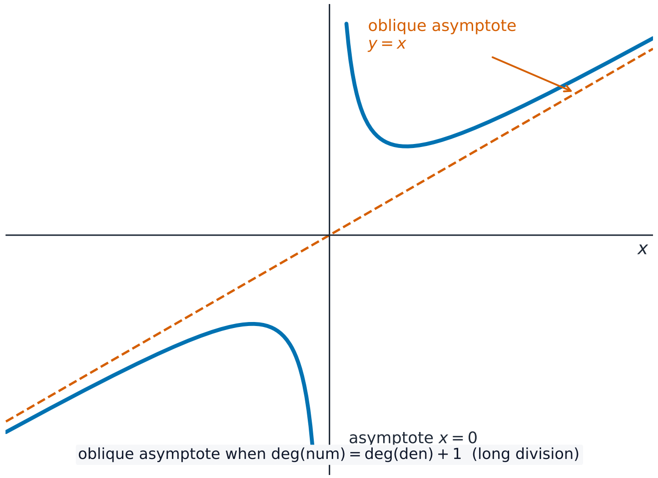
Worked example. The equation $x^2 - 5x + 6 = 0$ has roots α, β. Find the equation with roots $\alpha + 1, \beta + 1$.

Here $\alpha + \beta = 5$ and $\alpha\beta = 6$. The new sum is $(\alpha + 1) + (\beta + 1) = 7$, and the new product is $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$. So the new equation is

$$x^2 - 7x + 12 = 0.$$

Rational functions and graphs

A **rational function** 有理函数 is a fraction of two polynomials. When the top has degree exactly one higher than the bottom, the graph has an **oblique asymptote** 斜渐近线 (a slanted line the curve approaches); find it by dividing out. You should also be able to relate the graph of $y = f(x)$ to those of $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$. Find the set of values the function can take using the **discriminant** 判别式, and locate its **turning points** 驻点.



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertical asymptote.

Worked example. Find the oblique asymptote of $y = \dfrac{x^2+x+1}{x+1}$.

Divide out the fraction: $\dfrac{x^2+x+1}{x+1} = x + \dfrac{1}{x+1}$. As $x \to \pm\infty$ the remainder $\dfrac{1}{x+1} \to 0$, so the curve approaches the line $y = x$ — the oblique asymptote.

Summation of series

Learn the standard results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

The **method of differences** 差分法 sums a series by cancelling middle terms: if **partial fractions** 部分分式 write each term as $f(r) - f(r+1)$, almost everything cancels. From the sum to n terms you can see whether a series is **convergent** 收敛 and, if so, find its **sum to infinity** 无穷和.

Worked example. Find $\sum_{r=1}^{n}(2r+1)$.

$$\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = 2\cdot\tfrac{1}{2}n(n+1) + n = n(n+1) + n = n(n+2).$$

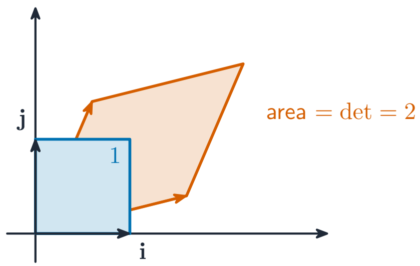
Matrices

A **matrix** 矩阵 is a rectangular block of numbers. **Matrix addition** adds entries in matching positions; you can also subtract and multiply matrices (multiplication is row × column). The **zero matrix** 零矩阵 has every entry 0, and the **identity matrix** 单位矩阵 (or **unit matrix**) I leaves any matrix unchanged under multiplication.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** 行列式 is $\det A = ad - bc$. If $\det A \neq 0$ the matrix is non-singular (otherwise it is a **singular matrix** 奇异矩阵), and the **inverse matrix** 逆矩阵 is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a product, $(AB)^{-1} = B^{-1}A^{-1}$. A 2×2 matrix can represent a **geometric transformation** 几何变换 of the plane — a **rotation** 旋转, **reflection** 反射, **enlargement** 放大, **stretch** 拉伸 or **shear** 切变: the determinant gives the area scale factor, and a product AB means "do B, then A". Points or lines that do not move are called **invariant points** 不变点 and **invariant lines** 不变直线.



The matrix sends the unit square to a parallelogram whose area is the determinant.

Worked example. Find the inverse of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \times 4 - 1 \times 2 = 10$, so

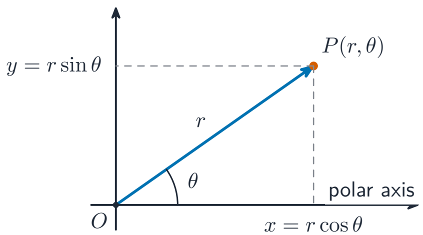
$$A^{-1} = \frac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

Polar coordinates

A spiral staircase: spirals are described naturally using polar coordinates.

Polar coordinates 极坐标 give a point by its distance r from the origin and its angle θ. They link to **Cartesian coordinates** 直角坐标 by

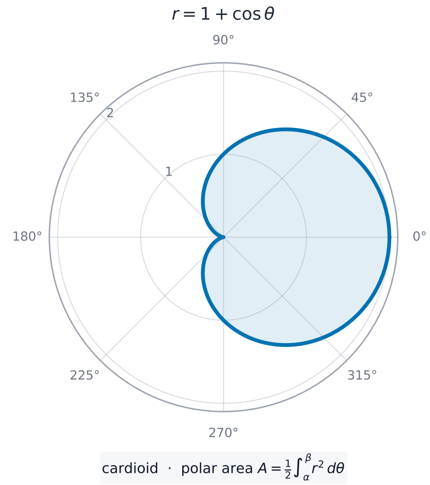
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2.$$



Polar coordinates (r,θ) convert to Cartesian by $x = r\cos\theta$ and $y = r\sin\theta$.

You should sketch simple **polar curves** 极坐标曲线, and find the area of a sector with

$$\text{area} = \tfrac{1}{2}\int r^2\, d\theta.$$



The cardioid $r = 1 + \cos\theta$, plotted straight from r as a function of θ.

Worked example. Convert the polar equation $r = 4\cos\theta$ to Cartesian form.

Multiply both sides by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$. Completing the square gives $(x-2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$.

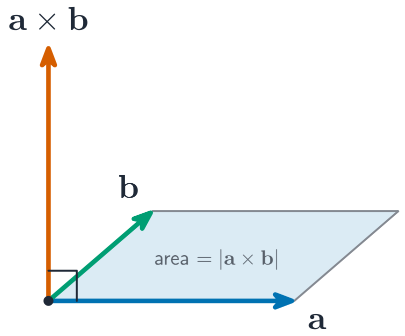
Vectors

Forces like wind and water are vectors — they have both size and direction.

In three dimensions a **plane** 平面 can be written as $ax+by+cz=d$, or $\mathbf{r}\cdot\mathbf{n}=p$, or $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. Besides the scalar product, there is the **vector product** 向量积 of two **vectors** 向量:

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

which gives a vector at right angles to both. With scalar and vector products you can find distances, angles, and where lines and planes meet — including the shortest distance between **skew lines** 异面直线.



$\mathbf{a}\times\mathbf{b}$ is perpendicular to both vectors; its length equals the area of the parallelogram they span.

Worked example. Find $\mathbf{a}\times\mathbf{b}$ for $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and $\mathbf{b}=\mathbf{j}+\mathbf{k}$.

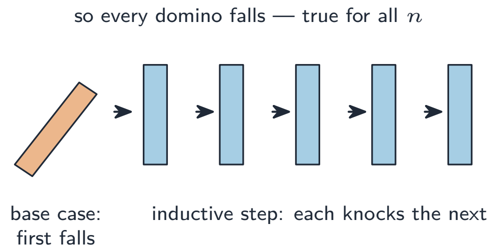
$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&1&0\\0&1&1\end{vmatrix}=\mathbf{i}(1)-\mathbf{j}(1)+\mathbf{k}(1)=\mathbf{i}-\mathbf{j}+\mathbf{k}.$$

Proof by induction

Mathematical induction 数学归纳法 proves a result for every positive integer n in two steps:

1. **Base case**: show the result is true for $n=1$.
2. **Inductive step**: assume it is true for $n=k$, then prove it for $n=k+1$.

If both steps work, the result is true for all n. Often you first make a **conjecture** 猜想 (a sensible guess) from a few cases, then confirm it by **inductive proof** 归纳证明.



Induction is like dominoes: the base case topples the first, and the inductive step makes each one knock over the next — so the result holds for every n

Worked example. Prove that $\displaystyle\sum_{r=1}^{n} r=\tfrac{1}{2}n(n+1)$.

Base case $n=1$: the left side is 1 and the right side is $\frac{1}{2}(1)(2)=1$. True. **Inductive step**: assume $\displaystyle\sum_{r=1}^{k} r=\tfrac{1}{2}k(k+1)$. Then

$$\sum_{r=1}^{k+1} r=\tfrac{1}{2}k(k+1)+(k+1)=(k+1)\left(\tfrac{k}{2}+1\right)=\tfrac{1}{2}(k+1)(k+2).$$

This is the formula with $n=k+1$, so by induction it holds for all n.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n=k$, prove it for $n=k+1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

Further Pure Mathematics 2

A-Level Further Mathematics

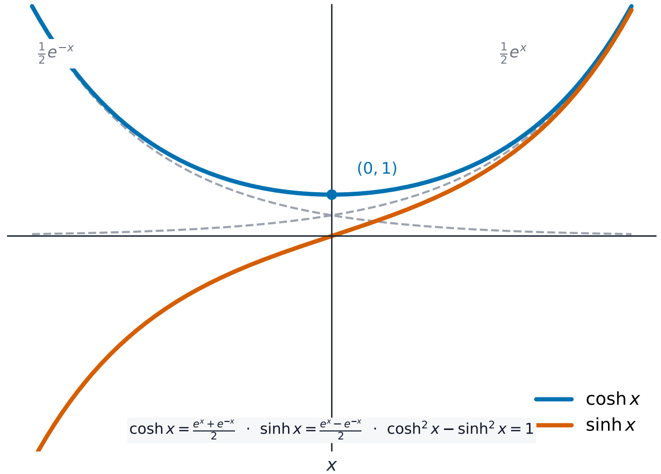
This handout covers Topic 2: **Further Pure Mathematics** 进阶纯数学 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

Hyperbolic functions

A hanging cable forms a catenary — the curve of the hyperbolic cosine.

The **hyperbolic functions** 双曲函数 are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}.$$



$\cosh$ *is the average of* e^x *and* e^{-x}*;* $\sinh$ *is half their difference.*

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The three **reciprocals** complete the set: $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{cosech} x = \dfrac{1}{\sinh x}$, and $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$ – all built from e^x too, and worth recognising in integrals and mark schemes. The **inverse hyperbolic functions** 反双曲函数 have a **logarithmic form** 对数形式, for example $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

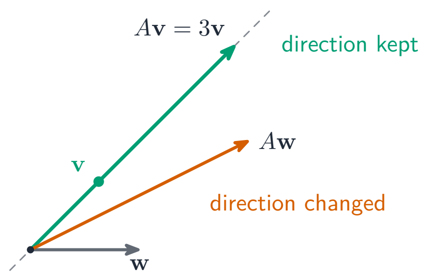
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{4} = \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** 矩阵方程 $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A, the **characteristic equation** 特征方程 is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** 特征值 λ, and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector** 特征向量. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** 对角矩阵 of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0 \;\Rightarrow\; 2-\lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

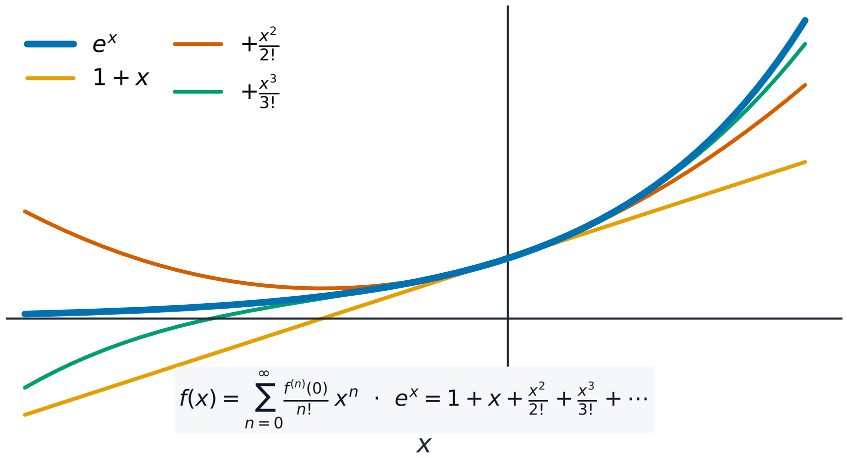
Differentiation

You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\dfrac{d^2y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series 麦克劳林级数** writes a function as a power series:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

For example $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$



Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Worked example. Find the Maclaurin series of $f(x) = \ln(1+x)$ up to the term in x^3.

$f(0) = 0$; $f'(x) = \dfrac{1}{1+x}$ so $f'(0) = 1$; $f''(x) = \dfrac{-1}{(1+x)^2}$ so $f''(0) = -1$; $f'''(x) = \dfrac{2}{(1+x)^3}$ so $f'''(0) = 2$. Substituting into the series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

A **trigonometric substitution**, a **hyperbolic substitution**, or **completing the square** 配方法 in the denominator handles related forms. A **reduction formula** 递推公式 links an integral I_n to I_{n-1}, so you can work down step by step. Integration also gives the **arc length** 弧长 of a curve and the **surface area of revolution** 旋转曲面面积 when a curve is turned about an axis.

Worked example. Find $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Bounding a sum by an integral. Draw rectangles of unit width under, or over, a decreasing curve such as $y = \frac{1}{x}$. Each rectangle of height $\frac{1}{r}$ is trapped between two integral strips, so summing gives

$$\ln(n+1) < \sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n.$$

Shrinking the rectangle width to $\frac{1}{n} \to 0$ turns such a sum into a **Riemann sum** 黎曼和 that equals the integral in the limit, e.g. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$.

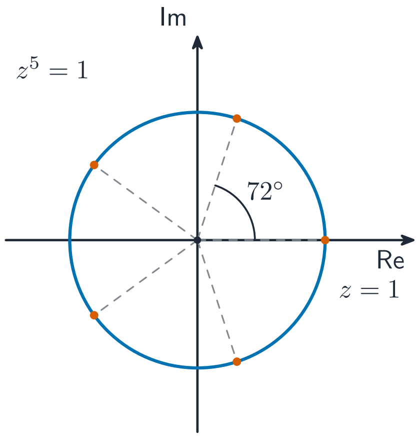
Complex numbers

Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

A **complex number** 复数 in polar form is $z = r(\cos\theta + i\sin\theta)$. **De Moivre's theorem** 棣莫弗定理 says that for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the n **roots of unity** 单位根 (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced $72°$ apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$.

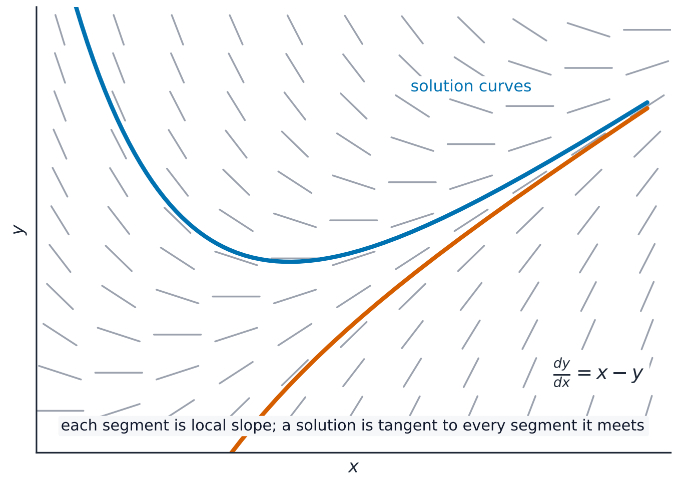
Take the real part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Differential equations

For a first order linear equation $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$. The left side then becomes $\dfrac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** 通解 is the sum of two parts: the **complementary function** 余函数 (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** 特积分 (any one solution of the full equation). **Initial conditions** then fix the constants to give the **particular solution** 特解; simpler equations with **separable variables** 可分离变量 are integrated directly.



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\dfrac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Multiplying through gives $\dfrac{d}{dx}(y\,e^{2x}) = e^{3x}$, so

$$y\,e^{2x} = \tfrac{1}{3}e^{3x} + C \;\Rightarrow\; y = \tfrac{1}{3}e^{x} + Ce^{-2x}.$$

Second order, constant coefficients. For $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$, solve the **auxiliary equation** 辅助方程 $m^2 + bm + c = 0$. The complementary function depends on its roots:

- two distinct real roots m_1, m_2: $y = Ae^{m_1 x} + Be^{m_2 x}$;
- a repeated root m: $y = (A + Bx)e^{mx}$;
- complex roots $p \pm qi$: $y = e^{px}(A \cos qx + B \sin qx)$.

Then add a particular integral by trying a form matching $f(x)$ (a polynomial, ae^{bx}, or $a \cos px + b \sin px$); if that trial already appears in the complementary function, **multiply it by** x.

Worked example. Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$. The auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ has a repeated root $m = 2$, so $y = (A + Bx)e^{2x}$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.

Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics 进阶力学**. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10$ m s^{-2}.

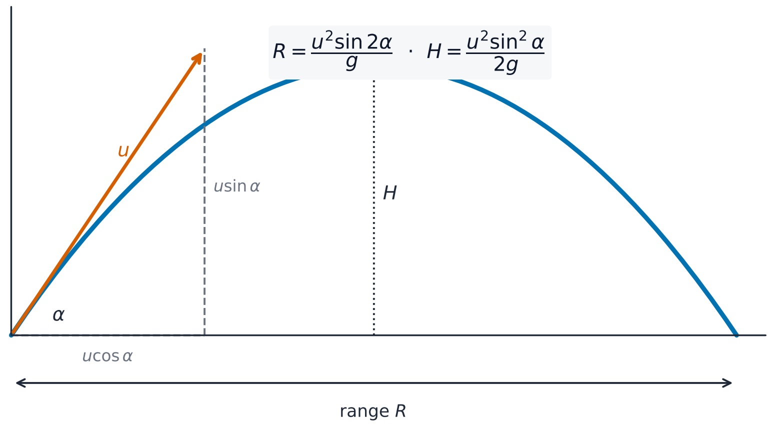
Motion of a projectile

Water jets follow parabolic paths — the classic projectile motion under gravity.

A **projectile 抛射体** moves freely under gravity, so it has **constant acceleration 匀加速** g downwards and no horizontal acceleration. Treat the horizontal and vertical motions separately. If it is launched at speed u and angle α:

$$\text{horizontal: } x = u\cos\alpha \cdot t, \qquad \text{vertical: } y = u\sin\alpha \cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t gives the **Cartesian equation 直角坐标方程** of the **trajectory 轨迹** (the path), which is a parabola. For a fixed launch speed, all the trajectories lie under a **bounding parabola 包络抛物线** (the envelope of reachable points). The range on level ground is $\dfrac{u^2 \sin 2\alpha}{g}$ and the greatest height is $\dfrac{u^2 \sin^2 \alpha}{2g}$.



The path is a parabola; the launch speed u splits into a steady horizontal part and a vertical part slowed by gravity.

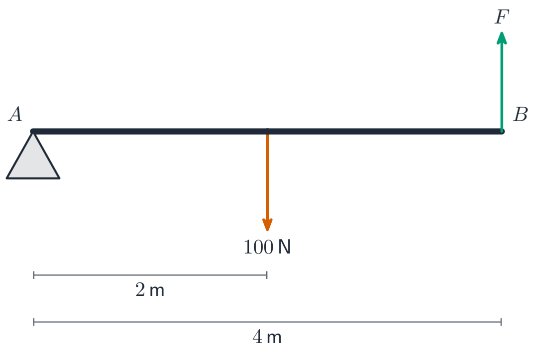
Worked example. A ball is thrown at $u = 20$ m s^{-1} at 30° to the horizontal. Find the range and greatest height.

$$\text{range} = \frac{20^2 \sin 60^\circ}{10} = 40 \times 0.866 = 34.6 \text{ m}, \qquad \text{height} = \frac{20^2 \sin^2 30^\circ}{2 \times 10} = \frac{400 \times 0.25}{20} = 5 \text{ m}.$$

Equilibrium of a rigid body

The **moment** 力矩 of a force about a point is force × perpendicular distance; it measures turning effect. The weight of a body acts at its **centre of mass** 质心, which you can find using **symmetry** 对称 for a uniform flat shape (a **lamina** 薄片), or by treating a composite body as a set of particles.

A rigid body under **coplanar forces** 共面力 is in **equilibrium** 平衡 when two conditions both hold: the vector sum of the forces is zero, and the sum of the moments about any point is zero. A body may also be on the edge of **toppling** 翻倒 (turning over) or **sliding** 滑动 (slipping).



Taking moments about the pivot A balances the turning effects: $F \times 4 = 100 \times 2$.

Worked example. A uniform beam AB of length 4 m and weight 100 N rests on a pivot at A. A vertical force F at B keeps it horizontal. Find F.

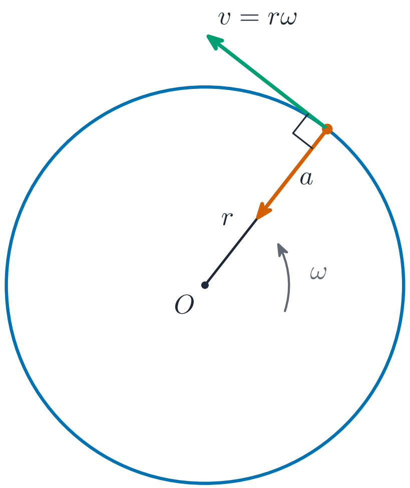
Take moments about A (the weight acts at the centre, 2 m from A):

$$F \times 4 = 100 \times 2 \;\Rightarrow\; F = 50 \text{ N}.$$

Circular motion

For a particle moving in a circle of radius r, the **angular speed** 角速度 ω links to the speed by $v = r\omega$. The acceleration points towards the centre — the **centripetal acceleration** 向心加速度 — with size

$$a = r\omega^2 = \frac{v^2}{r}.$$



The velocity points along the tangent; the acceleration points inward to the centre.

The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre — that inward pull is the centripetal force — while the ball's velocity points along the tangent. Let go, and with no inward force left the ball flies straight off that tangent

In a **horizontal circle** 水平圆 the speed is constant — as in a **conical pendulum** 圆锥摆 (a mass swung on a string). In a **vertical circle** 竖直圆 use energy conservation, because the speed changes with height; the **normal contact force** 法向接触力 or

string tension provides the centripetal force.

Worked example. A particle moves in a horizontal circle of radius 2 m with angular speed 3 rad s^{-1}. Find its speed and acceleration.

$$v = r\omega = 2 \times 3 = 6 \text{ m s}^{-1}, \qquad a = r\omega^2 = 2 \times 3^2 = 18 \text{ m s}^{-2}.$$

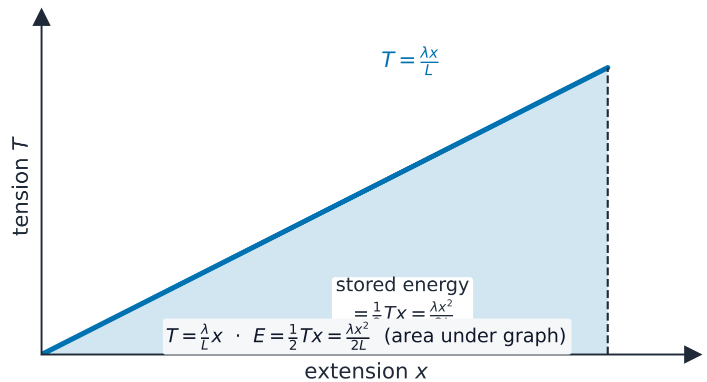
Hooke's law

Hooke's law 胡克定律 says the tension in an elastic string or spring is proportional to its extension x:

$$T = \frac{\lambda x}{L},$$

where L is the natural length and λ is the **modulus of elasticity** 弹性模量. Stretching stores **elastic potential energy** 弹性势能:

$$E = \frac{\lambda x^2}{2L}.$$



Tension rises in proportion to extension; the shaded triangle is the stored elastic energy.

Worked example. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension and the stored energy.

$$T = \frac{50 \times 0.5}{2} = 12.5 \text{ N}, \qquad E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

Linear motion under a variable force

When the force depends on position x, use acceleration in the form $a = v\dfrac{dv}{dx}$, which turns Newton's law into a **differential equation** 微分方程 relating v and x.

which form of acceleration?

both come from $a = \dfrac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

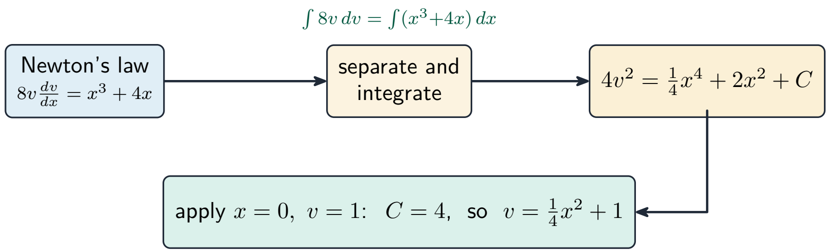
Use dv/dt when the force depends on time, v dv/dx when it depends on position

Worked example. A particle of mass 8 kg moves along a line under a **variable force** 变力 of size $(x^3 + 4x)$ N acting in the direction of motion. When $x = 0$, $v = 1$. Find v in terms of x.

Newton's law gives $8v\dfrac{dv}{dx} = x^3 + 4x$. Separating and integrating:

$$\int 8v\,dv = \int (x^3 + 4x)\,dx \;\Rightarrow\; 4v^2 = \tfrac{1}{4}x^4 + 2x^2 + C.$$

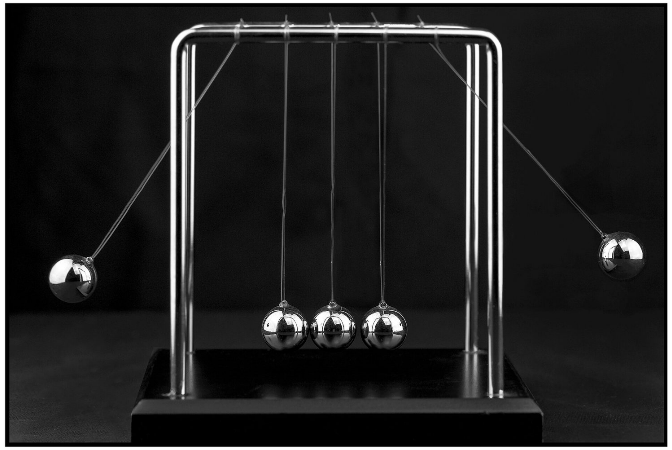
At $x = 0$, $v = 1$ gives $C = 4$, so $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4 = 4\left(\frac{x^2}{4} + 1\right)^2$. Hence $v = \frac{1}{4}x^2 + 1$.



a position-dependent force becomes a **differential equation** in v and x

A position-dependent force becomes a differential equation in v and x

Momentum

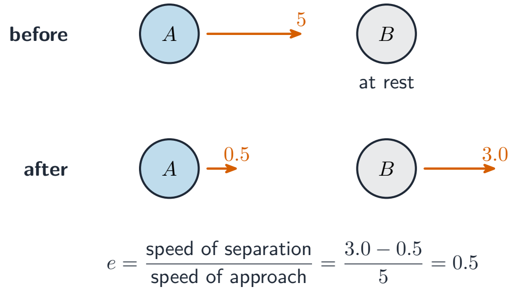


A Newton's cradle demonstrates conservation of momentum in collisions.

In a collision, total **linear momentum** is conserved — the **conservation of linear momentum** 动量守恒. The bounciness is measured by **Newton's experimental law** 牛顿实验定律, which defines the **coefficient of restitution** 恢复系数 e:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}, \qquad 0 \leqslant e \leqslant 1.$$

Here $e = 1$ is perfectly elastic (no energy lost) and $e = 0$ is inelastic (the bodies stay together). In an **oblique impact** 斜碰撞, resolve the velocities along and perpendicular to the line of impact, applying restitution along it.



The coefficient of restitution e compares how fast the bodies separate with how fast they approached.

Worked example. A sphere A of mass 2 kg moving at 5 m s^{-1} hits a stationary sphere B of mass 3 kg, with $e = 0.5$. Find the speeds afterwards.

Momentum: $2(5) = 2v_A + 3v_B$, so $2v_A + 3v_B = 10$. Restitution: $v_B - v_A = 0.5(5) = 2.5$. Solving together gives $v_A = 0.5$ m s^{-1} and $v_B = 3.0$ m s^{-1}.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v\,\frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.

Further Probability & Statistics

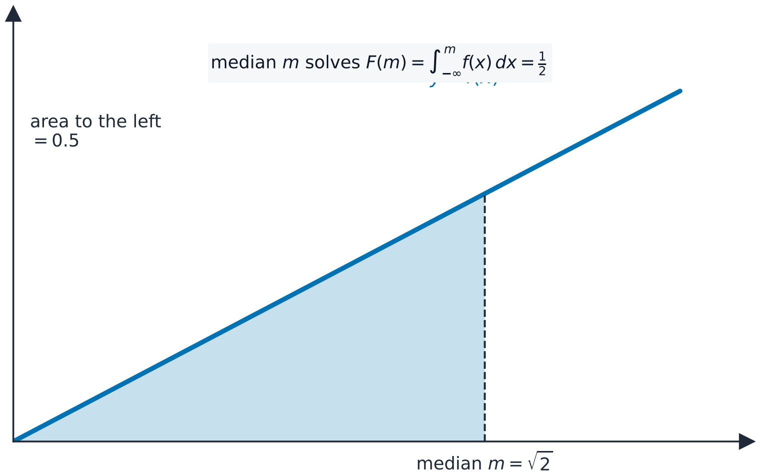
A-Level Further Mathematics

This handout covers Topic 4: **Further Probability & Statistics** 进阶概率统计. It adds continuous distributions, small-sample inference, the chi-squared and non-parametric tests, and probability generating functions.

Continuous random variables

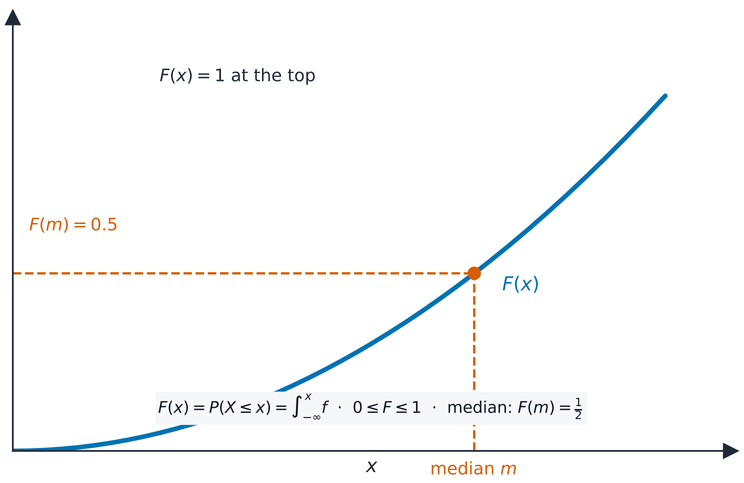
A continuous variable X is described by a **probability density function** 概率密度函数 $f(x)$, which may be defined piecewise. The probability over a range is the area under f, and the mean of any function of X is

$$E(g(X)) = \int g(x)\, f(x)\, dx.$$



The median sits where the area under $f(x)$ to its left is exactly 0.5.

The **cumulative distribution function** 累积分布函数 $F(x) = P(X \leqslant x)$ is the running total: $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\, dt$, and $f(x) = F'(x)$. Use F to find probabilities and **percentiles** 百分位数 (for example, the **median** 中位数 solves $F(x) = 0.5$).



The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

Worked example. A variable has $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$. Find the median.

The cumulative distribution function is $F(x) = \displaystyle\int_0^x \frac{1}{2}t\,dt = \frac{1}{4}x^2$. Set $F(m) = 0.5$:

$$\tfrac{1}{4}m^2 = 0.5 \;\Rightarrow\; m^2 = 2 \;\Rightarrow\; m = \sqrt{2} = 1.41.$$

The distribution of a related variable. If $Y = g(X)$, find Y's distribution through its cumulative function. For $Y = X^2$: $F_Y(y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, then differentiate for $f_Y = F_Y'$. When g is monotonic increasing there is a shortcut, $F_Y(y) = F_X\big(g^{-1}(y)\big)$.

Worked example. With $f_X(x) = \frac{1}{2}x$ on $[0, 2]$ (so $F_X(x) = \frac{1}{4}x^2$), let $Y = X^2$. For $0 \leqslant y \leqslant 4$, $F_Y(y) = F_X(\sqrt{y}) = \frac{1}{4}y$, so $f_Y(y) = F_Y'(y) = \frac{1}{4}$ – that is, Y is uniform on $[0, 4]$.

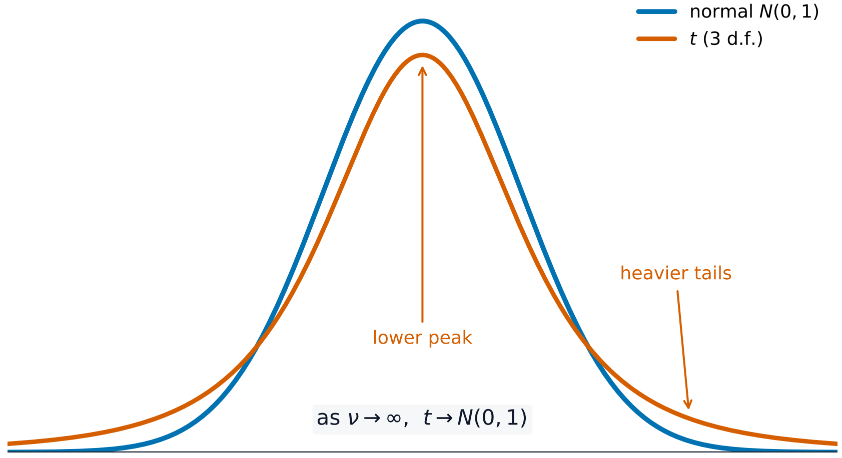
Inference using normal and t-distributions

A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

When a sample is small and the population variance is unknown, base your **hypothesis test** 假设检验 on the t-distribution instead of the normal. The same idea gives a **confidence interval** 置信区间 for the mean:

$$\bar{x} \pm t\,\frac{s}{\sqrt{n}},$$

where t comes from the t-tables with $n-1$ degrees of freedom. To compare two populations, use a **two-sample** (2-sample) or **paired-sample** t-test, after finding a **pooled estimate** 合并估计 of the shared variance when appropriate.



For a small sample the t-distribution is flatter with heavier tails, so its critical values are larger.

Worked example. A sample of $n = 10$ has mean $\bar{x} = 50$ and standard deviation $s = 4$. Find a 95% confidence interval for the mean (use $t = 2.262$ for 9 degrees of freedom).

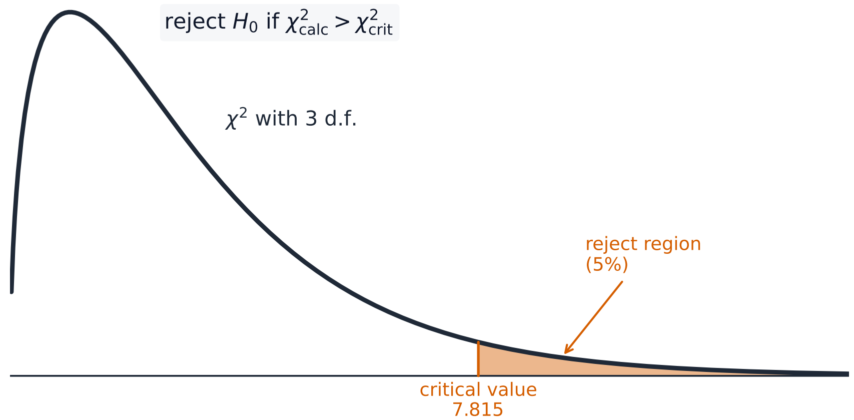
$$50 \pm 2.262 \times \frac{4}{\sqrt{10}} = 50 \pm 2.86 \Rightarrow (47.1,\ 52.9).$$

Chi-squared tests

A χ^2-test (**chi-squared test** 卡方检验) compares observed counts O with expected counts E from a **theoretical distribution** 理论分布:

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

Compare this with a table value for the right number of **degrees of freedom** 自由度. Two uses: a **goodness of fit** 拟合优度 test (does the data follow the proposed model?), and a test for **independence** 独立性 of two variables in a **contingency table** 列联表.



The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5% tail.

Worked example. Four equally likely categories give observed counts $20, 30, 25, 25$ (so each expected count is 25). Test the fit at the 5% level.

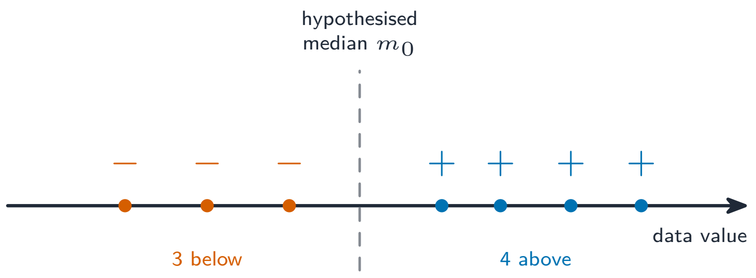
$$\chi^2 = \frac{(20-25)^2 + (30-25)^2 + 0 + 0}{25} = \frac{25+25}{25} = 2.$$

With $4 - 1 = 3$ degrees of freedom the table value is 7.815. Since $2 < 7.815$, do not reject the model.

Non-parametric tests

A **non-parametric test** 非参数检验 makes no assumption that the data is normal, so it is useful when that assumption fails. The basic ones are:

- the **sign test** 符号检验: count how many values fall above and below a proposed median, and test those counts with a binomial model;
- the **Wilcoxon signed-rank test** 威尔科克森符号秩检验 (the **matched-pairs** test for paired data), which also uses the sizes of the differences, not just their signs;
- the **Wilcoxon rank-sum test** 威尔科克森秩和检验, for comparing two separate samples.



The sign test counts how many values fall above and below the proposed median, then tests those counts with a binomial model

Worked example. Test whether a median is 5. In a sample of 10 values (none equal to 5), 9 lie **above** 5 and 1 lies below. Test at the 5% level (two-tailed).

Under H_0 (median $= 5$) the number above follows $B(10, 0.5)$. The observed result (9 above) is extreme, so find $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + (0.5)^{10} = \frac{11}{1024} = 0.0107$. For a two-tailed test compare with $\frac{1}{2}(5\%) = 0.025$. Since $0.0107 < 0.025$, **reject** H_0: there is evidence the median is not 5.

Probability generating functions

Dice: a starting point for probability and discrete random variables.

The **probability generating function** 概率母函数 of a discrete variable X is

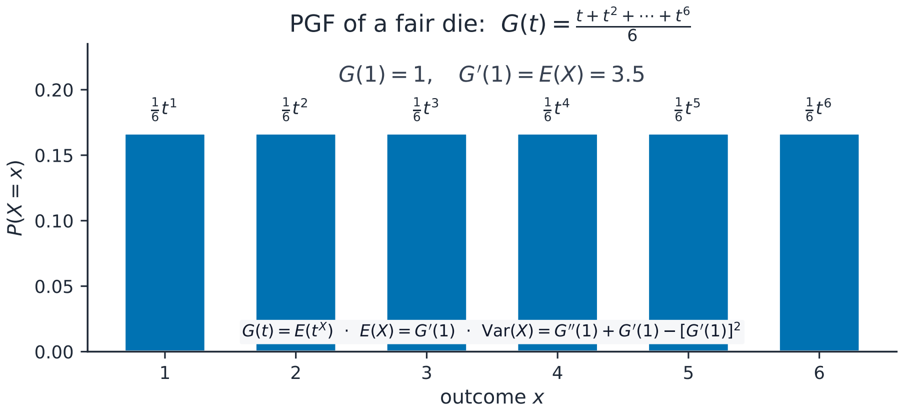
$$G(t) = E(t^X) = \sum_x P(X = x)\, t^x.$$

It packs the whole distribution into one function. The mean and variance come from its derivatives at $t = 1$: $E(X) = G'(1)$ and $\mathrm{Var}(X) = G''(1) + G'(1) - \big(G'(1)\big)^2$. Also, the PGF of a sum of independent variables is the product of their PGFs.

Worked example. X has $P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$. Find $E(X)$ using the PGF.

Here $G(t) = 0.5 + 0.3t + 0.2t^2$, so $G'(t) = 0.3 + 0.4t$ and

$$E(X) = G'(1) = 0.3 + 0.4 = 0.7.$$



PGF of a fair die packs equal masses at 1 through 6 into G(t)

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int x f(x)\, dx$.
- Use the t**-distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum (O - E)^2 / E$, compare with the critical value at the right degrees of freedom, and **combine classes with** $E < 5$.
- State H_0 and H_1 and give the conclusion **in context** for every test.