Course Companion

A-Level Further Mathematics

Every topic handout in one volume

全部专题讲义合集

exercises.lol

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Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics** 进阶纯数学 1. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

For a polynomial equation, the **roots** 根 are linked to the **coefficients** 系数. For $ax^2 + bx + c = 0$ with roots α, β:

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

For a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ:

$$\sum \alpha = -\frac{b}{a}, \qquad \sum \alpha\beta = \frac{c}{a}, \qquad \alpha\beta\gamma = -\frac{d}{a}.$$

Sums like $\sum \alpha$ and $\sum \alpha\beta$ are **symmetric functions** 对称函数 of the roots (unchanged if the roots are swapped). To find an equation whose roots are changed in a simple way, use a **substitution** 代换 (for example, put $w = \alpha + 1$).

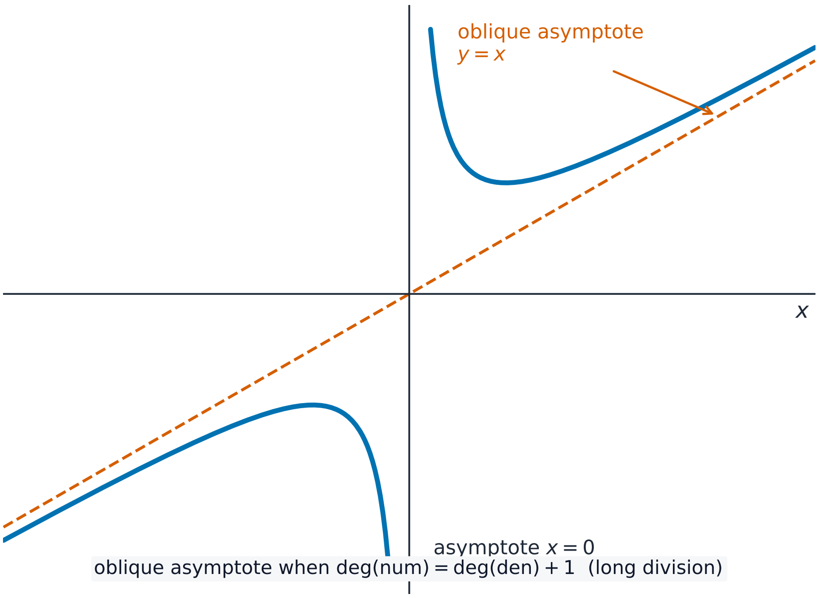
Worked example. The equation $x^2 - 5x + 6 = 0$ has roots α, β. Find the equation with roots $\alpha + 1, \beta + 1$.

Here $\alpha + \beta = 5$ and $\alpha\beta = 6$. The new sum is $(\alpha + 1) + (\beta + 1) = 7$, and the new product is $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$. So the new equation is

$$x^2 - 7x + 12 = 0.$$

Rational functions and graphs

A **rational function** 有理函数 is a fraction of two polynomials. When the top has degree exactly one higher than the bottom, the graph has an **oblique asymptote** 斜渐近线 (a slanted line the curve approaches); find it by dividing out. You should also be able to relate the graph of $y = f(x)$ to those of $y^2 = f(x)$, $y = \dfrac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$. Find the set of values the function can take using the **discriminant** 判别式, and locate its **turning points** 驻点.



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertical asymptote.

Worked example. Find the oblique asymptote of $y = \dfrac{x^2 + x + 1}{x + 1}$.

Divide out the fraction: $\dfrac{x^2 + x + 1}{x + 1} = x + \dfrac{1}{x + 1}$. As $x \to \pm\infty$ the remainder $\dfrac{1}{x + 1} \to 0$, so the curve approaches the line $y = x$ —the oblique asymptote.

Summation of series

Learn the standard results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

The **method of differences** 差分法 sums a series by cancelling middle terms: if **partial fractions** 部分分式 write each term as $f(r) - f(r+1)$, almost everything cancels. From the sum to n terms you can see whether a series is **convergent** 收敛 and, if so, find its **sum to infinity** 无穷和.

Worked example. Find $\displaystyle\sum_{r=1}^{n}(2r+1)$.

$$\sum_{r=1}^{n}(2r+1) = 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1 = 2 \cdot \tfrac{1}{2}n(n+1) + n = n(n+1) + n = n(n+2).$$

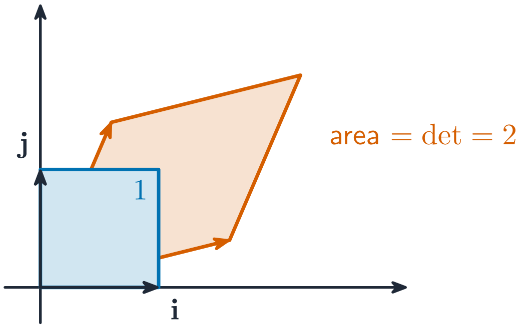
Matrices

A **matrix** 矩阵 is a rectangular block of numbers. **Matrix addition** adds entries in matching positions; you can also subtract and multiply matrices (multiplication is row $\times$ column). The **zero matrix** 零矩阵 has every entry 0, and the **identity matrix** 单位矩阵 (or **unit matrix**) I leaves any matrix unchanged under multiplication.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** 行列式 is $\det A = ad - bc$. If $\det A \neq 0$ the matrix is non-singular (otherwise it is a **singular matrix** 奇异矩阵), and the **inverse matrix** 逆矩阵 is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a product, $(AB)^{-1} = B^{-1}A^{-1}$. A 2×2 matrix can represent a **geometric transformation** 几何变换 of the plane —a **rotation** 旋转, **reflection** 反射, **enlargement** 放大, **stretch** 拉伸 or **shear** 切变: the determinant gives the area scale factor, and a product AB means "do B, then A". Points or lines that do not move are called **invariant points** 不变点 and **invariant lines** 不变直线.



The matrix sends the unit square to a parallelogram whose area is the determinant.

Worked example. Find the inverse of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \times 4 - 1 \times 2 = 10$, so

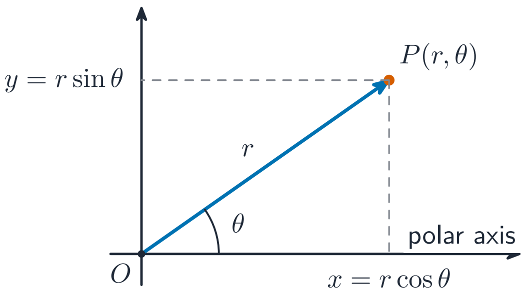
$$A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

Polar coordinates

A spiral staircase: spirals are described naturally using polar coordinates.

Polar coordinates 极坐标 give a point by its distance r from the origin and its angle θ. They link to **Cartesian coordinates** 直角坐标 by

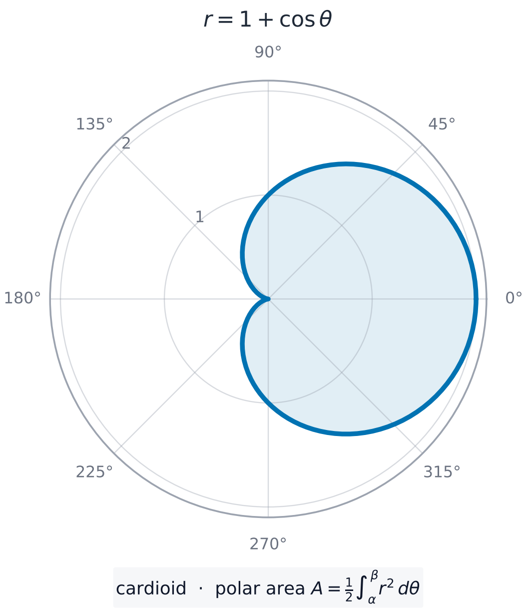
$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2.$$



Polar coordinates (r,θ) convert to Cartesian by $x = r\cos\theta$ and $y = r\sin\theta$.

You should sketch simple **polar curves** 极坐标曲线, and find the area of a sector with

$$\text{area} = \tfrac{1}{2}\int r^2\, d\theta.$$



The cardioid $r = 1 + \cos\theta$, plotted straight from r as a function of θ.

Worked example. Convert the polar equation $r = 4\cos\theta$ to Cartesian form.

Multiply both sides by r: $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$. Completing the square gives $(x-2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$.

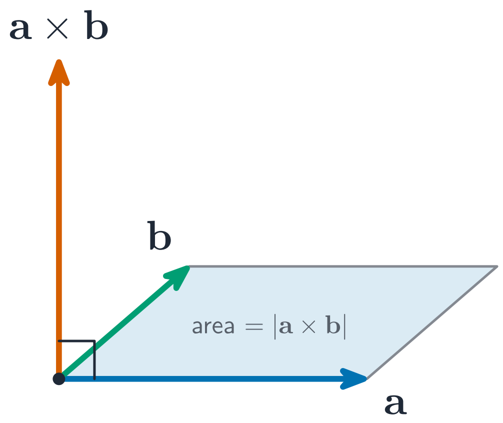
Vectors

Forces like wind and water are vectors —they have both size and direction.

In three dimensions a **plane** 平面 can be written as $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = p$, or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. Besides the scalar product, there is the **vector product** 向量积 of two **vectors** 向量:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

which gives a vector at right angles to both. With scalar and vector products you can find distances, angles, and where lines and planes meet —including the shortest distance between **skew lines** 异面直线.



$\mathbf{a} \times \mathbf{b}$ *is perpendicular to both vectors; its length equals the area of the parallelogram they span.*

Worked example. Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

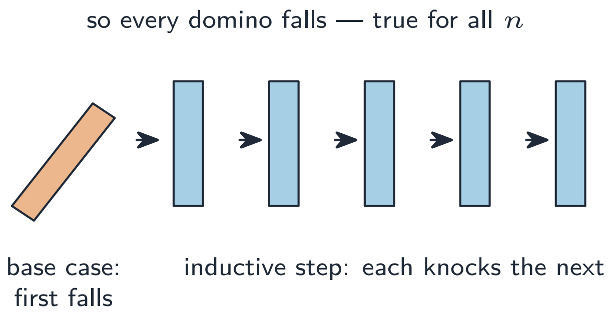
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(1) = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

Proof by induction

Mathematical induction 数学归纳法 proves a result for every positive integer n in two steps:

1. **Base case**: show the result is true for $n = 1$.
2. **Inductive step**: assume it is true for $n = k$, then prove it for $n = k + 1$.

If both steps work, the result is true for all n. Often you first make a **conjecture** 猜想 (a sensible guess) from a few cases, then confirm it by **inductive proof** 归纳证明.



Induction is like dominoes: the base case topples the first, and the inductive step makes each one knock over the next —so the result holds for every n

Worked example. Prove that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Base case $n = 1$: the left side is 1 and the right side is $\frac{1}{2}(1)(2) = 1$. True. **Inductive step**: assume $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$. Then

$$\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + (k+1) = (k+1)\left(\frac{k}{2}+1\right) = \frac{1}{2}(k+1)(k+2).$$

This is the formula with $n = k+1$, so by induction it holds for all n.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k+1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

Further Pure Mathematics 2

A-Level Further Mathematics

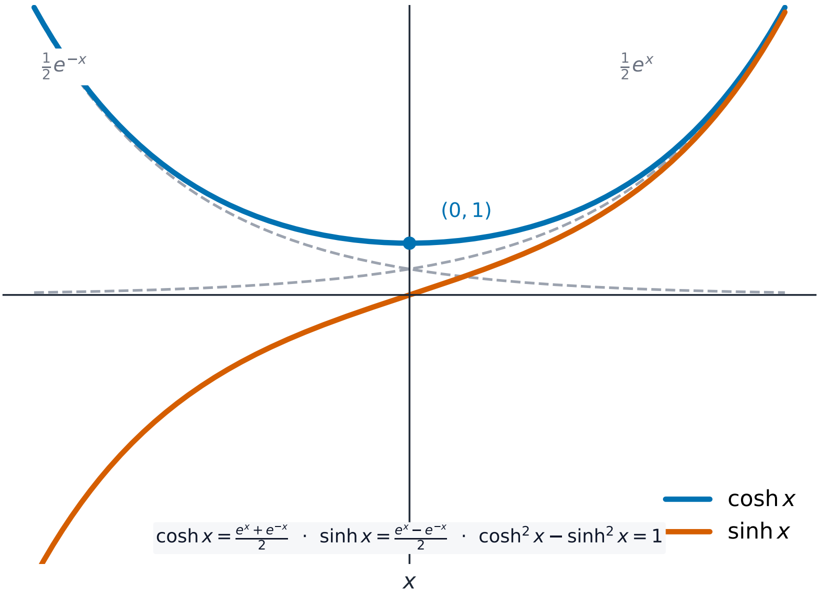
This handout covers Topic 2: **Further Pure Mathematics** 进阶纯数学 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

Hyperbolic functions

A hanging cable forms a catenary —the curve of the hyperbolic cosine.

The **hyperbolic functions** 双曲函数 are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}.$$



cosh *is the average of* e^x *and* e^{-x}*;* sinh *is half their difference.*

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The three **reciprocals** complete the set: $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{cosech} x = \dfrac{1}{\sinh x}$, and $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$ –all built from e^x too, and worth recognising in integrals and mark schemes. The **inverse hyperbolic functions** 反双曲函数 have a **logarithmic form** 对数形式, for example $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

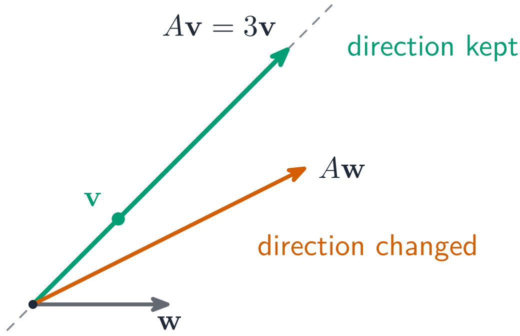
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x+e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** 矩阵方程 $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A, the **characteristic equation** 特征方程 is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** 特征值 λ, and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector** 特征向量. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** 对角矩阵 of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0 \;\Rightarrow\; 2-\lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

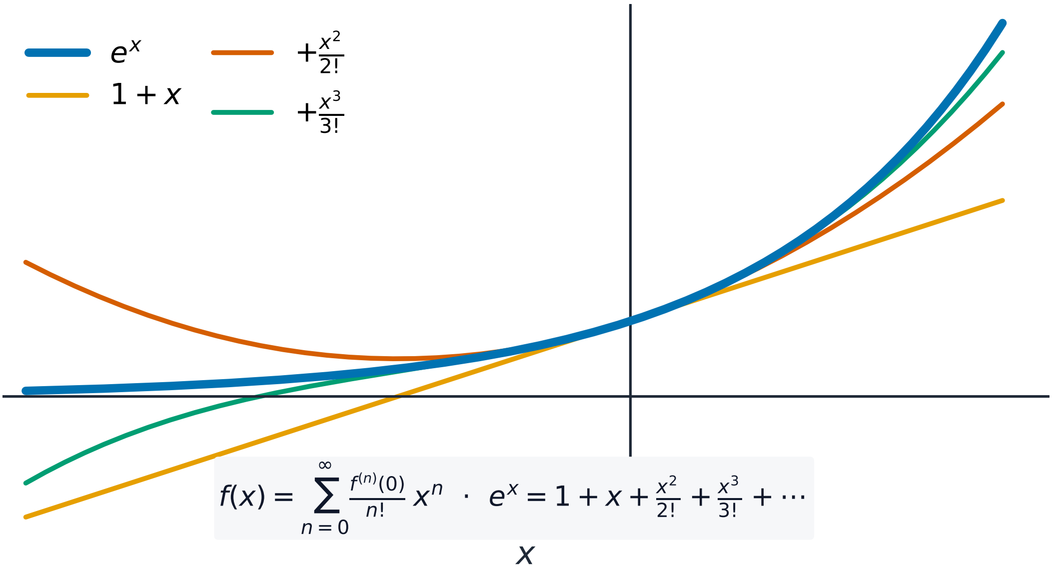
Differentiation

You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\dfrac{d^2y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series 麦克劳林级数** writes a function as a power series:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

For example $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$



Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Worked example. Find the Maclaurin series of $f(x) = \ln(1 + x)$ up to the term in x^3.

$f(0) = 0$; $f'(x) = \dfrac{1}{1+x}$ so $f'(0) = 1$; $f''(x) = \dfrac{-1}{(1+x)^2}$ so $f''(0) = -1$; $f'''(x) = \dfrac{2}{(1+x)^3}$ so $f'''(0) = 2$. Substituting into the series:

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

A **trigonometric substitution**, a **hyperbolic substitution**, or **completing the square** 配方法 in the denominator handles related forms. A **reduction formula** 递推公式 links an integral I_n to I_{n-1}, so you can work down step by step. Integration also gives the **arc length** 弧长 of a curve and the **surface area of revolution** 旋转曲面面积 when a curve is turned about an axis.

Worked example. Find $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Bounding a sum by an integral. Draw rectangles of unit width under, or over, a decreasing curve such as $y = \frac{1}{x}$. Each rectangle of height $\frac{1}{r}$ is trapped between two integral strips, so summing gives

$$\ln(n+1) < \sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n.$$

Shrinking the rectangle width to $\frac{1}{n} \to 0$ turns such a sum into a **Riemann sum** 黎曼和 that equals the integral in the limit, e.g. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$.

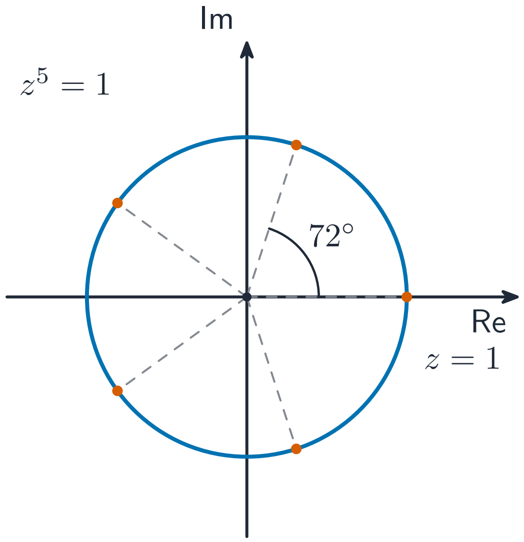
Complex numbers

Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

A **complex number** 复数 in polar form is $z = r(\cos\theta + i\sin\theta)$. **De Moivre's theorem** 棣莫弗定理 says that for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the n **roots of unity** 单位根 (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced $72°$ apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$.

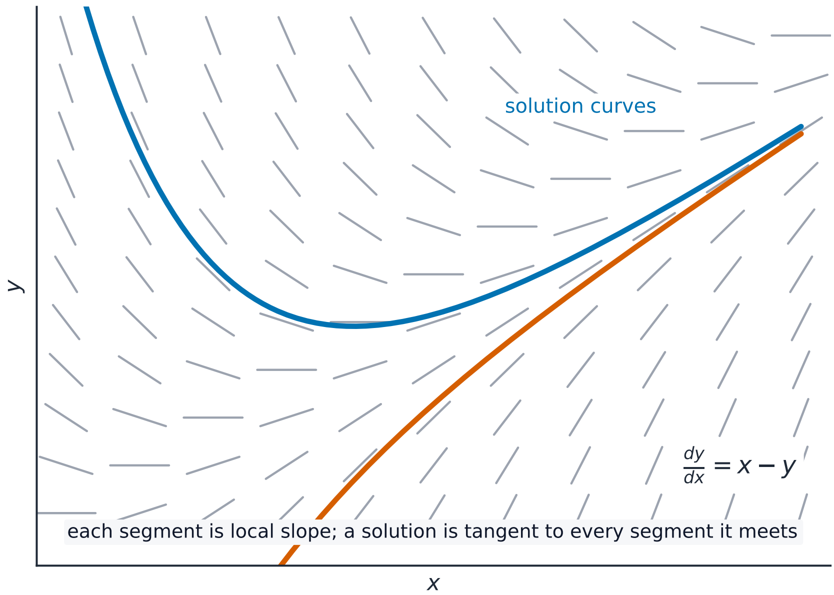
Take the real part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Differential equations

For a first order linear equation $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$. The left side then becomes $\dfrac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** 通解 is the sum of two parts: the **complementary function** 余函数 (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** 特积分 (any one solution of the full equation). **Initial conditions** then fix the constants to give the **particular solution** 特解; simpler equations with **separable variables** 可分离变量 are integrated directly.



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\dfrac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Multiplying through gives $\dfrac{d}{dx}(y\,e^{2x}) = e^{3x}$, so

$$y\,e^{2x} = \tfrac{1}{3}e^{3x} + C \;\Rightarrow\; y = \tfrac{1}{3}e^{x} + Ce^{-2x}.$$

Second order, constant coefficients. For $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$, solve the **auxiliary equation** 辅助方程 $m^2 + bm + c = 0$. The complementary function depends on its roots:

- two distinct real roots m_1, m_2: $y = Ae^{m_1 x} + Be^{m_2 x}$;
- a repeated root m: $y = (A + Bx)e^{mx}$;
- complex roots $p \pm qi$: $y = e^{px}(A\cos qx + B\sin qx)$.

Then add a particular integral by trying a form matching $f(x)$ (a polynomial, ae^{bx}, or $a\cos px + b\sin px$); if that trial already appears in the complementary function, **multiply it by** x.

Worked example. Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$. The auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ has a repeated root $m = 2$, so $y = (A + Bx)e^{2x}$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.

Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics** 进阶力学. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10 \text{ m s}^{-2}$.

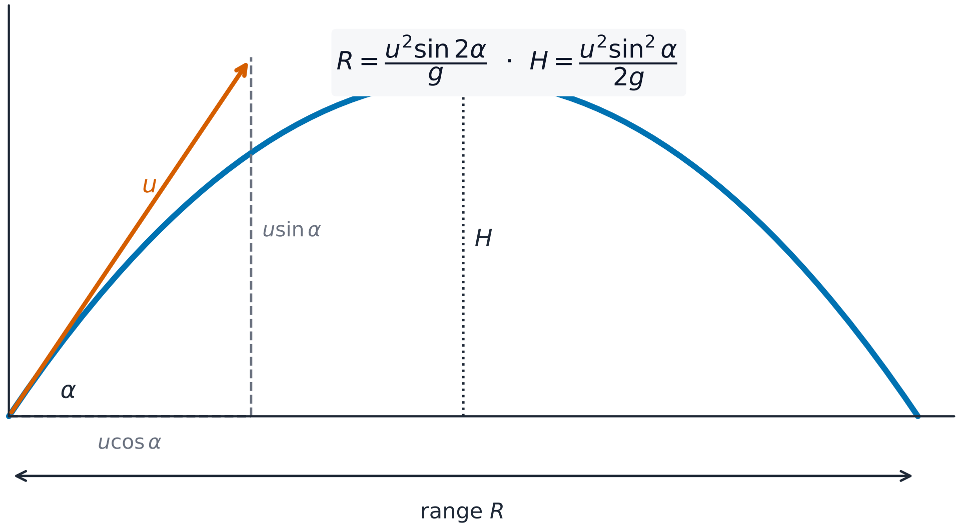
Motion of a projectile

Water jets follow parabolic paths —the classic projectile motion under gravity.

A **projectile** 抛射体 moves freely under gravity, so it has **constant acceleration** 匀加速 g downwards and no horizontal acceleration. Treat the horizontal and vertical motions separately. If it is launched at speed u and angle α:

$$\text{horizontal: } x = u\cos\alpha \cdot t, \qquad \text{vertical: } y = u\sin\alpha \cdot t - \tfrac{1}{2}gt^2.$$

Eliminating t gives the **Cartesian equation** 直角坐标方程 of the **trajectory** 轨迹 (the path), which is a parabola. For a fixed launch speed, all the trajectories lie under a **bounding parabola** 包络抛物线 (the envelope of reachable points). The range on level ground is $\dfrac{u^2 \sin 2\alpha}{g}$ and the greatest height is $\dfrac{u^2 \sin^2 \alpha}{2g}$.



The path is a parabola; the launch speed u splits into a steady horizontal part and a vertical part slowed by gravity.

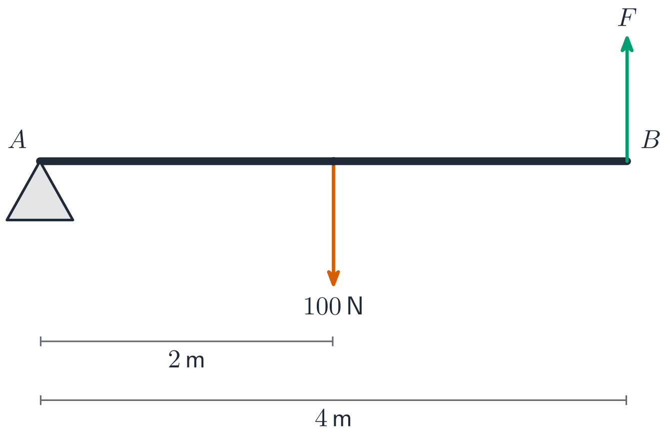
Worked example. A ball is thrown at $u = 20$ m s^{-1} at 30° to the horizontal. Find the range and greatest height.

$$\text{range} = \frac{20^2 \sin 60°}{10} = 40 \times 0.866 = 34.6 \text{ m}, \qquad \text{height} = \frac{20^2 \sin^2 30°}{2 \times 10} = \frac{400 \times 0.25}{20} = 5 \text{ m}.$$

Equilibrium of a rigid body

The **moment** 力矩 of a force about a point is force × perpendicular distance; it measures turning effect. The weight of a body acts at its **centre of mass** 质心, which you can find using **symmetry** 对称 for a uniform flat shape (a **lamina** 薄片), or by treating a composite body as a set of particles.

A rigid body under **coplanar forces** 共面力 is in **equilibrium** 平衡 when two conditions both hold: the vector sum of the forces is zero, and the sum of the moments about any point is zero. A body may also be on the edge of **toppling** 翻倒 (turning over) or **sliding** 滑动 (slipping).



Taking moments about the pivot A balances the turning effects: $F \times 4 = 100 \times 2$.

Worked example. A uniform beam AB of length 4 m and weight 100 N rests on a pivot at A. A vertical force F at B keeps it horizontal. Find F.

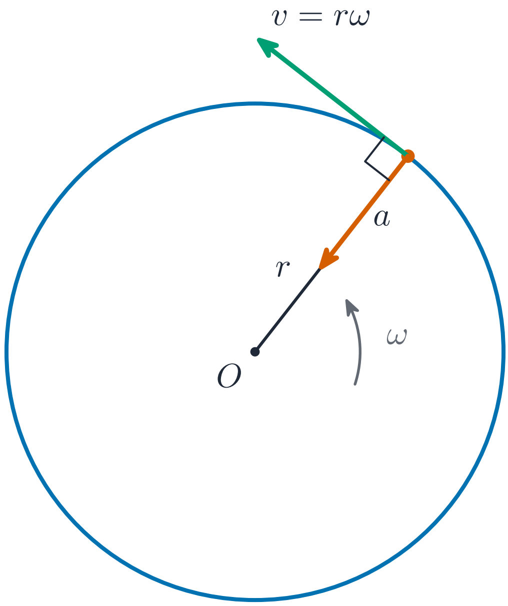
Take moments about A (the weight acts at the centre, 2 m from A):

$$F \times 4 = 100 \times 2 \;\Rightarrow\; F = 50 \text{ N}.$$

Circular motion

For a particle moving in a circle of radius r, the **angular speed** 角速度 ω links to the speed by $v = r\omega$. The acceleration points towards the centre —the **centripetal acceleration** 向心加速度—with size

$$a = r\omega^2 = \frac{v^2}{r}.$$



The velocity points along the tangent; the acceleration points inward to the centre.

The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre —that inward pull is the centripetal force —while the ball's velocity points along the tangent. Let go, and with no inward force left the ball flies straight off that tangent

In a **horizontal circle** 水平圆 the speed is constant —as in a **conical pendulum** 圆锥摆 (a mass swung on a string). In a **vertical circle** 竖直圆 use energy conservation, because the speed changes with height; the **normal contact force** 法向接触力 or string tension provides the centripetal force.

Worked example. A particle moves in a horizontal circle of radius 2 m with angular speed 3 rad s^{-1}. Find its speed and acceleration.

$$v = r\omega = 2 \times 3 = 6 \text{ m s}^{-1}, \qquad a = r\omega^2 = 2 \times 3^2 = 18 \text{ m s}^{-2}.$$

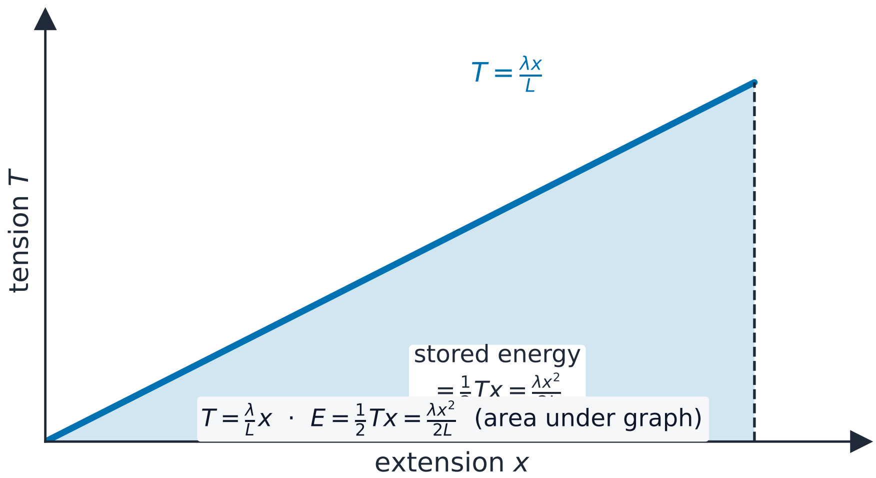
Hooke's law

Hooke's law 胡克定律 says the tension in an elastic string or spring is proportional to its extension x:

$$T = \frac{\lambda x}{L},$$

where L is the natural length and λ is the **modulus of elasticity** 弹性模量. Stretching stores **elastic potential energy** 弹性势能:

$$E = \frac{\lambda x^2}{2L}.$$



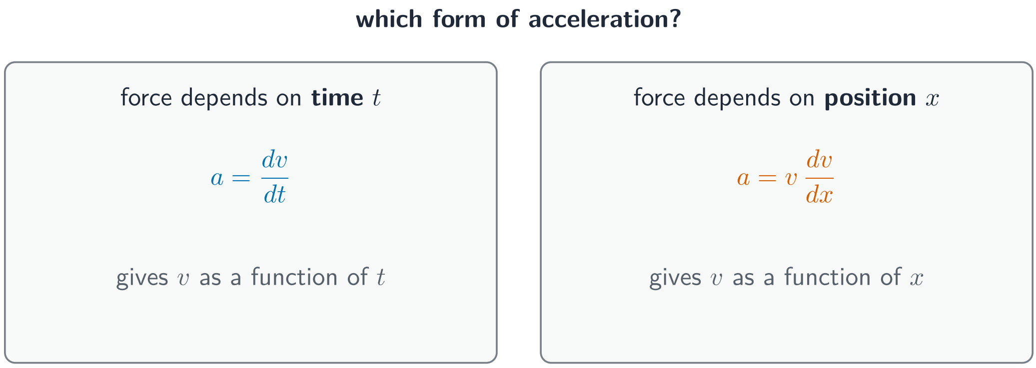
Tension rises in proportion to extension; the shaded triangle is the stored elastic energy.

Worked example. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension and the stored energy.

$$T = \frac{50 \times 0.5}{2} = 12.5 \text{ N}, \qquad E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

Linear motion under a variable force

When the force depends on position x, use acceleration in the form $a = v\dfrac{dv}{dx}$, which turns Newton's law into a **differential equation** 微分方程 relating v and x.



both come from $a = \dfrac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

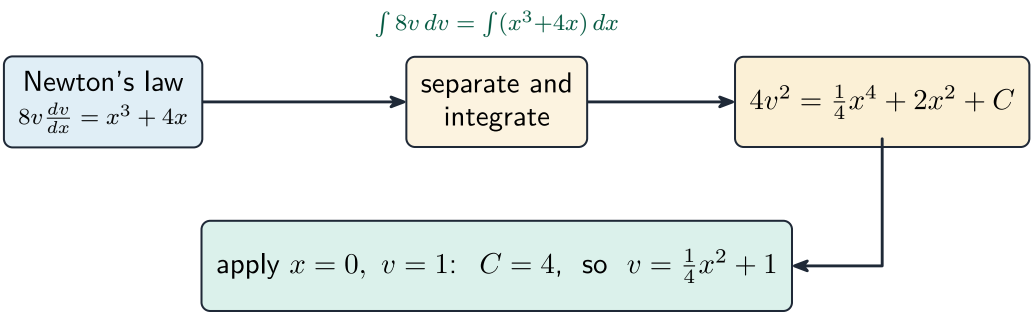
Use dv/dt when the force depends on time, v dv/dx when it depends on position

Worked example. A particle of mass 8 kg moves along a line under a **variable force** 变力 of size $(x^3 + 4x)$ N acting in the direction of motion. When $x = 0$, $v = 1$. Find v in terms of x.

Newton's law gives $8v\dfrac{dv}{dx} = x^3 + 4x$. Separating and integrating:

$$\int 8v\,dv = \int (x^3 + 4x)\,dx \;\Rightarrow\; 4v^2 = \tfrac{1}{4}x^4 + 2x^2 + C.$$

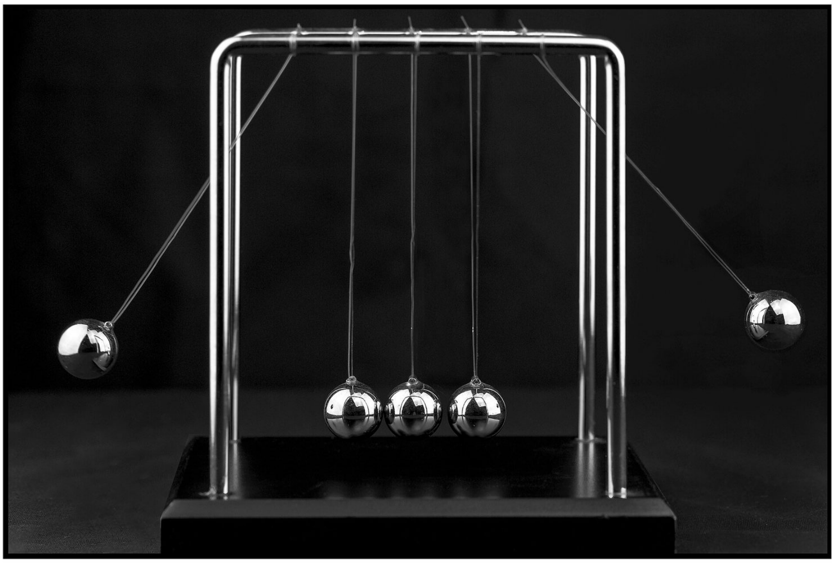
At $x = 0$, $v = 1$ gives $C = 4$, so $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4 = 4\left(\frac{x^2}{4} + 1\right)^2$. Hence $v = \frac{1}{4}x^2 + 1$.



a position-dependent force becomes a **differential equation** in v and x

A position-dependent force becomes a differential equation in v and x

Momentum

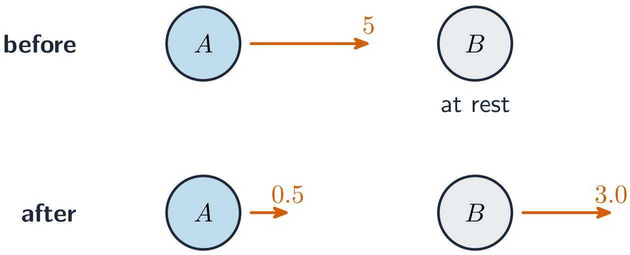


A Newton's cradle demonstrates conservation of momentum in collisions.

In a collision, total **linear momentum** is conserved —the **conservation of linear momentum** 动量守恒. The bounciness is measured by **Newton's experimental law** 牛顿实验定律, which defines the **coefficient of restitution** 恢复系数 e:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}, \qquad 0 \leqslant e \leqslant 1.$$

Here $e = 1$ is perfectly elastic (no energy lost) and $e = 0$ is inelastic (the bodies stay together). In an **oblique impact** 斜碰撞, resolve the velocities along and perpendicular to the line of impact, applying restitution along it.



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

The coefficient of restitution e compares how fast the bodies separate with how fast they approached.

Worked example. A sphere A of mass 2 kg moving at 5 m s^{-1} hits a stationary sphere B of mass 3 kg, with $e = 0.5$. Find the speeds afterwards.

Momentum: $2(5) = 2v_A + 3v_B$, so $2v_A + 3v_B = 10$. Restitution: $v_B - v_A = 0.5(5) = 2.5$. Solving together gives $v_A = 0.5$ m s^{-1} and $v_B = 3.0$ m s^{-1}.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v\,\frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.

Further Probability & Statistics

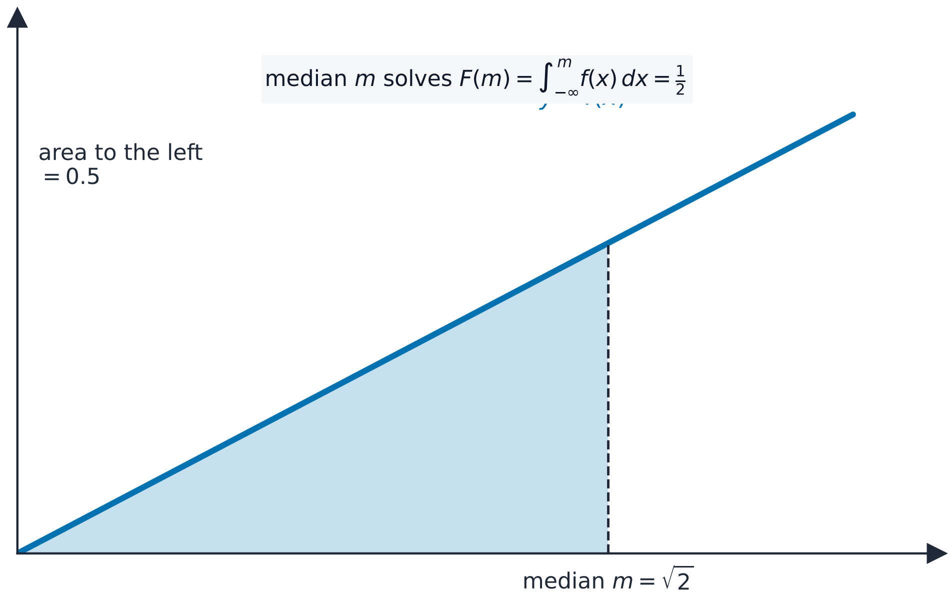
A-Level Further Mathematics

This handout covers Topic 4: **Further Probability & Statistics** 进阶概率统计. It adds continuous distributions, small-sample inference, the chi-squared and non-parametric tests, and probability generating functions.

Continuous random variables

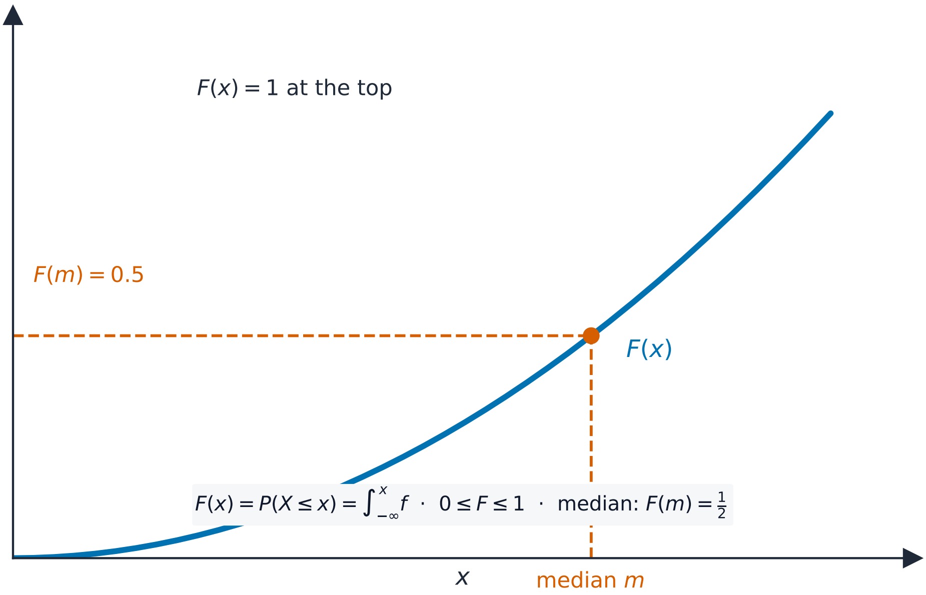
A continuous variable X is described by a **probability density function** 概率密度函数 $f(x)$, which may be defined piecewise. The probability over a range is the area under f, and the mean of any function of X is

$$E(g(X)) = \int g(x)\, f(x)\, dx.$$



The median sits where the area under $f(x)$ to its left is exactly 0.5.

The **cumulative distribution function** 累积分布函数 $F(x) = P(X \leqslant x)$ is the running total: $F(x) = \displaystyle\int_{-\infty}^{x} f(t)\, dt$, and $f(x) = F'(x)$. Use F to find probabilities and **percentiles** 百分位数 (for example, the **median** 中位数 solves $F(x) = 0.5$).



The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

Worked example. A variable has $f(x) = \frac{1}{2}x$ for $0 \leqslant x \leqslant 2$. Find the median.

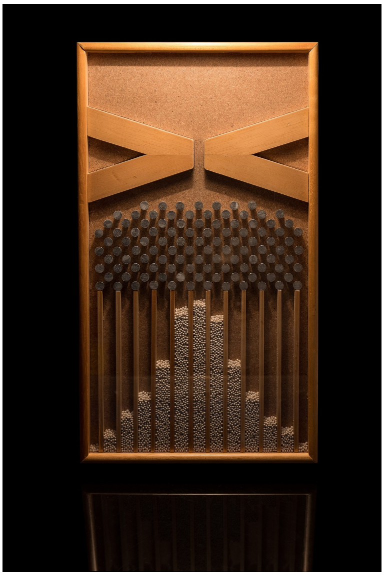
The cumulative distribution function is $F(x) = \displaystyle\int_0^x \tfrac{1}{2}t\,dt = \tfrac{1}{4}x^2$. Set $F(m) = 0.5$:

$$\tfrac{1}{4}m^2 = 0.5 \;\Rightarrow\; m^2 = 2 \;\Rightarrow\; m = \sqrt{2} = 1.41.$$

The distribution of a related variable. If $Y = g(X)$, find Y's distribution through its cumulative function. For $Y = X^2$: $F_Y(y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, then differentiate for $f_Y = F_Y'$. When g is monotonic increasing there is a shortcut, $F_Y(y) = F_X\big(g^{-1}(y)\big)$.

Worked example. With $f_X(x) = \frac{1}{2}x$ on $[0, 2]$ (so $F_X(x) = \frac{1}{4}x^2$), let $Y = X^2$. For $0 \leqslant y \leqslant 4$, $F_Y(y) = F_X(\sqrt{y}) = \frac{1}{4}y$, so $f_Y(y) = F_Y'(y) = \frac{1}{4}$ –that is, Y is uniform on $[0, 4]$.

Inference using normal and t-distributions

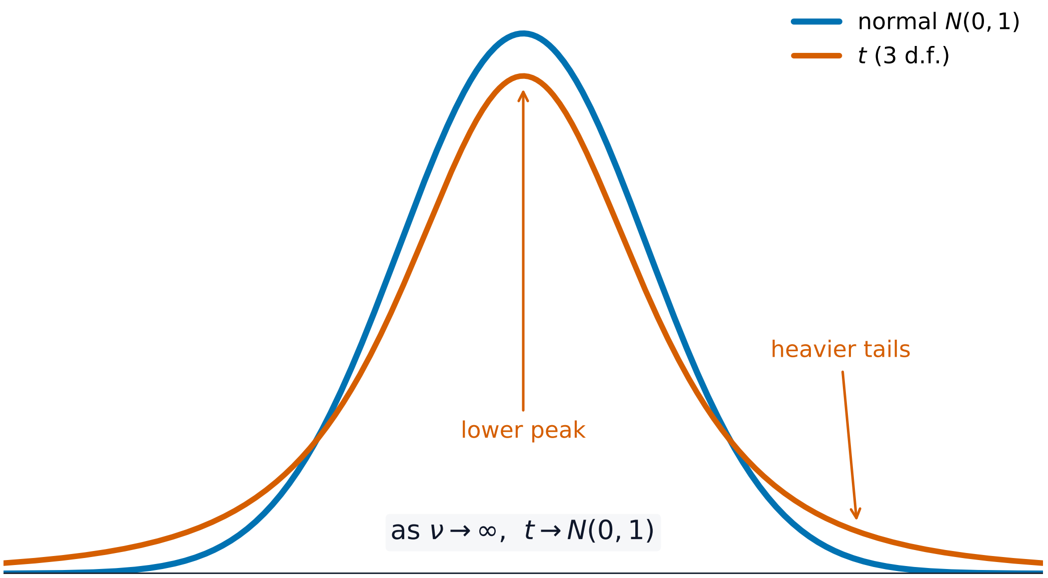


A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

When a sample is small and the population variance is unknown, base your **hypothesis test** 假设检验 on the t-distribution instead of the normal. The same idea gives a **confidence interval** 置信区间 for the mean:

$$\bar{x} \pm t\,\frac{s}{\sqrt{n}},$$

where t comes from the t-tables with $n-1$ degrees of freedom. To compare two populations, use a **two-sample** (2-sample) or **paired-sample** t-test, after finding a **pooled estimate** 合并估计 of the shared variance when appropriate.



For a small sample the t-distribution is flatter with heavier tails, so its critical values are larger.

Worked example. A sample of $n = 10$ has mean $\bar{x} = 50$ and standard deviation $s = 4$. Find a 95% confidence interval for the mean (use $t = 2.262$ for 9 degrees of freedom).

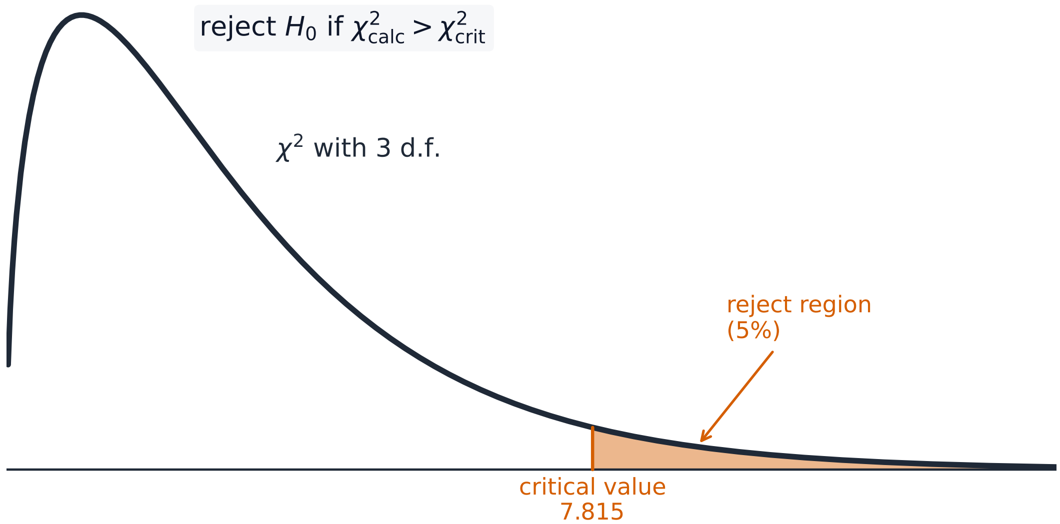
$$50 \pm 2.262 \times \frac{4}{\sqrt{10}} = 50 \pm 2.86 \;\Rightarrow\; (47.1,\; 52.9).$$

Chi-squared tests

A χ^2-test (**chi-squared test** 卡方检验) compares observed counts O with expected counts E from a **theoretical distribution** 理论分布:

$$\chi^2 = \sum \frac{(O-E)^2}{E}.$$

Compare this with a table value for the right number of **degrees of freedom** 自由度. Two uses: a **goodness of fit** 拟合优度 test (does the data follow the proposed model?), and a test for **independence** 独立性 of two variables in a **contingency table** 列联表.



The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5% tail.

Worked example. Four equally likely categories give observed counts $20, 30, 25, 25$ (so each expected count is 25). Test the fit at the 5% level.

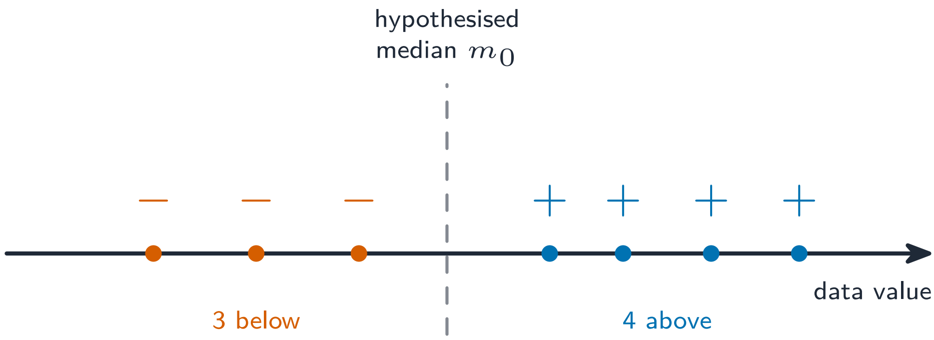
$$\chi^2 = \frac{(20-25)^2 + (30-25)^2 + 0 + 0}{25} = \frac{25+25}{25} = 2.$$

With $4-1=3$ degrees of freedom the table value is 7.815. Since $2 < 7.815$, do not reject the model.

Non-parametric tests

A **non-parametric test** 非参数检验 makes no assumption that the data is normal, so it is useful when that assumption fails. The basic ones are:

- the **sign test** 符号检验: count how many values fall above and below a proposed median, and test those counts with a binomial model;
- the **Wilcoxon signed-rank test** 威尔科克森符号秩检验 (the **matched-pairs** test for paired data), which also uses the sizes of the differences, not just their signs;
- the **Wilcoxon rank-sum test** 威尔科克森秩和检验, for comparing two separate samples.



The sign test counts how many values fall above and below the proposed median, then tests those counts with a binomial model

Worked example. Test whether a median is 5. In a sample of 10 values (none equal to 5), 9 lie **above** 5 and 1 lies below. Test at the 5% level (two-tailed).

Under H_0 (median $= 5$) the number above follows $B(10, 0.5)$. The observed result (9 above) is extreme, so find $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + (0.5)^{10} = \dfrac{11}{1024} = 0.0107$. For a two-tailed test compare with $\frac{1}{2}(5\%) = 0.025$. Since $0.0107 < 0.025$, **reject** H_0: there is evidence the median is not 5.

Probability generating functions

Dice: a starting point for probability and discrete random variables.

The **probability generating function** 概率母函数 of a discrete variable X is

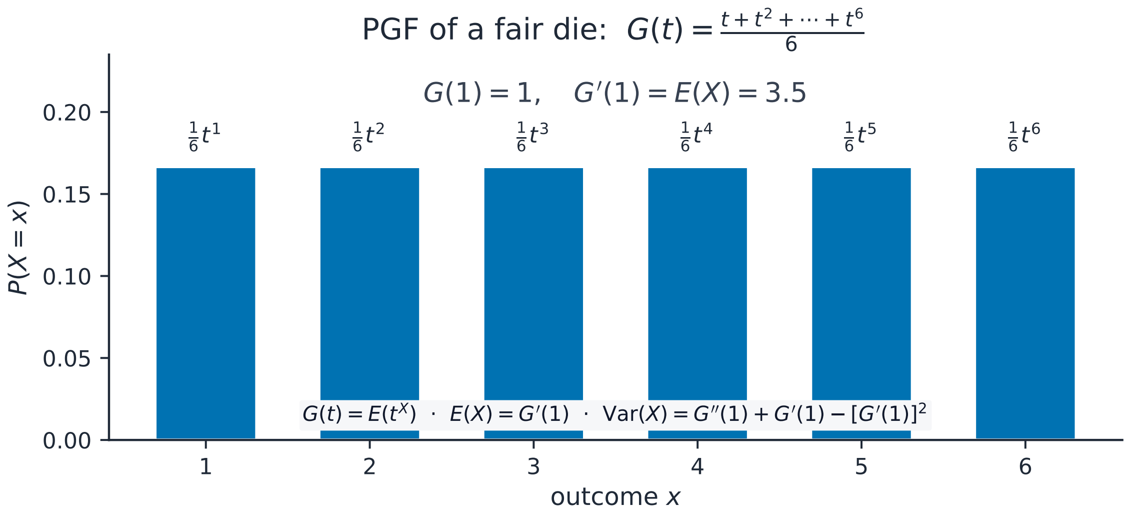
$$G(t) = E(t^X) = \sum_x P(X = x)\, t^x.$$

It packs the whole distribution into one function. The mean and variance come from its derivatives at $t = 1$: $E(X) = G'(1)$ and $\mathrm{Var}(X) = G''(1) + G'(1) - \big(G'(1)\big)^2$. Also, the PGF of a sum of independent variables is the product of their PGFs.

Worked example. X has $P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$. Find $E(X)$ using the PGF.

Here $G(t) = 0.5 + 0.3t + 0.2t^2$, so $G'(t) = 0.3 + 0.4t$ and

$$E(X) = G'(1) = 0.3 + 0.4 = 0.7.$$



PGF of a fair die packs equal masses at 1 through 6 into $G(t)$

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int x f(x)\,dx$.
- Use the t**-distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum(O - E)^2/E$, compare with the critical value at the right degrees of freedom, and **combine classes with** $E < 5$.
- State H_0 and H_1 and give the conclusion **in context** for every test.