| | |
|---|---|
| 5(a) | $\bar{x} = ka + \frac{1}{3}h$ |
| | $\bar{x} = \frac{1}{3}(3ka + h)$ |
| | $\bar{y} = a + \frac{1}{3}(h - a) = \frac{1}{3}(2a + h)$ |
| | **Alternative method for question 5(a)** |
| | Coordinates $A(ka, 0)$, $B(ka + h, h)$, $C(ka, 2a)$<br>$\bar{x} = \frac{1}{3}(ka + ka + h + ka) = \frac{1}{3}(3ka + h)$ |
| | $\bar{y} = \frac{1}{3}(0 + h + 2a) = \frac{1}{3}(h + 2a)$ |
| | |

| | |
|---|---|
| 5(b) | Taking moments about the $x$-axis:<br>$\bar{x}\left(2ka^2+ah\right)=2ka^2\times\frac{1}{2}ka+ah\left(ka+\frac{1}{3}h\right)$ |
| | For equilibrium, $\bar{x}\leqslant ka$, so<br>$\dfrac{a^2k^2+ahk+\frac{1}{3}h^2}{2ak+h}\leqslant ka$ |
| | $(0\leqslant)h\leqslant\sqrt{3}ka$ |
| | |
| 5(c) | Point of toppling means $h=\sqrt{3}ka\,[=a]$.<br>$\bar{x}=\frac{1}{3}a\left(\sqrt{3}+1\right)$ |
| | [ $h=a$ , meaning triangle $ABC$ is isosceles.]<br>$\bar{y}=a$ (by symmetry) |
| | |

| | |
|---|---|
| 3(a) | Vertical distance of COM of sector from $OA$: $\dfrac{\left(2a\cos\left(\frac{1}{4}\pi\right)\right)\sin\left(\frac{1}{4}\pi\right)}{3\left(\frac{1}{4}\pi\right)} = \dfrac{4a}{3\pi}$ |
| | $ka^2\left(\frac{1}{2}a\right) + \frac{1}{4}\pi a^2\left(a - \dfrac{4a}{3\pi}\right) = \left(ka^2 + \frac{1}{4}\pi a^2\right)\bar{y}$ <br> $\left[6ka + 3\pi a - 4a = 3(4k+\pi)\bar{y}\right]$ |
| | $\bar{y} = \dfrac{6k + 3\pi - 4}{3(4k+\pi)}a$ |
| | |
| 3(b) | $ka^2\left(\frac{1}{2}ka\right) = \frac{1}{4}\pi a^2\left(\dfrac{4a}{3\pi}\right)$ |
| | $\left[k^2 = \frac{2}{3}\right]$ $k = \sqrt{\frac{2}{3}} = 0.816$ |
| | **Alternative method for question 3(b)** |
| | On the point of toppling, $\bar{x} = ka$: $\dfrac{\left(6k^2 + 3k\pi + 4\right)a}{3(4k+\pi)} = ka$ |
| | $\left[6k^2 + 4 = 12k^2\right]$ $k = \sqrt{\frac{2}{3}} = 0.816$ |
| | |

| | | | | |
|---|---|---|---|---|
| 3(a) | | Area | From $AD$ | From $AB$ |
| | $ABCD$ | $4a^2$ | $a$ | $a$ |
| | $AED$ | $a^2$ | $\frac{1}{3}a$ | $\frac{2}{3}a$ |
| | $BCF$ | $ah$ | $2a-\frac{1}{3}h$ | $\frac{4}{3}a$ |
| | Shape $BFDE$ | $3a^2-ah$ | $\bar{x}$ | $\bar{y}$ |

Taking moments about $AD$:

$$\left(3a^2-ah\right)\bar{x}=4a^2\times a-a^2\times\tfrac{1}{3}a-ah\times\left(2a-\tfrac{1}{3}h\right)$$

$$\bar{x}=\frac{\frac{11}{3}a^3-2a^2h+\frac{1}{3}ah^2}{3a^2-ah}=\frac{\frac{1}{3}a(h^2-6ah+11a^2)}{a(3a-h)}=\frac{h^2-6ah+11a^2}{3\left(3a-h\right)}$$

Taking moments about $AB$:

$$\left(3a^2-ah\right)\bar{y}=4a^2\times a-a^2\times\tfrac{2}{3}a-ah\times\tfrac{4}{3}a$$

$$\bar{y}=\frac{2a\left(5a-2h\right)}{3\left(3a-h\right)}$$

| | |
|---|---|
| 3(a) | **Alternative method for question 3(a)** |
| | <table><tr><td></td><td>Area</td><td>From $AD$</td><td>From $AB$</td></tr><tr><td>$DEF$</td><td>$a(2a-h)$</td><td>$\frac{1}{3}(3a-h)$</td><td>$\frac{4}{3}a$</td></tr><tr><td>$EBF$</td><td>$a^2$</td><td>$\frac{1}{3}(5a-h)$</td><td>$\frac{2}{3}a$</td></tr><tr><td>Shape $BFDE$</td><td>$3a^2-ah$</td><td>$\bar{x}$</td><td>$\bar{y}$</td></tr></table> |
| | Taking moments about $AD$:<br>$\left(3a^2-ah\right)\bar{x}=a\left(2a-h\right)\times\frac{1}{3}(3a-h)+a^2\times\frac{1}{3}(5a-h)$ |
| | $\bar{x}=\dfrac{\frac{1}{3}a(6a^2-5ah+h^2+5a^2-ah)}{3a^2-ah}=\dfrac{\frac{1}{3}a(h^2-6ah+11a^2)}{a(3a-h)}=\dfrac{h^2-6ah+11a^2}{3\left(3a-h\right)}$ |
| | Taking moments about $AB$:<br>$\left(3a^2-ah\right)\bar{y}=a\left(2a-h\right)\times\frac{4}{3}a+a^2\times\frac{2}{3}a$ |
| | $\bar{y}=\dfrac{2a\left(5a-2h\right)}{3\left(3a-h\right)}$ |
| | |
| 3(b) | $\bar{x}\geqslant a$ meaning $h^2-3ah+2a^2\geqslant 0$ |
| | $[0<]\,h\leqslant a,\ \ h=2a$ . |
| | |

| | | |
|---|---|---|
| 4(a) | | |

| | Volume | CoM above base |
|---|---|---|
| Cone | $18\pi r^3$ | $\frac{3}{2}r$ |
| Hemisphere | $\frac{16}{3}\pi r^3$ | $\frac{3}{4}r$ |
| Total object | $18\pi r^3 - \frac{16}{3}\pi r^3$ | $\bar{y}$ |

$\left(18\pi r^3 - \frac{16}{3}\pi r^3\right)\bar{y} = 18\pi r^3\left(\frac{3}{2}r\right) - \frac{16}{3}\pi r^3\left(\frac{3}{4}r\right)$

$\bar{y} = \frac{69}{38}r$

4(b)

When on point of toppling:

$\tan\alpha = \dfrac{3r}{their\ \bar{y}} \quad \left[= \frac{38}{23}\right]$

$\alpha = 58.8^{\circ}$

4

| Areas | $2a^2$ | $\frac{1}{2}a(2a)$ | $2a^2+\frac{1}{2}a(2a)$ |
|---|---|---|---|
| Distances from $D$ | $-\frac{1}{2}a$ | $\frac{2}{3}a$ | $\bar{x}$ |

$$-2a^2\left(\tfrac{1}{2}a\right)+\tfrac{1}{2}\left(2a^2\right)\left(\tfrac{2}{3}a\right)=\left(3a^2\right)\bar{x}$$

$$m\left(\tfrac{1}{9}a\right)=km(2a)$$

$$k=\tfrac{1}{18}$$

**Alternative method for question 4**

| Ratio of masses | $2a^2$ | $\frac{1}{2}a(2a)$ | $k\left(2a^2+\frac{1}{2}a(2a)\right)$ |
|---|---|---|---|

$$2\left(\tfrac{1}{2}a\right)=1\left(\tfrac{2}{3}a\right)+3k(2a)$$

$$k=\tfrac{1}{18}$$