

7(a)	$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{2} \left(\frac{1-x}{1+x} \right) \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2}$
	$= \frac{1}{2} \left(\frac{1-x}{1+x} \right) \frac{2}{(1-x)^2} = \frac{1}{1-x^2}$
	Alternative method for question 7(a)
	$\tanh y = x \Rightarrow \operatorname{sech}^2 y \frac{dy}{dx} = 1$
	$\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}$
7(b)	$\frac{dy}{dx} - \frac{y}{x} = x \tanh^{-1} x$
	$e^{-\int x^{-1} d\theta} = e^{-\ln x} = x^{-1}$
	$\frac{d}{dx}(x^{-1}y) = \tanh^{-1} x$
	$x^{-1}y = x \tanh^{-1} x - \int \frac{x}{1-x^2} dx$
	$x^{-1}y = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) + C$
	$0 = \frac{1}{2} \tanh^{-1} \frac{1}{2} + \frac{1}{2} \ln \frac{3}{4} + C \Rightarrow C = -\frac{1}{4} \ln 3 - \frac{1}{2} \ln \frac{3}{4}$
	$x^{-1}y = x \tanh^{-1} x + \frac{1}{2} \ln(1-x^2) - \frac{1}{4} \ln \frac{27}{16}$

5(a)	$m^2 + 4m + 3 = 0 \Rightarrow m = -1, -3$
	$[y =] Ae^{-x} + Be^{-3x}$
	$y = p \sin x + q \cos x \Rightarrow y' = p \cos x - q \sin x \Rightarrow y'' = -p \sin x - q \cos x$
	$-p - 4q + 3p = 0 \quad -q + 4p + 3q = 5$
	$p = 1 \quad q = \frac{1}{2}$
	$y = Ae^{-x} + Be^{-3x} + \sin x + \frac{1}{2} \cos x$
5(b)	$y \approx \sin x + \frac{1}{2} \cos x = \frac{\sqrt{5}}{2} \left(\frac{2}{\sqrt{5}} \sin x + \frac{1}{\sqrt{5}} \cos x \right)$
	$\frac{\sqrt{5}}{2} \sin(x + 0.464...)$

	$5\theta = \frac{1}{3}\pi + 2k\pi \quad \text{or} \quad 5\theta = \frac{2}{3}\pi + 2k\pi$
	$\sin\left(\frac{1}{15}\pi\right)$
	$\sin\left(\frac{2}{15}\pi\right), \sin\left(-\frac{5}{15}\pi\right), \sin\left(-\frac{4}{15}\pi\right), \sin\left(\frac{7}{15}\pi\right)$
4(a)	$\int_0^1 \frac{1}{2}(3^x) dx < \frac{1}{2} \left(\left(\frac{1}{N}\right)3^{\frac{1}{N}} + \left(\frac{1}{N}\right)3^{\frac{2}{N}} + \dots + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}} + \left(\frac{1}{N}\right)3^{\frac{N}{N}} \right)$
	$= \frac{1}{2N} \sum_{n=1}^N \left(3^{\frac{1}{N}}\right)^n = \frac{3^{\frac{1}{N}}}{N(3^{\frac{1}{N}} - 1)}$
4(b)	$\int_0^1 \frac{1}{2}(3^x) dx > \frac{1}{2} \left(\left(\frac{1}{N}\right) + \left(\frac{1}{N}\right)3^{\frac{1}{N}} + \dots + \left(\frac{1}{N}\right)3^{\frac{N-2}{N}} + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}} \right)$
	$= \frac{1}{2N} \sum_{n=0}^{N-1} \left(3^{\frac{1}{N}}\right)^n = \frac{1}{N(3^{\frac{1}{N}} - 1)}$
4(c)	$\frac{3^{\frac{1}{N}}}{N(3^{\frac{1}{N}} - 1)} - \frac{1}{N(3^{\frac{1}{N}} - 1)} = \frac{1}{N}$
	$\frac{1}{N} \rightarrow 0 \text{ as } N \rightarrow \infty.$

5(a)	$m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$
	$y = e^{-\frac{1}{2}x} \left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right)$
	$y = p \sin 2x + q \cos 2x \Rightarrow y' = 2p \cos 2x - 2q \sin 2x \Rightarrow y'' = -4p \sin 2x - 4q \cos 2x$
	$-4p - 2q + p = 1 \qquad -4q + 2p + q = 1$
	$p = -\frac{1}{13} \qquad q = -\frac{5}{13}$
	$y = e^{-\frac{1}{2}x} \left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x \right) - \frac{1}{13} \sin 2x - \frac{5}{13} \cos 2x$
5(b)	$y \approx -\frac{1}{13} \sin 2x - \frac{5}{13} \cos 2x = \sqrt{\frac{2}{13}} \left(-\frac{1}{\sqrt{26}} \sin 2x - \frac{5}{\sqrt{26}} \cos 2x \right)$
	$\sqrt{\frac{2}{13}} \sin(2x - 1.77)$

6	$\int \frac{1+x}{1-2x-x^2} dx = -\frac{1}{2} \ln(1-2x-x^2)$
	$e^{-\frac{1}{2} \ln(1-2x-x^2)} = \frac{1}{\sqrt{1-2x-x^2}}$
	$\frac{d}{dx} \left(\frac{y}{\sqrt{1-2x-x^2}} \right) = \frac{1}{\sqrt{1-2x-x^2}}$
	$1-2x-x^2 = 2-(x+1)^2$
	$\int \frac{1}{\sqrt{2-(x+1)^2}} dx = \sin^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + C$
	$\pi = \sin^{-1} \frac{1}{\sqrt{2}} + C \Rightarrow C = \frac{3}{4} \pi$
	$\frac{y}{\sqrt{1-2x-x^2}} = \sin^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + \frac{3}{4} \pi$

$5 \int_{\frac{\pi}{6}}^{\frac{1}{2}\pi} \operatorname{cosec}^2 t + 1 dt = 5 \left[-\cot t + t \right]_{\frac{\pi}{6}}^{\frac{1}{2}\pi} = 5 \left(-\frac{1}{\sqrt{3}} + \frac{1}{3} \pi + \sqrt{3} - \frac{1}{6} \pi \right) = \frac{5}{6} \pi + \frac{10}{3} \sqrt{3}$

7(b)	$\frac{\dot{y}}{\dot{x}} = \frac{6 \operatorname{cosec} t - 4 \cot^2 t}{8 \operatorname{cosec} t + 3 \cot^2 t} = \frac{6 \sin t - 4 \cos^2 t}{8 \sin t + 3 \cos^2 t} = \frac{6 \sec t - 4 \cot t}{8 \sec t + 3 \cot t}$
	$\frac{d}{dt} \left(\frac{6 \operatorname{cosec} t - 4 \cot^2 t}{8 \operatorname{cosec} t + 3 \cot^2 t} \right)$ $= \frac{(8 \csc t + 3 \cot^2 t)(-6 \cot t \csc t + 8 \cot t \csc^2 t) + (6 \csc t - 4 \cot^2 t)(6 \csc^2 t \cot t + 8 \cot t \csc t)}{(8 \operatorname{cosec} t + 3 \cot^2 t)^2}$
	or $\frac{d}{dt} \left(\frac{6 \sin t - 4 \cos^2 t}{8 \sin t + 3 \cos^2 t} \right)$ $= \frac{(8 \sin t + 3 \cos^2 t)(6 \cos t + 8 \cos t \sin t) - (6 \sin t - 4 \cos^2 t)(8 \cos t - 6 \cos t \sin t)}{(8 \sin t + 3 \cos^2 t)^2}$
	$\frac{100 \cos t \sin^2 t + 50 \cos^3 t}{(8 \sin t + 3 \cos^2 t)^2} = \frac{50 \cos t (\sin^2 t + 1)}{(8 \sin t + 3 \cos^2 t)^2} = \frac{50 \cot t \operatorname{cosec} t (\operatorname{cosec}^2 t + 1)}{(8 \operatorname{cosec} t + 3 \cot^2 t)^2}$
	$\frac{d^2 y}{dx^2} = \frac{d}{dt} \left(\frac{6 \sin t - 4 \cos^2 t}{8 \sin t + 3 \cos^2 t} \right) \times \frac{dt}{dx} = \frac{50 \sin^2 t \cos t (\sin^2 t + 1)}{(8 \sin t + 3 \cos^2 t)^3}$ $= \frac{50 \cot t \operatorname{cosec} t (\operatorname{cosec}^2 t + 1)}{(8 \operatorname{cosec} t + 3 \cot^2 t)^3}$

8(a)	$\begin{vmatrix} 5 & a & 0 \end{vmatrix}$
6	$\int \frac{1+x}{1-2x-x^2} dx = -\frac{1}{2} \ln(1-2x-x^2)$
	$e^{-\frac{1}{2} \ln(1-2x-x^2)} = \frac{1}{\sqrt{1-2x-x^2}}$
	$\frac{d}{dx} \left(\frac{y}{\sqrt{1-2x-x^2}} \right) = \frac{1}{\sqrt{1-2x-x^2}}$

5

$$6m^2 + 3m + 6 = 0 \Rightarrow m = -\frac{1}{4} \pm \frac{1}{4}i\sqrt{15}$$

$$x = e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right)$$

$$x = ke^{-t} \Rightarrow x' = -ke^{-t} \Rightarrow x'' = ke^{-t}$$

$$6ke^{-t} - 3ke^{-t} + 6ke^{-t} = e^{-t}$$

$$k = \frac{1}{9}$$

$$x = e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right) + \frac{1}{9}e^{-t}$$

$$x' = \frac{\sqrt{15}}{4}e^{-\frac{1}{4}t} \left(-A \sin \frac{\sqrt{15}}{4}t + B \cos \frac{\sqrt{15}}{4}t \right) - \frac{1}{4}e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right) - \frac{1}{9}e^{-t}$$

$$0 = A + \frac{1}{9} \quad 0 = \frac{\sqrt{15}}{4}B - \frac{1}{4}A - \frac{1}{9} \Rightarrow A = -\frac{1}{9}, \quad B = \frac{1}{3\sqrt{15}}$$

$$x = \frac{1}{3}e^{-\frac{1}{4}t} \left(-\frac{1}{3}\cos \frac{\sqrt{15}}{4}t + \frac{1}{\sqrt{15}}\sin \frac{\sqrt{15}}{4}t \right) + \frac{1}{9}e^{-t}$$

7

$$\int \frac{x+5}{x^2+10x+61} dx = \frac{1}{2} \ln(x^2+10x+61)$$

$$e^{-\frac{1}{2}\ln(x^2+10x+61)} = \frac{1}{\sqrt{x^2+10x+61}}$$

$$\frac{d}{dx} \left(\frac{y}{\sqrt{x^2+10x+61}} \right) = \frac{1}{\sqrt{x^2+10x+61}}$$

$$x^2+10x+61 = (x+5)^2 + 36$$

$$\int \frac{1}{\sqrt{(x+5)^2+6^2}} dx = \sinh^{-1} \left(\frac{x+5}{6} \right) + C$$

$$C = -\sinh^{-1} \left(\frac{8}{6} \right) = -\ln \left(\frac{8}{6} + \frac{5}{3} \right) = -\ln 3$$

$$\frac{y}{\sqrt{x^2+10x+61}} = \sinh^{-1} \left(\frac{x+5}{6} \right) - \sinh^{-1} \left(\frac{4}{3} \right)$$

4	$m^2 + m - 2 = 0 \Rightarrow m = 1, -2$
	$x = Ae^t + Be^{-2t}$
	$x = pt^2 + qt + r \Rightarrow x' = 2pt + q \Rightarrow x'' = 2p$
	$p = -1 \quad 2p - 2q = 1 \quad 2p + q - 2r = -1$
	$p = -1 \quad q = -\frac{3}{2} \quad r = -\frac{5}{4}$
	$x = Ae^t + Be^{-2t} - t^2 - \frac{3}{2}t - \frac{5}{4}$
	$x' = Ae^t - 2Be^{-2t} - 2t - \frac{3}{2}$
	$A + B - \frac{5}{4} = 0 \quad A - 2B - \frac{3}{2} = 0 \Rightarrow A = \frac{4}{3}, B = -\frac{1}{12}$
	$x = \frac{4}{3}e^t - \frac{1}{12}e^{-2t} - t^2 - \frac{3}{2}t - \frac{5}{4}$

7	$\int \frac{2x+6}{x^2+6x+5} dx = \ln(x^2+6x+5)$
	$e^{-\ln(x^2+6x+5)} = \frac{1}{x^2+6x+5}$
	$\frac{d}{dx} \left(\frac{y}{x^2+6x+5} \right) = \frac{4}{x^2+6x+5}$
	$\frac{4}{x^2+6x+5} = \frac{1}{x+1} - \frac{1}{x+5}$
	$\int \frac{4}{x^2+6x+5} dx = \ln \left(\frac{x+1}{x+5} \right) + C$
	$0 = \ln \frac{1}{5} + C$
	$\frac{y}{x^2+6x+5} = \ln \left(\frac{x+1}{x+5} \right) - \ln \frac{1}{5} = \ln \left(\frac{5x+5}{x+5} \right)$