| | |
|---|---|
| 8(a) | $\dfrac{z^{n+1}-z}{z-1}$ |
| | |
| 8(b) | $\dfrac{z^{n+1}-z}{z-1}=\dfrac{\left(\frac{1}{2}\right)^{n+1}\cos(n+1)\theta-\frac{1}{2}\cos\theta+\mathrm{i}\left(\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta-\frac{1}{2}\sin\theta\right)}{\frac{1}{2}\cos\theta-1+\mathrm{i}\frac{1}{2}\sin\theta}$ |
| | $\dfrac{\left(\frac{1}{2}\right)^{n+1}\cos(n+1)\theta-\frac{1}{2}\cos\theta+\mathrm{i}\left(\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta-\frac{1}{2}\sin\theta\right)}{\frac{1}{2}\cos\theta-1+\mathrm{i}\frac{1}{2}\sin\theta}\times\dfrac{\frac{1}{2}\cos\theta-1-\mathrm{i}\frac{1}{2}\sin\theta}{\frac{1}{2}\cos\theta-1-\mathrm{i}\frac{1}{2}\sin\theta}$ |
| | $\dfrac{-\frac{1}{2}\sin\theta\left(\left(\frac{1}{2}\right)^{n+1}\cos(n+1)\theta-\frac{1}{2}\cos\theta\right)+\left(\frac{1}{2}\cos\theta-1\right)\left(\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta-\frac{1}{2}\sin\theta\right)}{\frac{5}{4}-\cos\theta}$ |
| | $\dfrac{-\left(\frac{1}{2}\right)^{n+2}\sin\theta\cos(n+1)\theta+\left(\frac{1}{2}\right)^{n+2}\cos\theta\sin(n+1)\theta-\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta+\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}$ $=\dfrac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta-\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta+\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}$ |





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| 8(b) | **Alternative method for question 8(b)** |
| | $z=\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ |
| | $\dfrac{z^{n+1}-z}{z-1}=\dfrac{\left(\frac{1}{2}\right)^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}-\left(\frac{1}{2}\right)\mathrm{e}^{\mathrm{i}\theta}}{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}-1}$ |
| | $\dfrac{\left(\frac{1}{2}\right)^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}-\left(\frac{1}{2}\right)\mathrm{e}^{\mathrm{i}\theta}}{\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}-1}\times\dfrac{\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}-1}{\frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}-1}$ |
| | $\mathrm{Im}\left(\dfrac{\left(\frac{1}{2}\right)^{n+2}\mathrm{e}^{\mathrm{i}n\theta}-\left(\frac{1}{2}\right)^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}+\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}-\frac{1}{4}}{\frac{5}{4}-\cos\theta}\right)$ $=\dfrac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta-\left(\frac{1}{2}\right)^{n+1}\sin\left(n+1\right)\theta+\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}$ |
| | |
| 8(c) | $\displaystyle\sum_{m=1}^{n}\left(\frac{1}{2}\right)^{m}m\cos m\theta=\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta-\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta+\frac{1}{2}\sin\theta}{\frac{5}{4}-\cos\theta}\right)$ |
| | $=\dfrac{\left(\frac{5}{4}-\cos\theta\right)\left(\left(\frac{1}{2}\right)^{n+2}n\cos n\theta-\left(\frac{1}{2}\right)^{n+1}(n+1)\cos\left(n+1\right)\theta+\frac{1}{2}\cos\theta\right)}{\left(\frac{5}{4}-\cos\theta\right)^{2}}$ $-\dfrac{\left(\left(\frac{1}{2}\right)^{n+2}\sin n\theta-\left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta+\frac{1}{2}\sin\theta\right)\sin\theta}{\left(\frac{5}{4}-\cos\theta\right)^{2}}$ |
| | |

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| 8(d) | $\displaystyle\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{m}m\cos m\theta=\frac{\left(\frac{5}{4}-\cos\theta\right)\left(\frac{1}{2}\cos\theta\right)-\left(\frac{1}{2}\sin\theta\right)\sin\theta}{\left(\frac{5}{4}-\cos\theta\right)^{2}}$ |
| | $=\dfrac{\frac{5}{8}\cos\theta-\frac{1}{2}}{\left(\frac{5}{4}-\cos\theta\right)^{2}}$ |
| | |

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| 2 | $z^4 = 8 - 8\mathrm{i}\sqrt{3} = 16\mathrm{e}^{-\mathrm{i}\frac{1}{3}\pi}$ |
| | $z_1 = 2\mathrm{e}^{-\mathrm{i}\frac{1}{12}\pi}$ |
| | $z_2 = 2\mathrm{e}^{\mathrm{i}\frac{5}{12}\pi}$ |
| | $z_3 = 2\mathrm{e}^{-\mathrm{i}\frac{7}{12}\pi}$, $z_4 = 2\mathrm{e}^{\mathrm{i}\frac{11}{12}\pi}$ |
| | |

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| 4(a) | $z + z^{-1} = 2\cos\theta$ |
| | $\left(z+z^{-1}\right)^6 = \left(z^6+z^{-6}\right) + 6\left(z^4+z^{-4}\right) + 15\left(z^2+z^{-2}\right) + 20$ |
| | $2^6\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ |
| | $\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ |
| | |
| 4(b) | $\int_0^{\frac{1}{2}\pi}\cos^6 2x\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi}\left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right)\mathrm{d}\theta$ |
| | $= \frac{1}{2}\left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\pi}$ |
| | $= \frac{5}{32}\pi$ |
| | |





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| 3(a) | $\sin 5\theta = \mathrm{Im}\left(\cos\theta + \mathrm{i}\sin\theta\right)^5 = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$ |
| | $= \sin^5\theta - 10\sin^3\theta\left(1-\sin^2\theta\right) + 5\sin\theta\left(1-\sin^2\theta\right)^2$ |
| | $= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ |
| | |
| 3(b) | $x = \sin\theta$, $\sin 5\theta = \frac{1}{2}\sqrt{3}$ |

A-Level Further Mathematics: **9231 Mathematics - Further June 2025 Question Paper 21**

| | |
|---|---|
| 1 | $z^3 = 27 - 27\mathrm{i} = 27\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{1}{4}\pi}$ |
| | $z_1 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\frac{1}{12}\pi}$ |
| | $z_2 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{\mathrm{i}\frac{7}{12}\pi}$, $z_3 = 3\times 2^{\frac{1}{6}}\mathrm{e}^{-\mathrm{i}\frac{9}{12}\pi}$ |
| | |
| 3 | $z - z^{-1} = 2\mathrm{i}\sin\theta$ |
| | $\left(z-z^{-1}\right)^5 = \left(z^5-z^{-5}\right) - 5\left(z^3-z^{-3}\right) + 10\left(z-z^{-1}\right)$ |
| | $2^5\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$ |
| | $\Rightarrow\ \mathrm{cosec}^5\theta = \dfrac{16}{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}.$ |
| | |

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| 4(b) | $\frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{2}}+\frac{1}{\sqrt{3}}\mathrm{e}^{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\mathrm{e}^{\sqrt{n}}$ |
| | $>\int_1^n \frac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x=\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n$ |
| | $\sum_{r=1}^{n}\frac{1}{\sqrt{r}}\mathrm{e}^{\sqrt{r}}>\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n+\mathrm{e}=2\mathrm{e}^{\sqrt{n}}-\mathrm{e}$ |
| | |





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| 5 | $6m^2+3m+6=0\Rightarrow m=-\frac{1}{4}\pm\frac{1}{4}\mathrm{i}\sqrt{15}$ |
| | $x=\mathrm{e}^{-\frac{1}{4}t}\left(A\cos\frac{\sqrt{15}}{4}t+B\sin\frac{\sqrt{15}}{4}t\right)$ |
| | $x=k\mathrm{e}^{-t}\Rightarrow x'=-k\mathrm{e}^{-t}\Rightarrow x''=k\mathrm{e}^{-t}$ |
| | $6k\mathrm{e}^{-t}-3k\mathrm{e}^{-t}+6k\mathrm{e}^{-t}=\mathrm{e}^{-t}$ |
| | $k=\frac{1}{9}$ |
| | $x=\mathrm{e}^{-\frac{1}{4}t}\left(A\cos\frac{\sqrt{15}}{4}t+B\sin\frac{\sqrt{15}}{4}t\right)+\frac{1}{9}\mathrm{e}^{-t}$ |
| | $x'=\frac{\sqrt{15}}{4}\mathrm{e}^{-\frac{1}{4}t}\left(-A\sin\frac{\sqrt{15}}{4}t+B\cos\frac{\sqrt{15}}{4}t\right)-\frac{1}{4}\mathrm{e}^{-\frac{1}{4}t}\left(A\cos\frac{\sqrt{15}}{4}t+B\sin\frac{\sqrt{15}}{4}t\right)-\frac{1}{9}\mathrm{e}^{-t}$ |
| | $0=A+\frac{1}{9}\quad 0=\frac{\sqrt{15}}{4}B-\frac{1}{4}A-\frac{1}{9}\Rightarrow A=-\frac{1}{9},\quad B=\frac{1}{3\sqrt{15}}$ |
| | $x=\frac{1}{3}\mathrm{e}^{-\frac{1}{4}t}\left(-\frac{1}{3}\cos\frac{\sqrt{15}}{4}t+\frac{1}{\sqrt{15}}\sin\frac{\sqrt{15}}{4}t\right)+\frac{1}{9}\mathrm{e}^{-t}$ |
| | |

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| | $1-\left(\frac{\mathrm{e}^u-\mathrm{e}^{-u}}{\mathrm{e}^u+\mathrm{e}^{-u}}\right)^2=\frac{\left(\mathrm{e}^u+\mathrm{e}^{-u}\right)^2-\left(\mathrm{e}^u-\mathrm{e}^{-u}\right)^2}{\left(\mathrm{e}^u+\mathrm{e}^{-u}\right)^2}=\frac{4}{\left(\mathrm{e}^u+\mathrm{e}^{-u}\right)^2}$ |
| | $=\operatorname{sech}^2 u$ |
| | |

| | |
|---|---|
| 3 | $z-z^{-1}=2\mathrm{i}\sin\theta$ |
| | $\left(z-z^{-1}\right)^5=\left(z^5-z^{-5}\right)-5\left(z^3-z^{-3}\right)+10\left(z-z^{-1}\right)$ |
| | $2^5\mathrm{i}\sin^5\theta=2\mathrm{i}\sin5\theta-5(2\mathrm{i}\sin3\theta)+10(2\mathrm{i}\sin\theta)$ |
| | $\Rightarrow\ \operatorname{cosec}^5\theta=\dfrac{16}{\sin5\theta-5\sin3\theta+10\sin\theta}.$ |
| | |





| | |
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| 5(a) | $\left(\cos5\theta+\mathrm{i}\sin5\theta\right)=\left(c+\mathrm{i}s\right)^5$ |
| | $\cos5\theta=c^5-10s^2c^3+5s^4c$ |
| | $=c^5-10\left(1-c^2\right)c^3+5\left(1-c^2\right)^2c$ |
| | $=16c^5-20c^3+5c$ |
| | $\sec5\theta=\dfrac{1}{16c^5-20c^3+5c}\times\dfrac{\sec^5\theta}{\sec^5\theta}$ |
| | $=\dfrac{\sec^5\theta}{5\sec^4\theta-20\sec^2\theta+16}$ |
| | |
| 5(b) | $x^5=\frac{2}{\sqrt3}\left(5x^4-20x^2+16\right)\Rightarrow\dfrac{x^5}{5x^4-20x^2+16}=\frac{2}{\sqrt3}$ |
| | $\sec5\theta=\frac{2}{\sqrt3}\Rightarrow\cos5\theta=\frac{\sqrt3}{2}$ |
| | $x=\sec\left(\frac{1}{30}\pi\right)$ |
| | $\sec\left(\frac{11}{30}\pi\right),\sec\left(\frac{13}{30}\pi\right),\sec\left(\frac{23}{30}\pi\right),\sec\left(\frac{25}{30}\pi\right)$ |
| | |