

8(a)	$\frac{z^{n+1} - z}{z - 1}$
8(b)	$\frac{z^{n+1} - z}{z - 1} = \frac{\left(\frac{1}{2}\right)^{n+1} \cos(n+1)\theta - \frac{1}{2} \cos \theta + i \left(\left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta - \frac{1}{2} \sin \theta\right)}{\frac{1}{2} \cos \theta - 1 + i \frac{1}{2} \sin \theta}$
	$\frac{\left(\frac{1}{2}\right)^{n+1} \cos(n+1)\theta - \frac{1}{2} \cos \theta + i \left(\left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta - \frac{1}{2} \sin \theta\right)}{\frac{1}{2} \cos \theta - 1 + i \frac{1}{2} \sin \theta} \times \frac{\frac{1}{2} \cos \theta - 1 - i \frac{1}{2} \sin \theta}{\frac{1}{2} \cos \theta - 1 - i \frac{1}{2} \sin \theta}$
	$\frac{-\frac{1}{2} \sin \theta \left(\left(\frac{1}{2}\right)^{n+1} \cos(n+1)\theta - \frac{1}{2} \cos \theta\right) + \left(\frac{1}{2} \cos \theta - 1\right) \left(\left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta - \frac{1}{2} \sin \theta\right)}{\frac{5}{4} - \cos \theta}$
	$\frac{-\left(\frac{1}{2}\right)^{n+2} \sin \theta \cos(n+1)\theta + \left(\frac{1}{2}\right)^{n+2} \cos \theta \sin(n+1)\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin \theta}{\frac{5}{4} - \cos \theta}$ $= \frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin \theta}{\frac{5}{4} - \cos \theta}$

8(b)	Alternative method for question 8(b)
	$z = \frac{1}{2}e^{i\theta}$
	$\frac{z^{n+1} - z}{z - 1} = \frac{\left(\frac{1}{2}\right)^{n+1} e^{i(n+1)\theta} - \left(\frac{1}{2}\right)e^{i\theta}}{\frac{1}{2}e^{i\theta} - 1}$
	$\frac{\left(\frac{1}{2}\right)^{n+1} e^{i(n+1)\theta} - \left(\frac{1}{2}\right)e^{i\theta}}{\frac{1}{2}e^{i\theta} - 1} \times \frac{\frac{1}{2}e^{-i\theta} - 1}{\frac{1}{2}e^{-i\theta} - 1}$
	$\operatorname{Im}\left(\frac{\left(\frac{1}{2}\right)^{n+2} e^{in\theta} - \left(\frac{1}{2}\right)^{n+1} e^{i(n+1)\theta} + \frac{1}{2}e^{i\theta} - \frac{1}{4}}{\frac{5}{4} - \cos\theta}\right)$ $= \frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin\theta}{\frac{5}{4} - \cos\theta}$
8(c)	$\sum_{m=1}^n \left(\frac{1}{2}\right)^m m \cos m\theta = \frac{d}{d\theta} \left(\frac{\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin\theta}{\frac{5}{4} - \cos\theta} \right)$
	$= \frac{\left(\frac{5}{4} - \cos\theta\right) \left(\left(\frac{1}{2}\right)^{n+2} n \cos n\theta - \left(\frac{1}{2}\right)^{n+1} (n+1) \cos(n+1)\theta + \frac{1}{2} \cos\theta \right)}{\left(\frac{5}{4} - \cos\theta\right)^2}$ $- \frac{\left(\left(\frac{1}{2}\right)^{n+2} \sin n\theta - \left(\frac{1}{2}\right)^{n+1} \sin(n+1)\theta + \frac{1}{2} \sin\theta \right) \sin\theta}{\left(\frac{5}{4} - \cos\theta\right)^2}$
8(d)	$\sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^m m \cos m\theta = \frac{\left(\frac{5}{4} - \cos\theta\right) \left(\frac{1}{2} \cos\theta\right) - \left(\frac{1}{2} \sin\theta\right) \sin\theta}{\left(\frac{5}{4} - \cos\theta\right)^2}$
	$= \frac{\frac{5}{8} \cos\theta - \frac{1}{2}}{\left(\frac{5}{4} - \cos\theta\right)^2}$

2	$z^4 = 8 - 8i\sqrt{3} = 16e^{-i\frac{1}{3}\pi}$
	$z_1 = 2e^{-i\frac{1}{12}\pi}$
	$z_2 = 2e^{i\frac{5}{12}\pi}$
	$z_3 = 2e^{-i\frac{7}{12}\pi}, z_4 = 2e^{i\frac{11}{12}\pi}$
4(a)	$z + z^{-1} = 2 \cos \theta$
	$(z + z^{-1})^6 = (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$
	$2^6 \cos^6 \theta = 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$
	$\cos^6 \theta = \frac{1}{32} \cos 6\theta + \frac{3}{16} \cos 4\theta + \frac{15}{32} \cos 2\theta + \frac{5}{16}$
4(b)	$\int_0^{\frac{1}{2}\pi} \cos^6 2x \, dx = \frac{1}{2} \int_0^{\pi} \left(\frac{1}{32} \cos 6\theta + \frac{3}{16} \cos 4\theta + \frac{15}{32} \cos 2\theta + \frac{5}{16} \right) d\theta$
	$= \frac{1}{2} \left[\frac{1}{192} \sin 6\theta + \frac{3}{64} \sin 4\theta + \frac{15}{64} \sin 2\theta + \frac{5}{16} \theta \right]_0^{\pi}$
	$= \frac{5}{32} \pi$

3(a)	$\sin 5\theta = \operatorname{Im}(\cos \theta + i \sin \theta)^5 = \sin^5 \theta - 10 \sin^3 \theta \cos^2 \theta + 5 \sin \theta \cos^4 \theta$
	$= \sin^5 \theta - 10 \sin^3 \theta (1 - \sin^2 \theta) + 5 \sin \theta (1 - \sin^2 \theta)^2$
	$= 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta$
3(b)	$x = \sin \theta, \quad \sin 5\theta = \frac{1}{2} \sqrt{3}$

1	$z^3 = 27 - 27i = 27\sqrt{2}e^{-i\frac{1}{4}\pi}$
	$z_1 = 3 \times 2^{\frac{1}{6}} e^{-i\frac{1}{12}\pi}$
	$z_2 = 3 \times 2^{\frac{1}{6}} e^{i\frac{7}{12}\pi}, \quad z_3 = 3 \times 2^{\frac{1}{6}} e^{-i\frac{9}{12}\pi}$
3	$z - z^{-1} = 2i \sin \theta$
	$(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$
	$2^5 i \sin^5 \theta = 2i \sin 5\theta - 5(2i \sin 3\theta) + 10(2i \sin \theta)$
	$\Rightarrow \operatorname{cosec}^5 \theta = \frac{16}{\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta}.$

4(b)

$$\frac{1}{\sqrt{2}}e^{\sqrt{2}} + \frac{1}{\sqrt{3}}e^{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}e^{\sqrt{n}}$$

$$> \int_1^n \frac{1}{\sqrt{x}}e^{\sqrt{x}} dx = \left[2e^{\sqrt{x}} \right]_1^n$$

$$\sum_{r=1}^n \frac{1}{\sqrt{r}}e^{\sqrt{r}} > \left[2e^{\sqrt{x}} \right]_1^n + e = 2e^{\sqrt{n}} - e$$

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$$6m^2 + 3m + 6 = 0 \Rightarrow m = -\frac{1}{4} \pm \frac{1}{4}i\sqrt{15}$$

$$x = e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right)$$

$$x = ke^{-t} \Rightarrow x' = -ke^{-t} \Rightarrow x'' = ke^{-t}$$

$$6ke^{-t} - 3ke^{-t} + 6ke^{-t} = e^{-t}$$

$$k = \frac{1}{9}$$

$$x = e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right) + \frac{1}{9}e^{-t}$$

$$x' = \frac{\sqrt{15}}{4}e^{-\frac{1}{4}t} \left(-A \sin \frac{\sqrt{15}}{4}t + B \cos \frac{\sqrt{15}}{4}t \right) - \frac{1}{4}e^{-\frac{1}{4}t} \left(A \cos \frac{\sqrt{15}}{4}t + B \sin \frac{\sqrt{15}}{4}t \right) - \frac{1}{9}e^{-t}$$

$$0 = A + \frac{1}{9} \quad 0 = \frac{\sqrt{15}}{4}B - \frac{1}{4}A - \frac{1}{9} \Rightarrow A = -\frac{1}{9}, \quad B = \frac{1}{3\sqrt{15}}$$

$$x = \frac{1}{3}e^{-\frac{1}{4}t} \left(-\frac{1}{3}\cos \frac{\sqrt{15}}{4}t + \frac{1}{\sqrt{15}}\sin \frac{\sqrt{15}}{4}t \right) + \frac{1}{9}e^{-t}$$

$$1 - \left(\frac{e^u - e^{-u}}{e^u + e^{-u}} \right)^2 = \frac{(e^u + e^{-u})^2 - (e^u - e^{-u})^2}{(e^u + e^{-u})^2} = \frac{4}{(e^u + e^{-u})^2}$$

$$= \operatorname{sech}^2 u$$

3	$z - z^{-1} = 2i \sin \theta$
	$(z - z^{-1})^5 = (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$
	$2^5 i \sin^5 \theta = 2i \sin 5\theta - 5(2i \sin 3\theta) + 10(2i \sin \theta)$
	$\Rightarrow \operatorname{cosec}^5 \theta = \frac{16}{\sin 5\theta - 5 \sin 3\theta + 10 \sin \theta}.$

5(a)	$(\cos 5\theta + i \sin 5\theta) = (c + is)^5$
	$\cos 5\theta = c^5 - 10s^2c^3 + 5s^4c$
	$= c^5 - 10(1 - c^2)c^3 + 5(1 - c^2)^2c$
	$= 16c^5 - 20c^3 + 5c$
	$\sec 5\theta = \frac{1}{16c^5 - 20c^3 + 5c} \times \frac{\sec^5 \theta}{\sec^5 \theta}$
	$= \frac{\sec^5 \theta}{5\sec^4 \theta - 20\sec^2 \theta + 16}$
5(b)	$x^5 = \frac{2}{\sqrt{3}}(5x^4 - 20x^2 + 16) \Rightarrow \frac{x^5}{5x^4 - 20x^2 + 16} = \frac{2}{\sqrt{3}}$
	$\sec 5\theta = \frac{2}{\sqrt{3}} \Rightarrow \cos 5\theta = \frac{\sqrt{3}}{2}$
	$x = \sec\left(\frac{1}{30}\pi\right)$
	$\sec\left(\frac{11}{30}\pi\right), \sec\left(\frac{13}{30}\pi\right), \sec\left(\frac{23}{30}\pi\right), \sec\left(\frac{25}{30}\pi\right)$