3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\left(1+x^2\right)^n\right)=2nx^2\left(1+x^2\right)^{n-1}+\left(1+x^2\right)^n$
	$=2n\left(1+x^2-1\right)\left(1+x^2\right)^{n-1}+\left(1+x^2\right)^n$
	$\left[x\left(1+x^2\right)^n\right]_0^1=2nI_n-2nI_{n-1}+I_n$
	$2^n=(2n+1)I_n-2nI_{n-1}\Rightarrow(2n+1)I_n=2^n+2nI_{n-1}$
3(b)	$I_{-1}=\left[\tan^{-1}x\right]_0^1=\frac{1}{4}\pi$
	$-I_{-1}=\frac{1}{2}-2I_{-2}$
	$-\frac{1}{4}\pi=\frac{1}{2}-2I_{-2}\Rightarrow I_{-2}=\frac{1}{4}+\frac{1}{8}\pi$

4	$5m^2+2m+1=0 \Rightarrow m=-\frac{1}{5}\pm\frac{2}{5}\mathrm{i}$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)$
	$y=px^2+qx+r \Rightarrow y'=2px+q \Rightarrow y''=2p$
	$p=1\quad 4p+q=5\quad 10p+2q+r=3$
	$q=1\quad r=-9$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)+x^2+x-9$
	$y'=\mathrm{e}^{-\frac{1}{5}x}\left(-\frac{2}{5}A\sin\frac{2}{5}x+\frac{2}{5}B\cos\frac{2}{5}x\right)-\frac{1}{5}\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)+2x+1$
	$A-9=0\quad \frac{2}{5}B-\frac{1}{5}A+1=0\quad \Rightarrow A=9,\ B=2$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(9\cos\frac{2}{5}x+2\sin\frac{2}{5}x\right)+x^2+x-9$

	$\frac{4}{3n}\to 0$ as $n\to\infty$

4	$5m^2+2m+1=0\Rightarrow m=-\frac{1}{5}\pm\frac{2}{5}\mathrm{i}$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)$
	$y=px^2+qx+r\Rightarrow y'=2px+q\Rightarrow y''=2p$
	$p=1\quad 4p+q=5\quad 10p+2q+r=3$
	$q=1\quad r=-9$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)+x^2+x-9$
	$y'=\mathrm{e}^{-\frac{1}{5}x}\left(-\frac{2}{5}A\sin\frac{2}{5}x+\frac{2}{5}B\cos\frac{2}{5}x\right)-\frac{1}{5}\mathrm{e}^{-\frac{1}{5}x}\left(A\cos\frac{2}{5}x+B\sin\frac{2}{5}x\right)+2x+1$
	$A-9=0\quad \frac{2}{5}B-\frac{1}{5}A+1=0\quad \Rightarrow A=9,\ B=2$
	$y=\mathrm{e}^{-\frac{1}{5}x}\left(9\cos\frac{2}{5}x+2\sin\frac{2}{5}x\right)+x^2+x-9$

5(a)	$\left[\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x < \right]\left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{2}{n}\right)^3 + \left(\frac{2}{n}\right)\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{n}{n}\right)^3 + \left(\frac{n}{n}\right)\right)$
	$= \frac{1}{3n^4}\sum_{r=1}^{n} r^3 + \frac{1}{n^2}\sum_{r=1}^{n} r = \frac{n^2(n+1)^2}{12n^4} + \frac{n(n+1)}{2n^2}$
	$= \frac{1}{12}\left(1+\frac{1}{n}\right)^2 + \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{12}\left(1+\frac{1}{n}\right)\left(7+\frac{1}{n}\right)$
5(b)	$\int_0^1 \left(\frac{1}{3}x^3 + x\right)\mathrm{d}x \geqslant \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{1}{n}\right)^3 + \left(\frac{1}{n}\right)\right) + \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{2}{n}\right)^3 + \left(\frac{2}{n}\right)\right) + \cdots + \left(\frac{1}{n}\right)\left(\frac{1}{3}\left(\frac{n-1}{n}\right)^3 + \left(\frac{n-1}{n}\right)\right)$
	$= \frac{1}{3n^4}\sum_{r=1}^{n-1} r^3 + \frac{1}{n^2}\sum_{r=1}^{n-1} r = \frac{(n-1)^2 n^2}{12n^4} + \frac{(n-1)n}{2n^2}$
	$= \frac{1}{12}\left(1-\frac{1}{n}\right)^2 + \frac{1}{2}\left(1-\frac{1}{n}\right) = \frac{1}{12}\left(1-\frac{1}{n}\right)\left(7-\frac{1}{n}\right)$
5(c)	$U_n - L_n = \frac{1}{12}\left(\frac{16}{n}\right) = \frac{4}{3n}$
	$\frac{4}{} \to 0$ as $n \to \infty$

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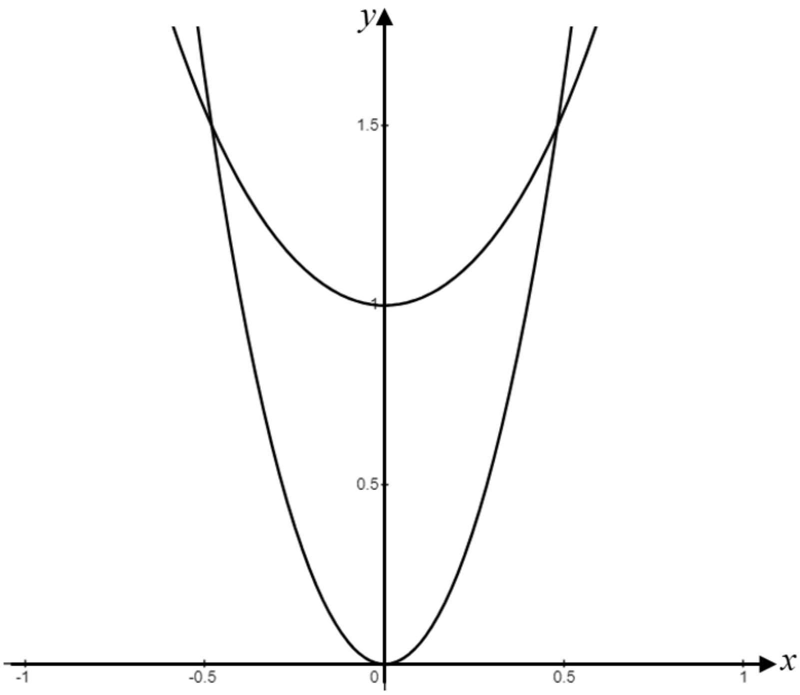
1	$s = \int_0^{\pi} \sqrt{\mathrm{e}^{\frac{8}{3}\theta} + \frac{16}{9}\mathrm{e}^{\frac{8}{3}\theta}}\,\mathrm{d}\theta = \frac{5}{3}\int_0^{\pi} \mathrm{e}^{\frac{4}{3}\theta}\,\mathrm{d}\theta$
	$s = \frac{5}{4}\left[\mathrm{e}^{\frac{4}{3}\theta}\right]_0^{\pi}$
	$= \frac{5}{4}\left(\mathrm{e}^{\frac{4}{3}\pi} - 1\right)$

6(a)	$\frac{dx}{du} = \frac{1}{2}\sqrt{2}\cosh u \Rightarrow \int \frac{1}{\sqrt{2x^2+1}}\,dx = \frac{1}{2}\sqrt{2}\int \frac{\cosh u}{\sqrt{\sinh^2 u+1}}\,du = \frac{1}{2}\sqrt{2}\int 1\,du$
	$= \frac{1}{2}\sqrt{2}\sinh^{-1}\left(x\sqrt{2}\right)[+C]$
6(b)	$\frac{1}{n}\left(\frac{1}{\sqrt{2\frac{1^2}{n^2}+1}} + \frac{1}{\sqrt{2\frac{2^2}{n^2}+1}} + \ldots + \frac{1}{\sqrt{2\frac{n^2}{n^2}+1}}\right)$ $= \frac{1}{n}\left(\frac{n}{\sqrt{2\times 1^2+n^2}} + \frac{n}{\sqrt{2\times 2^2+n^2}} + \ldots + \frac{n}{\sqrt{2\times n^2+n^2}}\right)$
	$< \int_0^1 \frac{1}{\sqrt{2x^2+1}}\,dx$
	$\int_0^1 \frac{1}{\sqrt{2x^2+1}}\,dx = \left[\frac{1}{2}\sqrt{2}\sinh^{-1}\left(x\sqrt{2}\right)\right]_0^1 = \frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$
	$\frac{1}{n}\sum_{r=1}^{n}\frac{n}{\sqrt{2r^2+n^2}} = \sum_{r=1}^{n}\frac{1}{\sqrt{2r^2+n^2}} < \frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$

6(c)	$\frac{1}{n}\left(\frac{1}{\sqrt{2\times 0^2+1}}+\frac{1}{\sqrt{2\frac{1^2}{n^2}+1}}+\ldots+\frac{1}{\sqrt{2\frac{(n-1)^2}{n^2}+1}}\right)$ $=\frac{1}{n}\left(1+\frac{n}{\sqrt{2\times 1^2+n^2}}+\ldots+\frac{n}{\sqrt{2(n-1)^2+n^2}}\right)$
	$\geq\int_0^1\frac{1}{\sqrt{2x^2+1}}\,\mathrm{d}x=\left[\frac{1}{2}\sqrt{2}\sinh^{-1}\left(x\sqrt{2}\right)\right]_0^1=\frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$
	$\frac{1}{n}+\sum_{r=1}^{n}\frac{1}{\sqrt{2r^2+n^2}}-\frac{1}{\sqrt{3n^2}}\geqslant\frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$ $\sum_{r=1}^{n}\frac{1}{\sqrt{2r^2+n^2}}\geqslant\frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)+\frac{1}{n}\left(\frac{1}{3}\sqrt{3}-1\right)$
6(d)	$\frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$

1(a)	$I_n=\left[x^n\cosh x\right]_0^1-n\int_0^1 x^{n-1}\cosh x\,\mathrm{d}x$
	$=\left[x^n\cosh x\right]_0^1-n\left(\left[x^{n-1}\sinh x\right]_0^1-(n-1)\int_0^1 x^{n-2}\sinh x\,\mathrm{d}x\right)$
	$I_n=\cosh 1-n\left(\sinh 1-(n-1)I_{n-2}\right)=\cosh 1-n\sinh 1+n(n-1)I_{n-2}$
1(b)	$I_2=\cosh 1-2\sinh 1+2I_0$
	$I_2=3\cosh 1-2\sinh 1-2$

2(a)	$1+2\sinh^2 x = 6\sinh^2 x \Rightarrow \sinh x = \pm\frac{1}{2}$
	$x = \sinh^{-1}\left(\pm\frac{1}{2}\right) = \ln\left(\pm\frac{1}{2}+\sqrt{\frac{1}{4}+1}\right)$
	$x = \ln\left(\pm\frac{1}{2}+\frac{1}{2}\sqrt{5}\right)$

2(b)	

3(a)	$\sin 5\theta = \mathrm{Im}\left(\cos\theta + \mathrm{i}\sin\theta\right)^5 = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$
	$= \sin^5\theta - 10\sin^3\theta\left(1-\sin^2\theta\right) + 5\sin\theta\left(1-\sin^2\theta\right)^2$
	$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$
3(b)	$x = \sin\theta, \quad \sin 5\theta = \frac{1}{2}\sqrt{3}$
	$5\theta = \frac{1}{3}\pi + 2k\pi$ or $5\theta = \frac{2}{3}\pi + 2k\pi$
	$\sin\left(\frac{1}{15}\pi\right)$
	$\sin(\frac{2}{15}\pi), \sin(-\frac{5}{15}\pi), \sin(-\frac{4}{15}\pi), \sin(\frac{7}{15}\pi)$

	$\dfrac{1}{N} \to 0$ as $N \to \infty$.

5(a)	$m^2+m+1=0\Rightarrow m=-\frac{1}{2}\pm\frac{1}{2}\sqrt{3}$
	$y=\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)$
	$y=p\sin 2x+q\cos 2x\Rightarrow y'=2p\cos 2x-2q\sin 2x\Rightarrow y''=-4p\sin 2x-4q\cos 2x$
	$-4p-2q+p=1$ $\quad$ $-4q+2p+q=1$
	$p=-\frac{1}{13}$ $\quad$ $q=-\frac{5}{13}$
	$y=\mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)-\frac{1}{13}\sin 2x-\frac{5}{13}\cos 2x$
5(b)	$y\approx-\frac{1}{13}\sin 2x-\frac{5}{13}\cos 2x=\sqrt{\frac{2}{13}}\left(-\frac{1}{\sqrt{26}}\sin 2x-\frac{5}{\sqrt{26}}\cos 2x\right)$
	$\sqrt{\frac{2}{13}}\sin(2x-1.77)$

	$1-2x-x^2=2-(x+1)^2$
	$\int\frac{1}{\sqrt{2-(x+1)^2}}\,\mathrm{d}x=\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$
	$\pi=\sin^{-1}\frac{1}{\sqrt{2}}+C\Rightarrow C=\frac{3}{4}\pi$
	$\frac{y}{\sqrt{1-2x-x^2}}=\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+\frac{3}{4}\pi$

4(a)	$\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x < \frac{1}{2}\left(\left(\frac{1}{N}\right)3^{\frac{1}{N}} + \left(\frac{1}{N}\right)3^{\frac{2}{N}} + \cdots + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}} + \left(\frac{1}{N}\right)3^{\frac{N}{N}}\right)$
	$= \frac{1}{2N}\sum_{n=1}^{N}\left(3^{\frac{1}{N}}\right)^n = \frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}}-1\right)}$
4(b)	$\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x > \frac{1}{2}\left(\left(\frac{1}{N}\right) + \left(\frac{1}{N}\right)3^{\frac{1}{N}} + \cdots + \left(\frac{1}{N}\right)3^{\frac{N-2}{N}} + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}}\right)$
	$= \frac{1}{2N}\sum_{n=0}^{N-1}\left(3^{\frac{1}{N}}\right)^n = \frac{1}{N\left(3^{\frac{1}{N}}-1\right)}$
4(c)	$\frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}}-1\right)} - \frac{1}{N\left(3^{\frac{1}{N}}-1\right)} = \frac{1}{N}$

2(a)	$I_n = \int_0^1 (1-x)^n \sinh x\,\mathrm{d}x = \left[(1-x)^n \cosh x\right]_0^1 + \int_0^1 n(1-x)^{n-1}\cosh x\,\mathrm{d}x$
	$I_n = -1 + n\left[(1-x)^{n-1}\sinh x\right]_0^1 + n\left(n-1\right)\int_0^1 \left(1-x\right)^{n-2}\sinh x\,\mathrm{d}x$
	$I_n = -1 + n\left(n-1\right)I_{n-2}$
2(b)	$I_0 = \int_0^1 \sinh x\,\mathrm{d}x = \left[\cosh x\right]_0^1 = \cosh 1 - 1\,.$
	$I_2 = -1 + 2I_0$
	$I_2 = 2\cosh 1 - 3$

4(a)	$\frac{1}{\sqrt{1}}\mathrm{e}^{\sqrt{1}}+\frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n-1}}\mathrm{e}^{\sqrt{n-1}}$
	$<\int_1^n \frac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x$
	$\int_1^n \frac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x=\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n$
	$\sum_{r=1}^{n}\frac{1}{\sqrt{r}}\mathrm{e}^{\sqrt{r}}<\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n+\frac{1}{\sqrt{n}}\mathrm{e}^{\sqrt{n}}=\left(2+\frac{1}{\sqrt{n}}\right)\mathrm{e}^{\sqrt{n}}-2\mathrm{e}$
4(b)	$\frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{2}}+\frac{1}{\sqrt{3}}\mathrm{e}^{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}\mathrm{e}^{\sqrt{n}}$
	$>\int_1^n \frac{1}{\sqrt{x}}\mathrm{e}^{\sqrt{x}}\,\mathrm{d}x=\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n$
	$\sum_{r=1}^{n}\frac{1}{\sqrt{r}}\mathrm{e}^{\sqrt{r}}>\left[2\mathrm{e}^{\sqrt{x}}\right]_1^n+\mathrm{e}=2\mathrm{e}^{\sqrt{n}}-\mathrm{e}$

6(a)	$\frac{1}{n}\left(\frac{1}{\frac{1^2}{n^2}+1}+\frac{1}{\frac{2^2}{n^2}+1}+\ldots+\frac{1}{\frac{n^2}{n^2}+1}\right)=\frac{1}{n}\left(\frac{n^2}{1^2+n^2}+\frac{n^2}{2^2+n^2}+\ldots+\frac{n^2}{n^2+n^2}\right)$
	$\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x=\left[\tan^{-1}x\right]_0^1=\frac{1}{4}\pi$
	$\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}=\sum_{r=1}^{n}\frac{n}{n^2+r^2}<\frac{1}{4}\pi$
6(b)	$\frac{1}{n}\left(\frac{1}{0^2+1}+\frac{1}{\frac{1^2}{n^2}+1}+\ldots+\frac{1}{\frac{(n-1)^2}{n^2}+1}\right)=\frac{1}{n}\left(\frac{n^2}{0^2+n^2}+\frac{n^2}{1^2+n^2}+\ldots+\frac{n^2}{(n-1)^2+n^2}\right)$
	$\frac{1}{n}+\sum_{r=1}^{n}\frac{n}{n^2+r^2}-\frac{1}{2n}\geq\frac{1}{4}\pi\Rightarrow\sum_{r=1}^{n}\frac{n}{n^2+r^2}\geq\frac{1}{4}\pi-\frac{1}{2n}$
6(c)	$\frac{1}{4}\pi$