3 The integral I_n is defined by $I_n = \int_0 \left(1+x^2\right)\, \mathrm{d}x$.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^n\right)$, or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$ [5]

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(b) Find the exact value of I_{-2}. [4]

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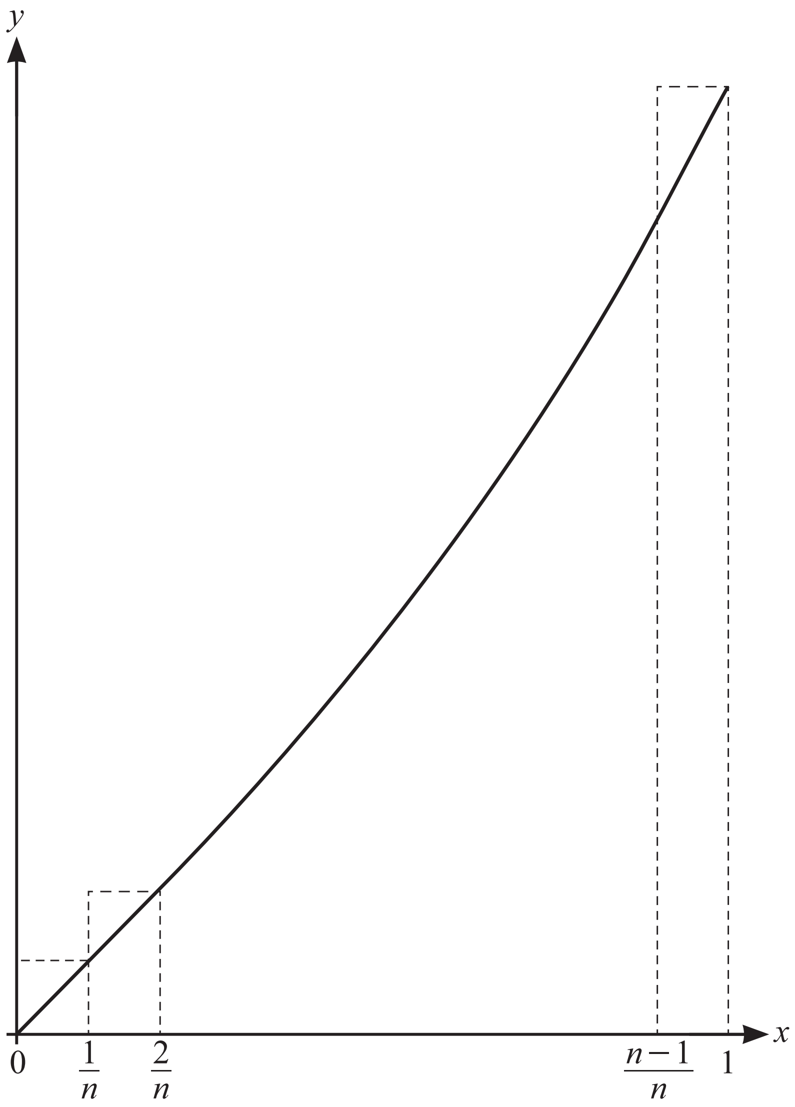

The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx < U_n$, where

$$U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right).$$ [5]

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(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx$. [4]

(c) Show that $\lim_{n \to \infty}(U_n - L_n) = 0$. [2]

1 The curve C has polar equation $r = e^{\frac{4}{3}\theta}$ for $0 \leqslant \theta \leqslant \pi$.

Find the length of C. Give your answer in an exact form. [4]

6 **(a)** Use the substitution $x = \frac{1}{2}\sqrt{2}\sinh u$ to find $\int \frac{1}{\sqrt{2x^2+1}}\,dx$. [3]

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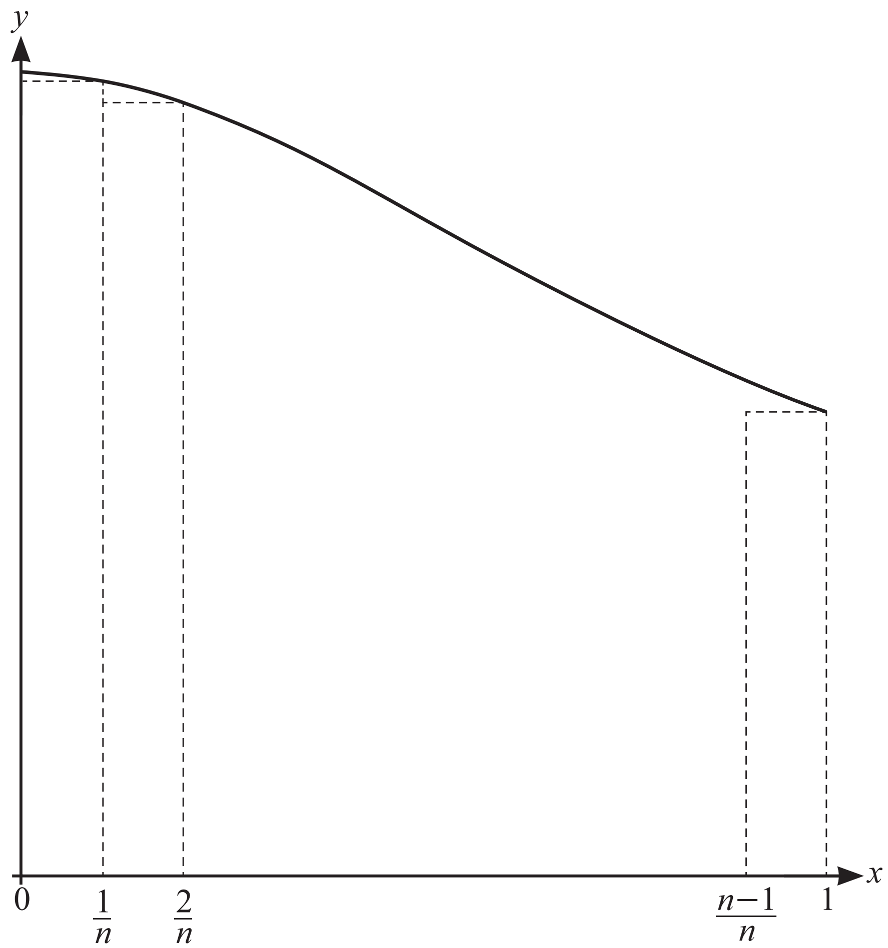

The diagram shows the curve with equation $y = \frac{1}{\sqrt{2x^2+1}}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \tfrac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right).$$ [5]

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(c) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [4]

(d) Deduce the exact value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}}$. [1]

1 It is given that $I_n = \int_0 x^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$,

$$I_n = \cosh 1 - n \sinh 1 + n(n-1) I_{n-2}.$$ [3]

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(b) Find the exact value of I_2. [2]

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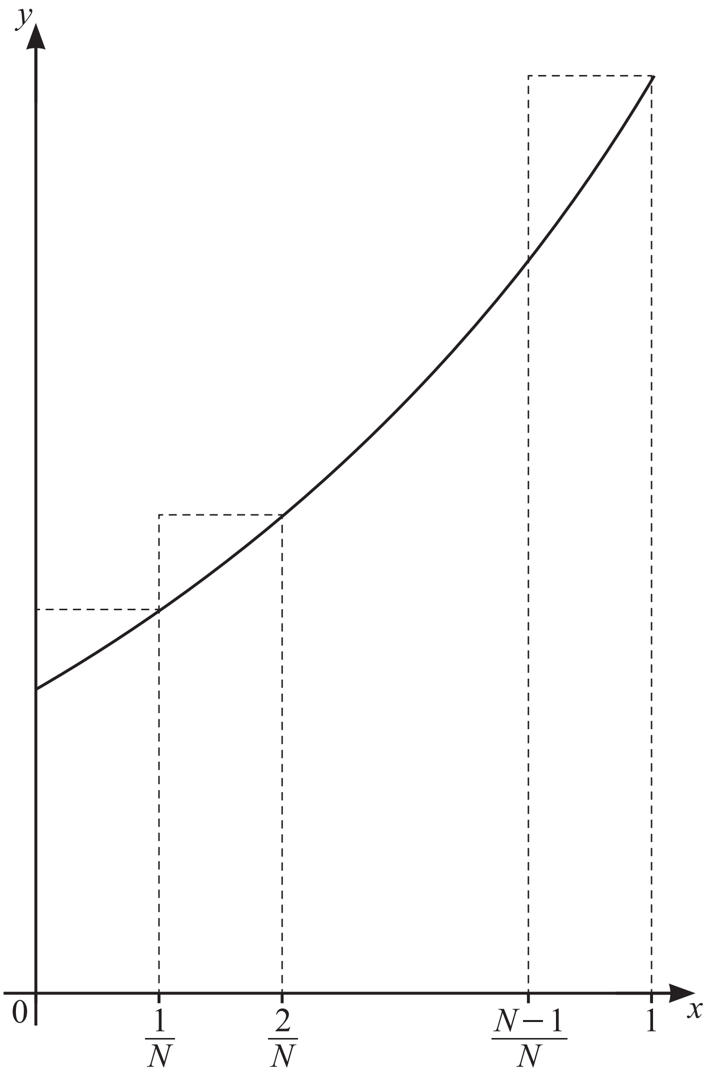

The diagram shows the curve with equation $y = \frac{1}{2}(3^x)$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \frac{1}{2}(3^x)\,dx < U_N$, where

$$U_N = \frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}} - 1\right)}.$$ [4]

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(b) Use a similar method to find, in terms of N, a lower bound L_{N} for $\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x$. [4]

(c) By simplifying $U_N - L_N$, show that $\lim_{N \to \infty}\left(U_N - L_N\right) = 0$. [2]

2 Let $I_n = \int_0^1 (1-x)^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$, $I_n = -1 + n(n-1)I_{n-2}$. [4]

(b) Find the exact value of I_2. [3]

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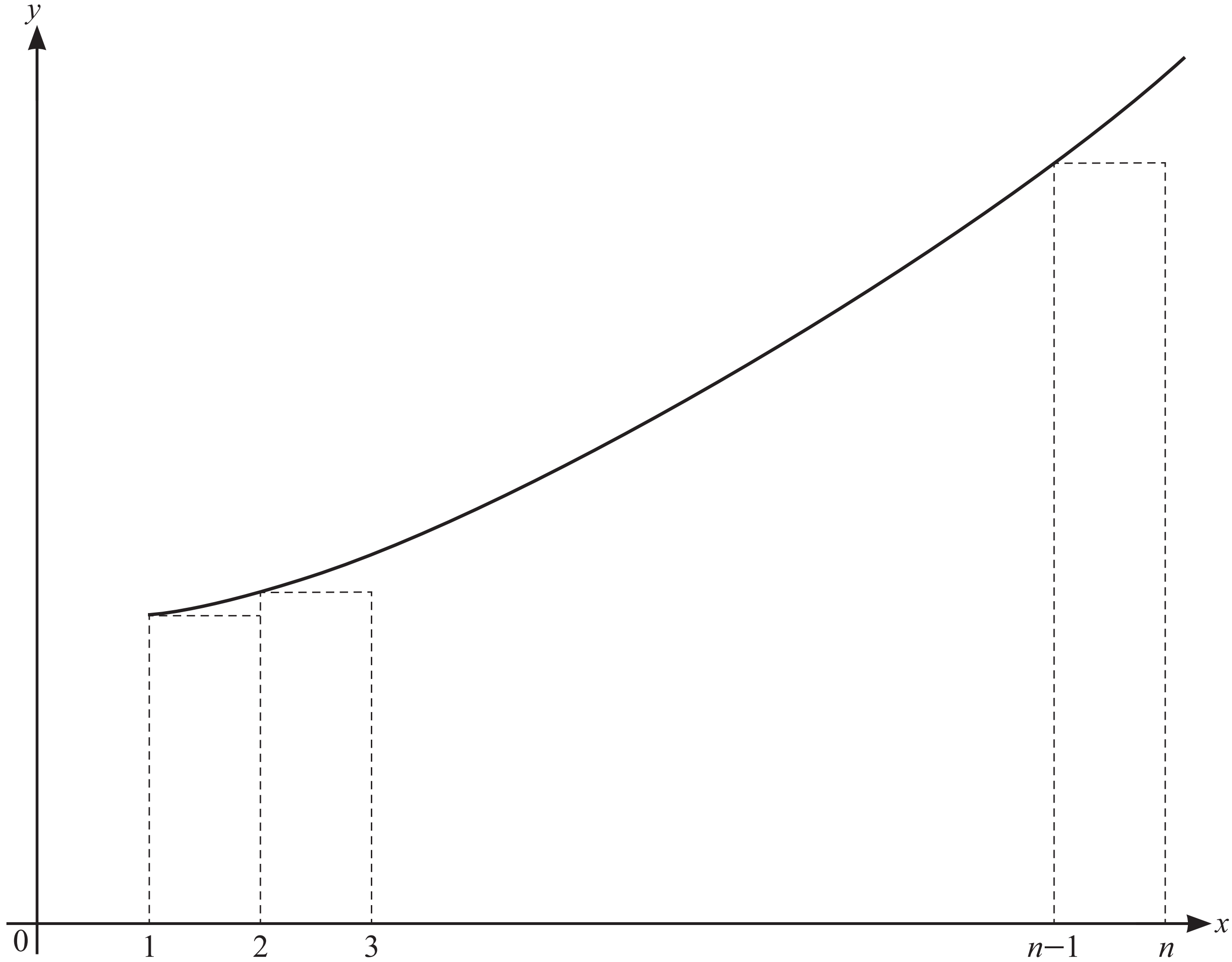

The diagram shows the curve with equation $y = \frac{1}{\sqrt{x}}e^{\sqrt{x}}$ for $x \geqslant 1$, together with a set of $n-1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)e^{\sqrt{n}} - 2e.$$ [5]

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(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} e^{\sqrt{r}}$. [4]

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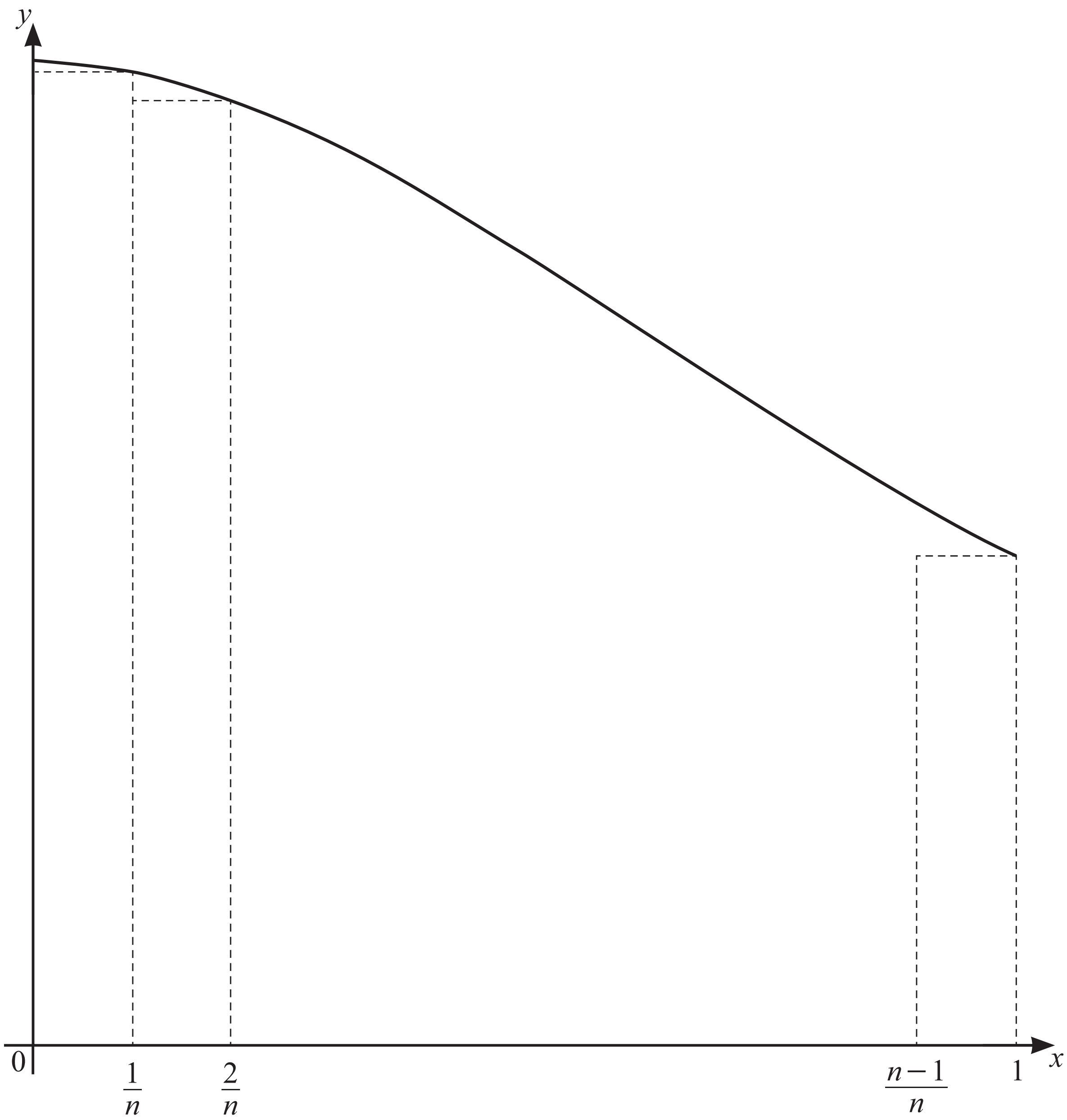

The diagram shows the curve with equation $y = \frac{1}{x^2+1}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{n}{n^2+r^2} < \frac{1}{4}\pi.$$ [5]

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(b) Use a similar method to find a lower bound for $\sum_{r=1}^{n} \frac{n}{n^2+r^2}$. Give your answer in terms of n and π. [4]

(c) Deduce the exact value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$. [1]