1	$\begin{vmatrix} 1 & -1 & 1 \\ 1 & k & 3 \\ 1 & 2 & k \end{vmatrix} = \begin{vmatrix} k & 3 \\ 2 & k \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & k \end{vmatrix} + \begin{vmatrix} 1 & k \\ 1 & 2 \end{vmatrix} = k^2 - 7$
	$k = \pm\sqrt{7}$

6(a)	$a, \quad 2a, \quad -3a$
6(b)(i)	$\mathbf{P}^2 = \begin{pmatrix} a^2 & 3a & -2a-1 \\ 0 & 4a^2 & a \\ 0 & 0 & 9a^2 \end{pmatrix}$
6(b)(ii)	$\mathbf{P}^3 - 7a^2\mathbf{P} + 6a^3\mathbf{I} = 0$
	$6a^3\mathbf{P}^{-1} = 7a^2\mathbf{I} - \mathbf{P}^2$
	$\mathbf{P}^2 = \begin{pmatrix} a^2 & 3a & -2a-1 \\ 0 & 4a^2 & a \\ 0 & 0 & 9a^2 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \frac{1}{6a^3}\begin{pmatrix} 6a^2 & -3a & 2a+1 \\ 0 & 3a^2 & -a \\ 0 & 0 & -2a^2 \end{pmatrix}$

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6(c)	$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
	$\mathbf{A} = \mathbf{P}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\mathbf{P}^{-1}$
	$= \frac{1}{6a^3}\begin{pmatrix} a & 2 & 3 \\ 0 & 4a & -3 \\ 0 & 0 & -9a \end{pmatrix}\begin{pmatrix} 6a^2 & -3a & 2a+1 \\ 0 & 3a^2 & -a \\ 0 & 0 & -2a^2 \end{pmatrix}$
	$\begin{pmatrix} 1 & \frac{1}{2a} & -\frac{4a+1}{6a^2} \\ 0 & 2 & \frac{1}{3a} \\ 0 & 0 & 3 \end{pmatrix}$

8(a)	$\begin{vmatrix} 1 & -1 & 2k \\ k & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 0 \Rightarrow 2k^2 + 3k + 1 = 0$
	$2k^2 + 3k + 1 = 0 \Rightarrow k = -\frac{1}{2}, -1$
8(b)	$\begin{aligned} x - y - z &= 1, \\ -\tfrac{1}{2}x + y + 2z &= 2, \\ 2x - y + z &= 3, \end{aligned} \Rightarrow \begin{aligned} \tfrac{1}{2}x + z &= 3, \\ x + 2z &= 2, \end{aligned}$
	Triangular prism.
8(c)	$\begin{aligned} x - y - 2z &= 1, \\ -x + y + 2z &= 2, \\ 2x - y + z &= 3, \end{aligned} \Rightarrow 1 = -2$
	Two parallel planes. Other plane not parallel.
	Parallel planes not identical.
8(d)	$\det\begin{pmatrix} 1-\lambda & -1 & -2 \\ -1 & 1-\lambda & 2 \\ 2 & -1 & 1-\lambda \end{pmatrix} = 0$
	$-\lambda^3 + 3\lambda^2 - 8\lambda = 0$
	$\mathbf{A}^3 = 3\mathbf{A}^2 - 8\mathbf{A}$
	$\mathbf{A}^4 = 3\mathbf{A}^3 - 8\mathbf{A}^2 = 3(3\mathbf{A}^2 - 8\mathbf{A}) - 8\mathbf{A}^2$
	$\mathbf{A}^4 = \mathbf{A}^2 - 24\mathbf{A}$

8(a)	$\begin{vmatrix} 5 & a & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{vmatrix} = -20a - 25$
	$a \neq -\frac{5}{4}$
	Three planes intersect at a *single* point.
8(b)	If $a = -\frac{5}{4}$ then $\begin{aligned} 5x - \tfrac{5}{4}y &= 3, \\ 20x - 5y &= 2, \\ 2x - 3y + z &= 1, \end{aligned} \Rightarrow 2 = 12$
	Two *distinct* parallel planes.
	Other plane not parallel.
8(c)	Eigenvalues of **A** are $5,\ -5$ and 1.
	$\lambda = 5: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & -10 & 0 \\ 2 & -3 & -4 \end{vmatrix} = \begin{pmatrix} 40 \\ 80 \\ -40 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$
	$\lambda = -5: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20 & 0 & 0 \\ 2 & -3 & 6 \end{vmatrix} = \begin{pmatrix} 0 \\ -120 \\ -60 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$
	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & 0 \\ 20 & -6 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 64 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 16 \end{pmatrix}$

8(a)	$\mathbf{Ae} = \lambda \mathbf{e} \Rightarrow \mathbf{A}^3\mathbf{e} = \lambda \mathbf{A}(\mathbf{Ae})$
	$\Rightarrow \mathbf{A}^3\mathbf{e} = \lambda \mathbf{A}(\lambda \mathbf{e}) = \lambda^2(\mathbf{Ae}) = \lambda^3 \mathbf{e}$
8(b)	$\begin{vmatrix} -1-\lambda & 3 & 4 \\ 0 & 1-\lambda & 0 \\ 0 & -2 & 5-\lambda \end{vmatrix} = 0$ $\Rightarrow (-1-\lambda)((1-\lambda)(5-\lambda)-0)-3(0)+4(0)=0$
	$\Rightarrow (-1-\lambda)(1-\lambda)(5-\lambda)=0 \Rightarrow \lambda = -1, 1$ and 5.
8(c)	$\lambda = -1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ 0 & 2 & 0 \end{vmatrix} = \begin{pmatrix} -8 \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 4 \\ 0 & -2 & 4 \end{vmatrix} = \begin{pmatrix} 20 \\ 8 \\ 4 \end{pmatrix} \sim \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ $\lambda = 5: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -6 & 3 & 4 \\ 0 & -4 & 0 \end{vmatrix} = \begin{pmatrix} 16 \\ 0 \\ 24 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 5 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

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8(d)	$(\mathbf{A}-2\mathbf{I})^3 = \mathbf{A}^3 - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I}$
	$= 5\mathbf{A}^2 + \mathbf{A} - 5\mathbf{I} - 6\mathbf{A}^2 + 12\mathbf{A} - 8\mathbf{I}$
	$= -\mathbf{A}^2 + 13\mathbf{A} - 13\mathbf{I}$
	Alternative method for 8(d)
	$(\mu+3)(\mu+1)(\mu-3) = \mu^3 + \mu^2 - 9\mu - 9 = 0$
	$(\mathbf{A}-2\mathbf{I})^3 = -(\mathbf{A}-2\mathbf{I})^2 + 9(\mathbf{A}-2\mathbf{I}) + 9\mathbf{I}$
	$= -\mathbf{A}^2 + 13\mathbf{A} - 13\mathbf{I}$

8(a)	$\begin{vmatrix} \frac{3}{2} & 3 & 8 \\ a & 3 & 4 \\ 0 & a & -1 \end{vmatrix} = 0 \Rightarrow 8a^2 - 3a - \frac{9}{2} = 0$
	$a = \frac{3}{16}\left(1 \pm \sqrt{17}\right)$
8(b)	$\left(\lambda - \frac{3}{2}\right)(\lambda-3)(\lambda+1) = \lambda^3 - \frac{7}{2}\lambda^2 + \frac{9}{2} = 0$
	$\frac{9}{2}\mathbf{I} = \frac{7}{2}\mathbf{A}^2 - \mathbf{A}^3 \Rightarrow \frac{9}{2}\mathbf{B} = \frac{7}{2}\mathbf{A} - \mathbf{A}^2$
	$\mathbf{B}^2 = \frac{7}{9}\mathbf{I} - \frac{2}{9}\mathbf{A}$

8(c)	Eigenvalues of $\mathbf{A}$ are $\frac{3}{2}, 3$ and -1.
	$\lambda = \frac{3}{2}: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 8 \\ 0 & \frac{3}{2} & \frac{5}{2} \end{vmatrix} = \begin{pmatrix} -\frac{9}{2} \\ 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
	$\lambda = 3: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{3}{2} & 3 & 8 \\ 0 & 0 & 4 \end{vmatrix} = \begin{pmatrix} 12 \\ 6 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\lambda = -1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{5}{2} & 3 & 8 \\ 0 & 4 & 4 \end{vmatrix} = \begin{pmatrix} -20 \\ -10 \\ 10 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$
	Thus $\mathbf{P} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix}$