1 Find the values of k for which the system of equations

$$\begin{aligned} x - y + z &= 0, \\ x + ky + 3z &= 0, \\ x + 2y + kz &= 0, \end{aligned}$$

does not have a unique solution. [3]

6 The matrix $\mathbf{P}$ has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

(a) State, in terms of a, the eigenvalues of $\mathbf{P}$. [1]

(b) **(i)** Find $\mathbf{P}^2$ in terms of a. [1]

(ii) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$ in terms of a. [3]

The 3×3 matrix **A** has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

(c) Find **A** in terms of a. [5]

8 **(a)** Find the values of k for which the system of equations

$$\begin{aligned} x - y + 2kz &= 1, \\ kx + y + 2z &= 2, \\ 2x - y + z &= 3, \end{aligned}$$

does not have a unique solution. [3]

(b) Given that $k = -\frac{1}{2}$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [3]

(c) Given instead that $k = -1$, show that the system of equations in part **(a)** is also inconsistent. Interpret this situation geometrically. [4]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}.$$

(d) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A}$ where p and q are integers to be determined. [5]

Additional page

If you use the following page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

8 **(a)** Find the set of values of a for which the system of equations

$$\begin{aligned} 5x + ay &= 3, \\ 20x - 5y &= 2, \\ 2x - 3y + z &= 1, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [3]

(b) Given that $a = -\frac{5}{4}$, show that the system of equations in part **(a)** is inconsistent and interpret this situation geometrically. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix}.$$

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A} + 3\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

8 **(a)** It is given that λ is an eigenvalue of the non-singular square matrix **A**, with corresponding eigenvector **e**.

Show that **e** is an eigenvector of $\mathbf{A}^3$ with corresponding eigenvalue λ^3. [2]

The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{pmatrix}.$$

(b) Show that the eigenvalues of **A** are -1, 1 and 5. [2]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}-2\mathbf{I}=\mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of $\mathbf{A}$ to show that $(\mathbf{A}-2\mathbf{I})^3 = a\mathbf{A}^2+b\mathbf{A}+c\mathbf{I}$ where a, b and c are constants to be determined. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

8 **(a)** Find the values of a for which the system of equations

$$\begin{aligned} \tfrac{3}{2}x + 3y + 8z &= 1, \\ ax + 3y + 4z &= 2, \\ ay - z &= 3, \end{aligned}$$

does not have a unique solution. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Given that $\mathbf{B} = \mathbf{A}^{-1}$, use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{B}^2 = p\mathbf{I} + q\mathbf{A}$, where p and q are constants to be determined. [4]

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.