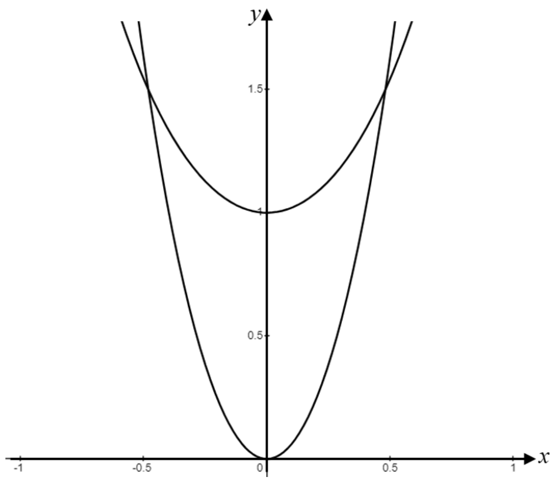
7(a)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}x\right)=\tfrac{1}{2}\left(\dfrac{1-x}{1+x}\right)\dfrac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$
	$=\tfrac{1}{2}\left(\dfrac{1-x}{1+x}\right)\dfrac{2}{(1-x)^2}=\dfrac{1}{1-x^2}$
	Alternative method for question 7(a)
	$\tanh y=x\Rightarrow\operatorname{sech}^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$
	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{1-\tanh^2 y}=\dfrac{1}{1-x^2}$
7(b)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}-\dfrac{y}{x}=x\tanh^{-1}x$
	$\mathrm{e}^{-\int x^{-1}\,\mathrm{d}\theta}=\mathrm{e}^{-\ln x}=x^{-1}$
	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^{-1}y\right)=\tanh^{-1}x$
	$x^{-1}y=x\tanh^{-1}x-\displaystyle\int\frac{x}{1-x^2}\,\mathrm{d}x$
	$x^{-1}y=x\tanh^{-1}x+\tfrac{1}{2}\ln\left(1-x^2\right)+C$
	$0=\tfrac{1}{2}\tanh^{-1}\tfrac{1}{2}+\tfrac{1}{2}\ln\tfrac{3}{4}+C\Rightarrow C=-\tfrac{1}{4}\ln 3-\tfrac{1}{2}\ln\tfrac{3}{4}$
	$x^{-1}y=x\tanh^{-1}x+\tfrac{1}{2}\ln\left(1-x^2\right)-\tfrac{1}{4}\ln\tfrac{27}{16}$

1(b)	$I_2=\cosh 1-2\sinh 1+2I_0$
	$I_2=3\cosh 1-2\sinh 1-2$
2(a)	$1+2\sinh^2 x=6\sinh^2 x\Rightarrow\sinh x=\pm\tfrac{1}{2}$
	$x=\sinh^{-1}\left(\pm\tfrac{1}{2}\right)=\ln\left(\pm\tfrac{1}{2}+\sqrt{\tfrac{1}{4}+1}\right)$
	$x=\ln\left(\pm\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}\right)$

2(b)	

3(a)	$\sin 5\theta = \mathrm{Im}\left(\cos\theta + \mathrm{i}\sin\theta\right)^5 = \sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta$
	$= \sin^5\theta - 10\sin^3\theta\left(1-\sin^2\theta\right) + 5\sin\theta\left(1-\sin^2\theta\right)^2$
	$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$
3(b)	$x = \sin\theta, \quad \sin 5\theta = \frac{1}{2}\sqrt{3}$
	$5\theta = \frac{1}{3}\pi + 2k\pi$ or $5\theta = \frac{2}{3}\pi + 2k\pi$
	$\sin\left(\frac{1}{15}\pi\right)$
	$\sin(\frac{2}{15}\pi), \sin(-\frac{5}{15}\pi), \sin(-\frac{4}{15}\pi), \sin(\frac{7}{15}\pi)$

4(a)	$\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x < \frac{1}{2}\left(\left(\frac{1}{N}\right)3^{\frac{1}{N}} + \left(\frac{1}{N}\right)3^{\frac{2}{N}} + \cdots + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}} + \left(\frac{1}{N}\right)3^{\frac{N}{N}}\right)$
	$= \dfrac{1}{2N}\sum_{n=1}^{N}\left(3^{\frac{1}{N}}\right)^n = \dfrac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}}-1\right)}$
4(b)	$\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x > \frac{1}{2}\left(\left(\frac{1}{N}\right) + \left(\frac{1}{N}\right)3^{\frac{1}{N}} + \cdots + \left(\frac{1}{N}\right)3^{\frac{N-2}{N}} + \left(\frac{1}{N}\right)3^{\frac{N-1}{N}}\right)$
	$= \dfrac{1}{2N}\sum_{n=0}^{N-1}\left(3^{\frac{1}{N}}\right)^n = \dfrac{1}{N\left(3^{\frac{1}{N}}-1\right)}$
4(c)	$\dfrac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}}-1\right)} - \dfrac{1}{N\left(3^{\frac{1}{N}}-1\right)} = \dfrac{1}{N}$
	$\dfrac{1}{N} \to 0$ as $N \to \infty$.

5(a)	$m^2 + m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$
	$y = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$
	$y = p\sin 2x + q\cos 2x \Rightarrow y' = 2p\cos 2x - 2q\sin 2x \Rightarrow y'' = -4p\sin 2x - 4q\cos 2x$
	$-4p - 2q + p = 1$ $\quad$ $-4q + 2p + q = 1$
	$p = -\frac{1}{13}$ $\quad$ $q = -\frac{5}{13}$
	$y = \mathrm{e}^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}\sin 2x - \frac{5}{13}\cos 2x$
5(b)	$y \approx -\frac{1}{13}\sin 2x - \frac{5}{13}\cos 2x = \sqrt{\frac{2}{13}}\left(-\frac{1}{\sqrt{26}}\sin 2x - \frac{5}{\sqrt{26}}\cos 2x\right)$
	$\sqrt{\frac{2}{13}}\sin(2x - 1.77)$

6	$\int \frac{1+x}{1-2x-x^2}\,dx = -\frac{1}{2}\ln\left(1-2x-x^2\right)$
	$e^{-\frac{1}{2}\ln\left(1-2x-x^2\right)} = \frac{1}{\sqrt{1-2x-x^2}}$
	$\frac{d}{dx}\left(\frac{y}{\sqrt{1-2x-x^2}}\right) = \frac{1}{\sqrt{1-2x-x^2}}$
	$1-2x-x^2 = 2-\left(x+1\right)^2$
	$\int \frac{1}{\sqrt{2-\left(x+1\right)^2}}\,dx = \sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C$
	$\pi = \sin^{-1}\frac{1}{\sqrt{2}}+C \Rightarrow C = \frac{3}{4}\pi$
	$\frac{y}{\sqrt{1-2x-x^2}} = \sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+\frac{3}{4}\pi$

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7(a)(i)	$\dot{x} = 4\frac{\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}+3\operatorname{cosec}^2 t-3$ $\dot{x} = 8\operatorname{cosec} t+3\operatorname{cosec}^2 t-3 = 8\operatorname{cosec} t+3\cot^2 t$
7(a)(ii)	$\dot{y} = 6\operatorname{cosec} t-4\cot^2 t$
7(a)(iii)	$\dot{x}^2+\dot{y}^2 = \left(8\operatorname{cosec} t+3\cot^2 t\right)^2+\left(6\operatorname{cosec} t-4\cot^2 t\right)^2$ $=100\operatorname{cosec}^2 t+25\cot^4 t = 100\operatorname{cosec}^2 t+25\left(\operatorname{cosec}^2 t-1\right)^2$ $=25\operatorname{cosec}^4 t+50\operatorname{cosec}^2 t+25$
	$=25\left(\operatorname{cosec}^2 t+1\right)^2$
	$5\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\operatorname{cosec}^2 t+1\,dt = 5\left[-\cot t+t\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} = 5\left(-\frac{1}{\sqrt{3}}+\frac{1}{3}\pi+\sqrt{3}-\frac{1}{6}\pi\right) = \frac{5}{6}\pi+\frac{10}{3}\sqrt{3}$

7(b)	$\frac{\dot{y}}{\dot{x}}=\frac{6\operatorname{cosec}t-4\cot^2 t}{8\operatorname{cosec}t+3\cot^2 t}=\frac{6\sin t-4\cos^2 t}{8\sin t+3\cos^2 t}=\frac{6\sec t-4\cot t}{8\sec t+3\cot t}$
	$\frac{d}{dt}\left(\frac{6\operatorname{cosec}t-4\cot^2 t}{8\operatorname{cosec}t+3\cot^2 t}\right)$ $=\frac{(8\csc t+3\cot^2 t)(-6\cot t\csc t+8\cot t\csc^2 t)+(6\csc t-4\cot^2 t)(6\csc^2 t\cot t+8\cot t\csc t)}{(8\operatorname{cosec}t+3\cot^2 t)^2}$ or $\frac{d}{dt}\left(\frac{6\sin t-4\cos^2 t}{8\sin t+3\cos^2 t}\right)$ $=\frac{(8\sin t+3\cos^2 t)(6\cos t+8\cos t\sin t)-(6\sin t-4\cos^2 t)(8\cos t-6\cos t\sin t)}{(8\sin t+3\cos^2 t)^2}$
	$\frac{100\cos t\sin^2 t+50\cos^3 t}{(8\sin t+3\cos^2 t)^2}=\frac{50\cos t(\sin^2 t+1)}{(8\sin t+3\cos^2 t)^2}=\frac{50\cot t\operatorname{cosec}t(\operatorname{cosec}^2 t+1)}{(8\operatorname{cosec}t+3\cot^2 t)^2}$
	$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{6\sin t-4\cos^2 t}{8\sin t+3\cos^2 t}\right)\times\frac{dt}{dx}=\frac{50\sin^2 t\cos t(\sin^2 t+1)}{(8\sin t+3\cos^2 t)^3}$ $=\frac{50\cot t\operatorname{cosec}t(\operatorname{cosec}^2 t+1)}{(8\operatorname{cosec}t+3\cot^2 t)^3}$

8(a)	$\begin{vmatrix}5 & a & 0\\ 20 & -5 & 0\\ 2 & -3 & 1\end{vmatrix}=-20a-25$
	$a\neq-\frac{5}{4}$
	Three planes intersect at a *single* point.

6(a)	$\tanh u=\frac{e^u-e^{-u}}{e^u+e^{-u}}\quad \operatorname{sech}u=\frac{2}{e^u+e^{-u}}$
	$1-\left(\frac{e^u-e^{-u}}{e^u+e^{-u}}\right)^2=\frac{(e^u+e^{-u})^2-(e^u-e^{-u})^2}{(e^u+e^{-u})^2}=\frac{4}{(e^u+e^{-u})^2}$
	$=\operatorname{sech}^2 u$
6(b)	$u=\operatorname{sech}^{-1}t\Rightarrow\operatorname{sech}u=t\Rightarrow-\tanh u\operatorname{sech}u\frac{du}{dt}=1$
	$\frac{du}{dt}=-\frac{1}{\tanh u\operatorname{sech}u}=-\frac{1}{t\sqrt{1-t^2}}$
	First alternative method for question 6(b)
	$u=\operatorname{sech}^{-1}t\Rightarrow\cosh u=\frac{1}{t}\Rightarrow\sinh u\frac{du}{dt}=-\frac{1}{t^2}$
	$\frac{du}{dt}=-\frac{1}{t^2\sqrt{\cosh^2 u-1}}=-\frac{1}{t^2\sqrt{t^{-2}-1}}=-\frac{1}{t\sqrt{1-t^2}}$

6(b)	**Second alternative method for question 6(b)**
	$u=\mathrm{sech}^{-1}t \Rightarrow \cosh u=\frac{1}{t} \Rightarrow u=\cosh^{-1}\left(\frac{1}{t}\right)$
	$\frac{\mathrm{d}u}{\mathrm{d}t}=\frac{1}{\sqrt{\left(\frac{1}{t}\right)^2-1}}\times\left(-t^{-2}\right)=-\frac{1}{t\sqrt{1-t^2}}$
	Third alternative method for question 6(b)
	$\mathrm{sech}\,u=t \Rightarrow \cosh u=\frac{1}{t} \Rightarrow u=\ln\left(\frac{1}{t}+\sqrt{\frac{1}{t^2}-1}\right)=\ln\left(\frac{1+\sqrt{1-t^2}}{t}\right)$
	$\frac{\mathrm{d}u}{\mathrm{d}t}=\left(\frac{-1-\sqrt{1-t^2}}{t^2\sqrt{1-t^2}}\right)\left(\frac{t}{1+\sqrt{1-t^2}}\right)=-\frac{1}{t\sqrt{1-t^2}}$
6(c)	$\frac{\mathrm{d}y}{\mathrm{d}t}=\mathrm{sech}^{-1}t-\frac{1}{\sqrt{1-t^2}}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{1-t^2}$
	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=-\sqrt{1-t^2}+\left(1-t^2\right)\mathrm{sech}^{-1}t$

6(d)	$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{t}{\sqrt{1-t^2}}-\frac{1-t^2}{t\sqrt{1-t^2}}-2t\,\mathrm{sech}^{-1}t$
	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=t\sqrt{1-t^2}-\frac{\left(1-t^2\right)^{\frac{3}{2}}}{t}-2t\left(1-t^2\right)\mathrm{sech}^{-1}t$

2(a)	$\tanh t=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{\mathrm{e}^t+\mathrm{e}^{-t}}$ $\quad \mathrm{sech}\,t=\frac{2}{\mathrm{e}^t+\mathrm{e}^{-t}}$
	$\left(\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{\mathrm{e}^t+\mathrm{e}^{-t}}\right)^2+\frac{4}{\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2}=\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}+2}{\left(\mathrm{e}^t+\mathrm{e}^{-t}\right)^2}=1$
2(b)	$\frac{\mathrm{d}x}{\mathrm{d}t}=\tanh t$
	$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\cosh t}{1+\sinh^2 t}=\mathrm{sech}\,t$
	$\int_0^1\sqrt{\tanh^2 t+\mathrm{sech}^2 t}\,\mathrm{d}t=\int_0^1 1\,\mathrm{d}t$
	$=1$

3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(9y^2\right)=18yy'$
	$\frac{\mathrm{d}}{\mathrm{d}x}\left(-3\sinh^{-1}(xy)\right)=-3\frac{xy'+y}{\sqrt{1+x^2y^2}}$
	$\frac{18}{3}y'-3\frac{4y'+\frac{1}{3}}{\frac{5}{3}}=0 \Rightarrow y'=-\frac{1}{2}$

3(b)	$18yy''+18(y')^2$
	$-3\left(\dfrac{\sqrt{1+x^2y^2}\,(xy''+2y')-(xy'+y)^2\left(1+x^2y^2\right)^{-\frac{1}{2}}(xy)}{1+x^2y^2}\right)$
	$\frac{18}{3}y''+\frac{18}{4}-3\dfrac{\frac{5}{3}(4y''-1)-\left(\frac{5}{3}\right)^2\left(\frac{3}{5}\right)\left(\frac{4}{3}\right)}{\frac{25}{9}}=0$ $\frac{18}{3}y''+\frac{18}{4}-\dfrac{45(4y''-1)-60}{25}=0$
	$-\frac{6}{5}y''+\frac{87}{10}=0 \Rightarrow y''=\frac{29}{4}$