4 The quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Show that a quartic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1, 2\delta + 1$ is

$$y^4 - 2y^3 + 4y^2 + 2y + 11 = 0.$$ [4]

The sum $(2\alpha+1)^n+(2\beta+1)^n+(2\gamma+1)^n+(2\delta+1)^n$ is denoted by S_n.

(b) Find the value of S_2. [2]

(c) Given that $S_3=-22$, find the value of S_4. [2]

2 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 3,$$
$$\alpha^4 + \beta^4 + \gamma^4 = 5.$$

(a) Find the values of b and c. [3]

(b) Find the value of d. [5]

2 The cubic equation $x^3 + bx^2 + cx + d = 0$, where $d \neq 0$, has roots α, β, γ such that $\gamma = \frac{1}{\beta}$.

(a) Show that $\beta + \frac{1}{\beta} = d - b$. [3]

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(b) Show also that $\beta + \frac{1}{\beta} = \frac{1-c}{d}$. [2]

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(c) It is given that $b = 3$, $c = -3$ and $d > 0$.

(i) Find the value of d. [2]

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

2 The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1$, $\beta^3 - 1$, $\gamma^3 - 1$. [3]

(b) Find the value of $\left(\alpha^3-1\right)^2+\left(\beta^3-1\right)^2+\left(\gamma^3-1\right)^2$. [2]

(c) Find the value of $\left(\alpha^3-1\right)^3+\left(\beta^3-1\right)^3+\left(\gamma^3-1\right)^3$. [2]

3 The quartic equation $x^4 + 7x^2 + 3x + 22 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [2]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

(c) Use standard results from the list of formulae (MF19) to find the value of

$$\sum_{r=1}^{10}\left(\left(\alpha^2+r\right)^2+\left(\beta^2+r\right)^2+\left(\gamma^2+r\right)^2+\left(\delta^2+r\right)^2\right). \quad [5]$$