Week 31

A-Level Further Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

A-Level Further Mathematics Name: _____ Score: _____

1. A ball is thrown at u = 20 m/s at 30°. Using height = $u^2\sin^2\alpha/(2g)$ with g = 10, find the greatest height (m).

   _____

2. A uniform 4 m beam of weight 100 N is pivoted at A; a force F at B (4 m away) balances it. Taking moments about A (weight acts at the 2 m centre), find F (N).

   _____

3. A particle moves in a circle of radius 2 m with angular speed 3 rad/s. What is its speed $v = r\omega$ (m/s)?

   _____

4. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension $T = \lambda x/L$ (N).

   _____

5. When a force depends on position x, acceleration is best written as:

   ☐ v dv/dx

   ☐ dv/dt

   ☐ x dx/dv

   ☐ dx/dt

6. A 2 kg body at 3 m/s hits a stationary 1 kg body and they stick together. By momentum conservation 2(3) = (2+1)v, find v (m/s).

   _____

7. During the flight, the horizontal velocity of a projectile (ignoring air resistance):

   ☐ stays constant

   ☐ increases

   ☐ decreases to zero

   ☐ reverses

8. The moment of a force about a point is:

   ☐ force × perpendicular distance

   ☐ force × time

   ☐ force ÷ distance

☐ mass × acceleration

9. Substituting a = v dv/dx into F = ma gives a separable differential equation in v and x.

☐ True ☐ False

10. The coefficient of restitution e always lies in which range?

☐ $0 \leq e \leq 1$

☐ $e \geq 1$

☐ $-1 \leq e \leq 0$

☐ any real number

A-Level Further Mathematics

1. **5**

   $400 \times \sin^2 30° / 20 = 400 \times 0.25 / 20 = 5$ m.

2. **50**

   $F \times 4 = 100 \times 2$, so $F = 200/4 = 50$ N.

3. **6**

   $v = r\omega = 2 \times 3 = 6$ m/s.

4. **12.5**

   $T = 50 \times 0.5 / 2 = 12.5$ N.

5. **v dv/dx**

   $a = dv/dt = (dv/dx)(dx/dt) = v\ dv/dx$ — convenient when force depends on x.

6. **2**

   $6 = 3v$, so $v = 2$ m/s.

7. **stays constant**

   There is no horizontal force, so u cos α is constant throughout.

8. **force × perpendicular distance**

   Moment = force × perpendicular distance — the turning effect.

9. **True**

   $F(x) = mv\ dv/dx$ separates into v dv and an x-expression, then integrate.

10. **$0 \leq e \leq 1$**

    e runs from 0 (perfectly inelastic) to 1 (perfectly elastic).

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

3.1 Motion of a projectile

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS motion of a projectile 抛射体运动 • projectile 抛射体 • trajectory 轨迹

Objective

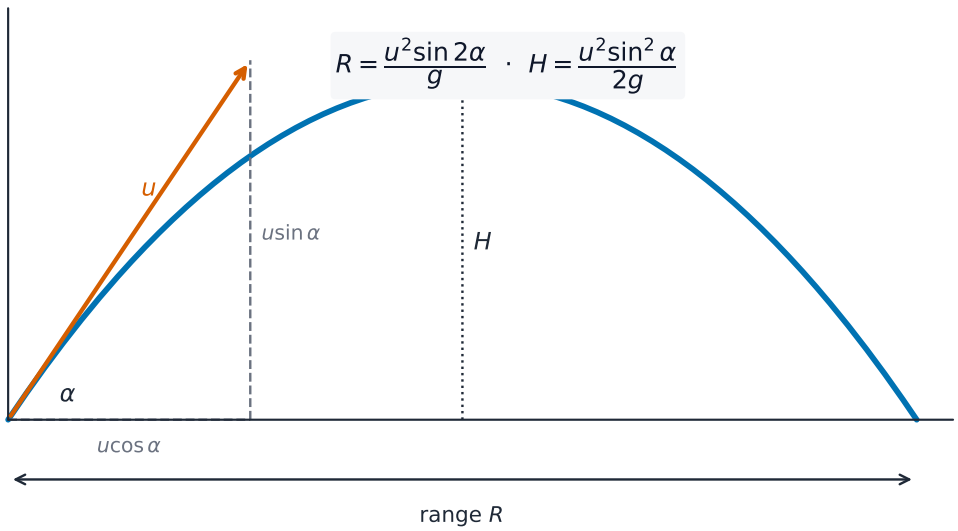
Build the skills to answer exam questions on the **motion of a projectile** 抛射体运动 — separating horizontal/vertical motion, the **trajectory** 轨迹 equation, range and greatest height. Take $g = 10\ \text{m s}^{-2}$.

You must be able to:

- use $x = u\cos\alpha\, t$, $y = u\sin\alpha\, t - \frac{1}{2}gt^2$
- eliminate t to get the Cartesian trajectory
- find range $\dfrac{u^2\sin 2\alpha}{g}$ and greatest height $\dfrac{u^2\sin^2\alpha}{2g}$

1 Worked examples

■ Range and height



The path is a parabola; u splits into a steady horizontal part and a vertical part slowed by gravity

$u = 20$, $\alpha = 30°$: range $= \dfrac{400 \sin 60°}{10} = 34.6$ m; greatest height $= \dfrac{400 \sin^2 30°}{20} = 5$ m.

2 Practice

2.1 A projectile is launched at 25 m s^{-1} at 40°. Find its initial vertical velocity component. [2]

2.2 Write down the formula for the range on level ground. [1]

3 Exam-style questions

3.1 A ball is projected horizontally at 15 m s^{-1} from a height. After 2 s its horizontal distance is: [1]

- **A** 15 m
- **B** 30 m
- **C** 20 m
- **D** 7.5 m

3.2 A particle is projected at 30 m s^{-1} at 60° to the horizontal.

(a) Find the time to reach the greatest height. [2]

(b) Find the greatest height reached. [3]

3.3 A stone is thrown at 20 m s^{-1} at 45° from the top of a cliff. It lands in the sea 50 m horizontally away.

(a) Find the time of flight. [3]

(b) Find the height of the cliff above the sea. [3]

Solutions

2.1 $25\sin 40° = 25 \times 0.643 = 16.1$ m s^{-1}.

2.2 $\dfrac{u^2 \sin 2\alpha}{g}$.

3.1 B. $x = 15 \times 2 = 30$ m.

3.2 (a) at greatest height the vertical velocity is 0: $0 = 30\sin 60° - 10t \Rightarrow t = \dfrac{25.98}{10} = 2.60$ s.
(b) height $= \dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{900 \times 0.75}{20} = 33.75$ m.

3.3 (a) horizontal: $50 = 20\cos 45°\, t = 14.14t \Rightarrow t = 3.54$ s.
(b) take up positive with launch at $y = 0$; at landing $y = -h$: $-h = 20\sin 45°(3.54) - 5(3.54)^2$; $= 50.0 - 62.5 = -12.5$, so the cliff is $h = 12.5$ m above the sea.

7 A particle P is projected from a point O on a horizontal plane and moves freely under gravity. The initial velocity of P is $25\,\mathrm{m\,s^{-1}}$ at an angle θ above the horizontal, where $\tan\theta = \frac{4}{3}$. At point A, the direction of motion of P makes an angle of $45°$ with the downward vertical through A.

(a) By differentiating the equation of the trajectory or otherwise, find the coordinates of A. [5]

At point A, the particle strikes a fixed smooth barrier, rebounds, and lands on the horizontal plane. The barrier is inclined at an angle of 45° to the horizontal.

(b) Find the speed of P immediately before it collides with the barrier. [3]

(c) Given that the coefficient of restitution between the barrier and the particle is $\frac{1}{9}$, find the horizontal distance travelled by P after it strikes the barrier. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

4 A particle Q is initially positioned at a distance d vertically above a particle P. Particle P is projected with speed U at an angle α above the horizontal. At the same time, Q is projected at an angle β below the horizontal. Both particles move freely under gravity. The particles collide at time T after the projections.

(a) Show that $d = UT(\sin\alpha + \cos\alpha\tan\beta)$. [4]

The particles collide when P is at its maximum height.

(b) Given that $\alpha = 30°$ and $\beta = 60°$, find d in terms of U and g. [3]

2 A particle P is projected with speed $u\,\text{m s}^{-1}$ at an angle θ, where $\tan\theta = 2$, above the horizontal from a point O on a horizontal plane and moves freely under gravity. The horizontal and vertical displacements of P from O at a time t s are denoted by x m and y m respectively.

(a) Use the equation of the trajectory given in the list of formulae (MF 19) to show that

$$y = 2x - \frac{25x^2}{u^2}\ .$$

[1]

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(b) In the subsequent motion, P passes through the point with coordinates (8,12). The particle then hits a fixed vertical barrier 7 m high that is at a horizontal distance of D m from the point of projection.

Find the set of possible values of D. [5]

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2 A particle P is projected with speed $24\,\text{m}\,\text{s}^{-1}$ at an angle $\theta°$ above the horizontal from a point O on a horizontal plane, and moves freely under gravity. At a horizontal distance 35 m from O, there is a vertical wall of height 10 m which is perpendicular to the vertical plane of motion of P.

(a) Determine the two values of θ for which P just clears the wall. [4]

(b) Given that P clears the wall, find the minimum distance from point O where P can land. [2]

7 A particle P is projected from a point O with speed U at an angle 45° above the horizontal and moves freely under gravity.

(a) State the vertical and horizontal components of velocity at time t. [1]

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At time T, particle P is moving at an angle of 60° below the horizontal.

(b) Show that $T = \frac{U}{2g}\left(\sqrt{2} + \sqrt{6}\right)$. [3]

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At time T, the particle strikes a smooth horizontal plane at a point which is a horizontal distance D from O and a vertical distance H below O.

(c) Find the ratio $H : D$. [4]

After striking the horizontal plane, P rebounds with speed w. The coefficient of restitution between P and the plane is $\frac{2}{3}$.

(d) Find w in terms of U. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

3.2 Equilibrium of a rigid body

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS centre of mass 质心 • toppling 翻倒 • equilibrium 平衡 • sliding 滑动 • moment 力矩

Objective

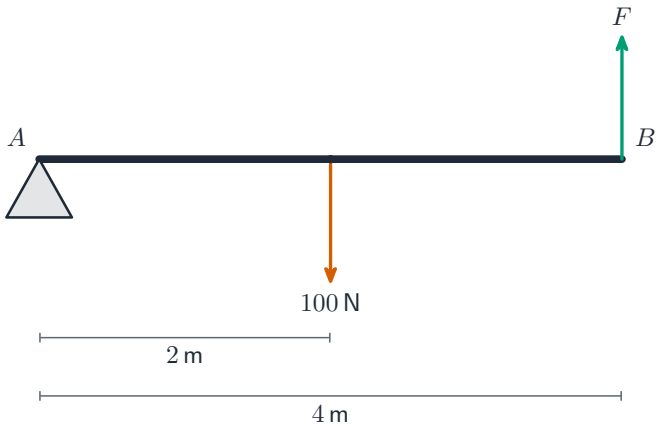
Build the skills to answer exam questions on the **equilibrium of a rigid body** 刚体平衡 — **moments** 力矩, the **centre of mass** 质心, and conditions for **toppling** 翻倒 or **sliding** 滑动. Take $g = 10$ m s^{-2}.

You must be able to:

- take moments about a point (force × perpendicular distance)
- apply the two equilibrium conditions (forces and moments both balance)
- decide whether a body topples or slides first

1 Worked examples

■ Taking moments



Taking moments about the pivot A balances the turning effects

Uniform beam AB, length 4 m, weight 100 N, pivot at A, force F at B: moments about A → $F \times 4 = 100 \times 2 \Rightarrow F = 50$ N.

2 Practice

2.1 A force of 20 N acts 3 m from a pivot, at right angles. Find its moment. [2]

2.2 State the two conditions for a rigid body to be in equilibrium. [2]

3 Exam-style questions

3.1 The moment of a force about a point is force multiplied by: [1]

- **A** time
- **B** the perpendicular distance
- **C** mass
- **D** velocity

3.2 A uniform ladder AB of weight 200 N and length 6 m rests with A on rough ground and B against a smooth vertical wall, at 60° to the horizontal.

(a) By taking moments about A, find the reaction at the wall. [4]

(b) Hence find the friction force at the ground. [2]

3.3 A uniform rod AB of length 2 m and weight 40 N rests horizontally on supports at A and at C, where C is 1.5 m from A. Find the reaction at each support. [4]

Solutions

2.1 moment $= 20 \times 3 = 60$ N m.

2.2 the resultant force is zero; the resultant moment about any point is zero.

3.1 B. Moment = force $\times$ perpendicular distance.

3.2 (a) wall reaction S is horizontal at B; weight 200 N acts at the midpoint (3 m along). Moments about A: $S \times 6\sin 60° = 200 \times 3\cos 60°$; $S \times 5.196 = 300 \Rightarrow S = 57.7$ N.
(b) horizontal equilibrium: friction $= S = 57.7$ N.

3.3 moments about A: $R_C \times 1.5 = 40 \times 1 \Rightarrow R_C = 26.7$ N; vertical: $R_A + R_C = 40 \Rightarrow R_A = 13.3$ N.

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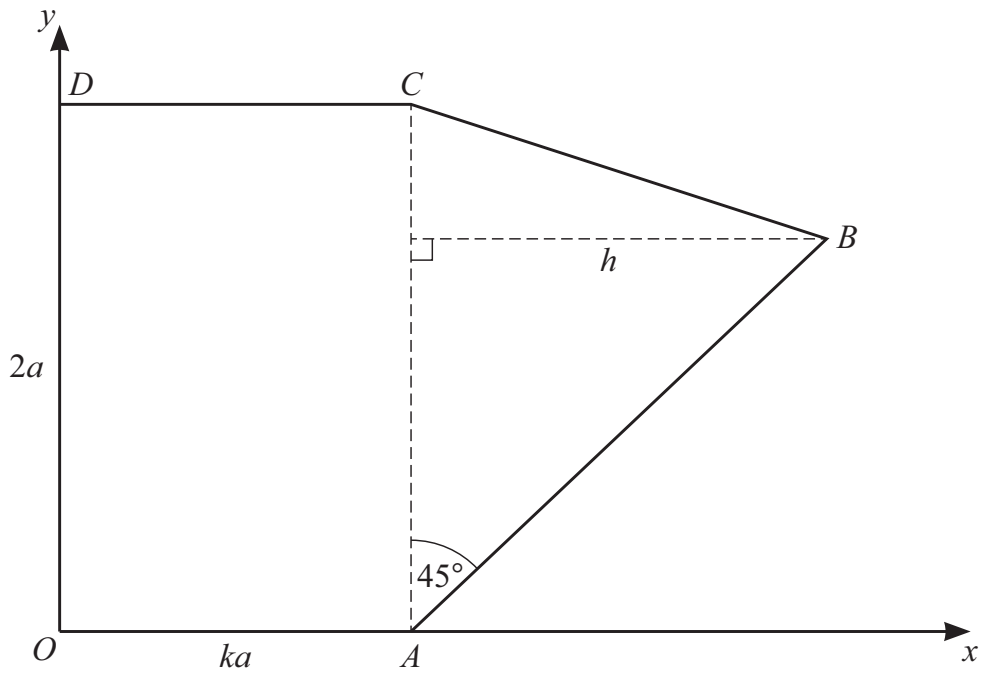

A uniform lamina $OABCD$ consists of a rectangle $OACD$ and a triangle ABC. The length of OA is ka, the length of OD is $2a$, the height of triangle ABC is h and angle CAB is 45° (see diagram). Relative to axes through O, parallel and perpendicular to OA as shown, the centre of mass of triangle ABC is $(\overline{x}, \overline{y})$.

(a) Show that $\overline{x}$ is $\frac{1}{3}(3ka+h)$, and find an expression for $\overline{y}$. [3]

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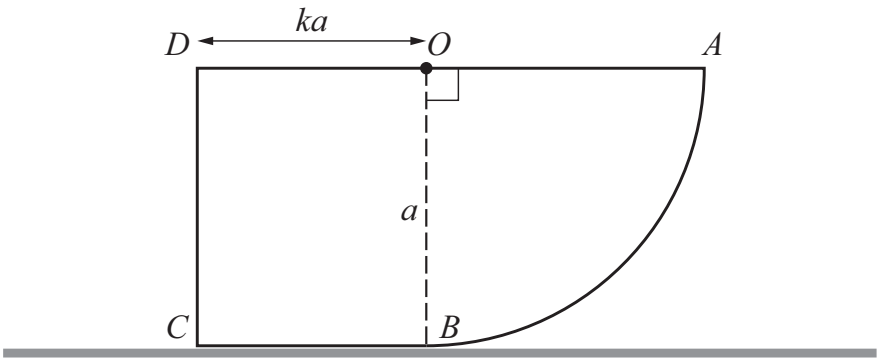
The lamina $OABCD$ is placed vertically on its edge OA on a horizontal plane.

(b) Find, in terms of a and k, the set of values of h for which the lamina is in equilibrium. [4]

It is now given that $k = \frac{\sqrt{3}}{3}$ and that the lamina is on the point of toppling.

(c) Find, in terms of a, the coordinates of the centre of mass of the triangle ABC. [2]

3 A uniform lamina *OABCD* is in the form of a rectangle, *OBCD*, joined along the edge *OB* to a quarter circle *OAB*. The length of *DO* is *ka* and the length of *OB* is *a*. The lamina rests in a vertical plane with its edge *CB* on a horizontal surface (see diagram).



(a) Find, in terms of k, a and π, an expression for the distance of the centre of mass above the horizontal surface. [4]

[You may use without proof the result for the centre of mass of a circular sector in the list of formulae (MF19).]

The lamina is on the point of toppling about B.

(b) Find the value of k. [2]

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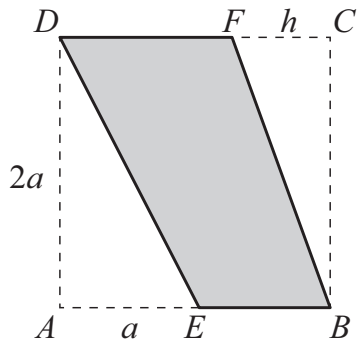

The lamina $BFDE$ is obtained by removing triangles AED and BCF from a uniform square lamina $ABCD$ of side $2a$. The length of side AE is a and the length of side FC is h (see diagram). The centre of mass of $BFDE$ is at a distance $\overline{x}$ from AD, and at a distance $\overline{y}$ from AB.

(a) Show that $\overline{x} = \dfrac{h^2 - 6ah + 11a^2}{3(3a - h)}$ and find a corresponding expression for $\overline{y}$. [5]

(b) The lamina *BFDE* is placed vertically on its edge *EB* on a smooth horizontal surface.

Find, in terms of a, the set of possible values of h for which the lamina remains in equilibrium. [2]

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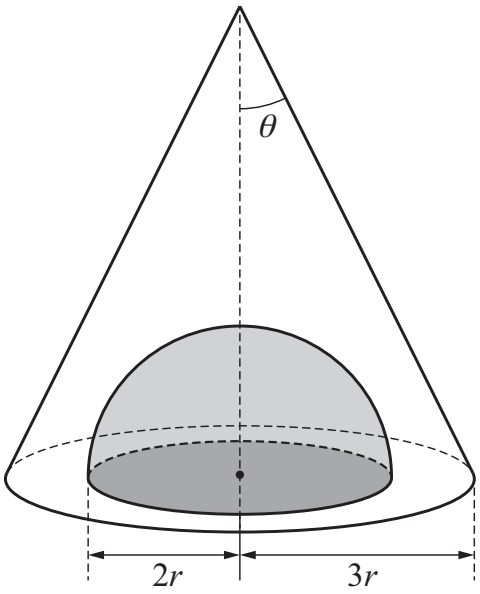

An object is formed by removing a solid hemisphere, radius $2r$, from a uniform solid cone, radius $3r$ and semi-vertical angle θ, where $\tan\theta = \frac{1}{2}$. The axes of symmetry of the cone and the hemisphere coincide. The base of the cone and the base of the hemisphere are in the same plane as each other (see diagram).

(a) Find, in terms of r, the distance of the centre of mass of the object from its base. [4]

The object is placed such that its circular base makes contact with a rough plane which is inclined to the horizontal at an angle α. The object is on the point of toppling. The plane is sufficiently rough to prevent sliding.

(b) Find the value of α. [3]

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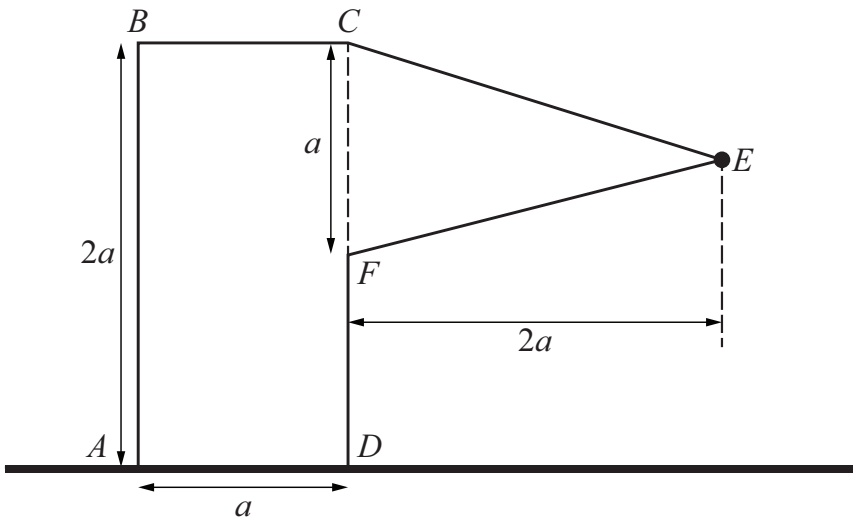


An object consists of a uniform lamina with a particle attached. The uniform lamina $ABCEFD$ of mass m is formed from a rectangle $ABCD$ and an isosceles triangle CEF, where F is the midpoint of CD. The rectangle has sides $AB = 2a$ and $AD = a$. The triangle CEF has base a and height $2a$. The particle of mass km is attached to the lamina at E. The object rests in a vertical plane with its edge AD on horizontal ground (see diagram).

Given that the object is on the point of toppling in its vertical plane about the vertex D, find the value of k. [4]

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

3.3 Circular motion

Name: _____ Class: _____ Date: _____ **Total: 16 marks**

KEY WORDS angular speed 角速度 • centripetal acceleration 向心加速度 • vertical circle 竖直圆 • horizontal circle • circular motion 圆周运动

Objective

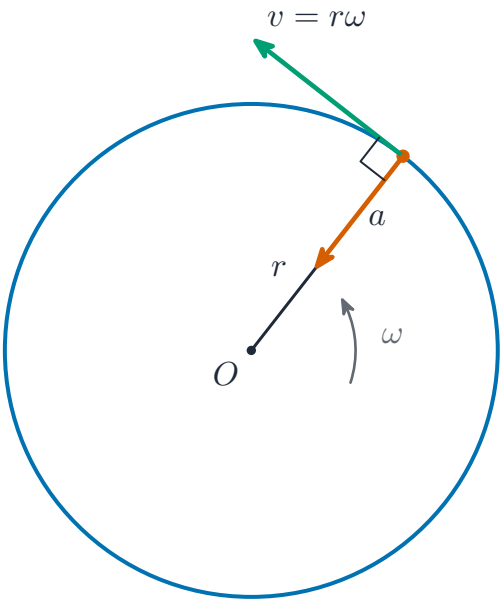
Build the skills to answer exam questions on **circular motion** 圆周运动 — **angular speed** 角速度, **centripetal acceleration** 向心加速度, the **conical pendulum** 圆锥摆, and motion in a **vertical circle** 竖直圆. Take $g = 10$ m s^{-2}.

You must be able to:

- use $v = r\omega$ and $a = r\omega^2 = \dfrac{v^2}{r}$
- analyse a horizontal circle (conical pendulum)
- use energy + the centripetal equation for a vertical circle

1 Worked examples

■ Centripetal force



Velocity points along the tangent; acceleration points inward to the centre

$r = 2$ **m,** $\omega = 3$ **rad s^{-1}:** $v = r\omega = 6$ m s^{-1}; $a = r\omega^2 = 18$ m s^{-2}.

Centripetal force $= ma = \dfrac{mv^2}{r}$, provided by tension, friction or a component of weight.

2 Practice

2.1 A particle moves in a circle of radius 0.5 m at 4 m s^{-1}. Find its acceleration. [2]

2.2 Convert an angular speed of 120 rev/min to rad s^{-1}. [2]

3 Exam-style questions

3.1 The centripetal acceleration of a particle moving at speed v in a circle of radius r is: [1]

- **A** $\dfrac{v}{r}$
- **B** $\dfrac{v^2}{r}$

- **C** vr
- **D** $\dfrac{r}{v^2}$

3.2 A particle of mass 0.2 kg is attached to a string of length 0.5 m and whirled in a horizontal circle on a smooth table at 3 rev s^{-1}.

(a) Find the angular speed in rad s^{-1}. [2]

(b) Find the tension in the string. [3]

3.3 A conical pendulum has a bob of mass 0.4 kg on a string of length 1 m, which makes an angle of 30° with the vertical.

(a) Find the tension in the string. [3]

(b) Find the angular speed of the bob. [3]

Solutions

2.1 $a = \dfrac{v^2}{r} = \dfrac{16}{0.5} = 32$ m s^{-2}.

2.2 120 rev/min = 2 rev/s = $2 \times 2\pi = 4\pi = 12.6$ rad s^{-1}.

3.1 B. $a = \dfrac{v^2}{r}$ (also $= r\omega^2$).

3.2 (a) $\omega = 3 \times 2\pi = 6\pi = 18.8$ rad s^{-1}.
(b) $T = mr\omega^2 = 0.2 \times 0.5 \times (6\pi)^2 = 0.2 \times 0.5 \times 355.3 = 35.5$ N.

3.3 (a) vertical: $T\cos 30^\circ = mg = 4 \Rightarrow T = \dfrac{4}{0.866} = 4.62$ N.
(b) horizontal: $T\sin 30^\circ = mr\omega^2$ with $r = 1 \times \sin 30^\circ = 0.5$; $4.62 \times 0.5 = 0.4 \times 0.5 \times \omega^2 \Rightarrow \omega^2 = 11.55 \Rightarrow \omega = 3.40$ rad s^{-1}.

2 A particle P of mass m is moving in a horizontal circle with angular speed ω_1 on the smooth inner surface of a hemispherical shell of radius r. The angle between the upward vertical and the normal reaction of the surface on P is θ_1, where $\tan\theta_1 = \frac{3}{4}$.

When the angular speed is increased to ω_2, the angle between the upward vertical and the normal reaction of the surface on P becomes θ_2, where $\tan\theta_2 = \frac{4}{3}$.

Find the ratio $\frac{\omega_1}{\omega_2}$. [4]

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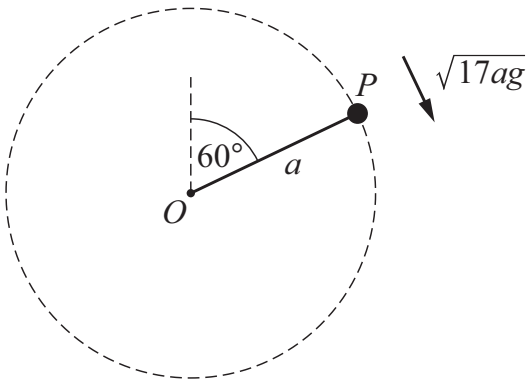

A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. Initially P is held with the string taut and making an angle of 60° with the upward vertical through O. The particle P is projected perpendicular to the string in a downwards direction with speed $\sqrt{17ag}$. It then starts to move along a circular path in a vertical plane with centre O (see diagram). At the lowest point of its path, vertically below O, the particle P collides with a stationary particle Q.

(a) Find, in terms of a and g, an expression for the speed of P immediately before the collision with Q. [2]

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As a result of the collision, P rebounds and moves back along a circular path with centre O. The string becomes slack when P reaches the point on the circle vertically above O.

(b) Find, in terms of a and g, an expression for the speed of P immediately after the collision with Q. [3]

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The mass of particle Q is km and the collision between P and Q is perfectly elastic.

(c) Find the value of k. [3]

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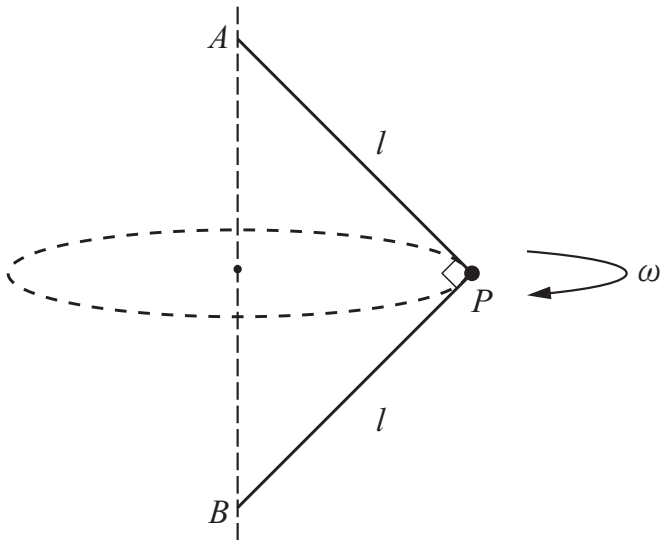
1 A particle P of mass m is attached to two light inextensible strings each of length l. The end of one string is attached to a fixed point A and the end of the other string is attached to a fixed point B, with A vertically above B. Angle APB is a right angle. The particle P rotates in a horizontal circle at a constant angular speed ω with both strings taut (see diagram).



Find the tension in string AP in terms of m, g, l and ω. [4]

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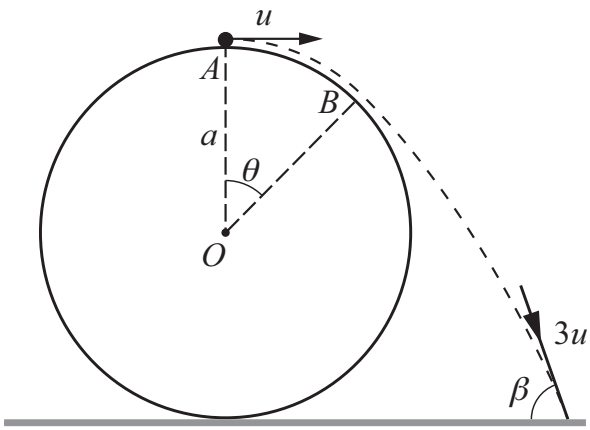

A fixed smooth sphere with radius a and centre O rests on horizontal ground. A particle is projected horizontally from the highest point, A, of the sphere with speed u. The particle begins to move in a vertical circle along the surface of the sphere. The particle loses contact with the sphere at the point B, where the angle AOB is θ.

After leaving the surface of the sphere, the particle moves freely under gravity before striking the horizontal ground with speed $3u$ at an angle β to the horizontal (see diagram).

Find the value of β. [8]

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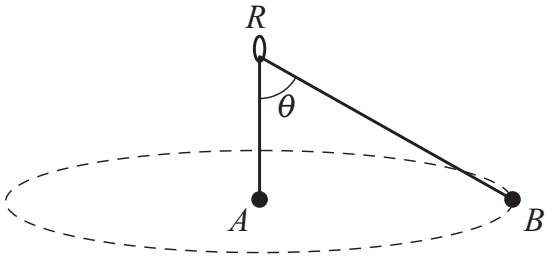

A light inextensible string of length $12a$ is threaded through a fixed smooth ring R. One end of the string is attached to a particle A of mass m. The other end of the string is attached to a particle B of mass $0.5m$. Particle A hangs in equilibrium vertically below the ring. Particle B moves with constant angular speed ω in a horizontal circle with particle A at its centre. The angle between AR and BR is θ (see diagram).

Express ω in terms of g and a. [5]

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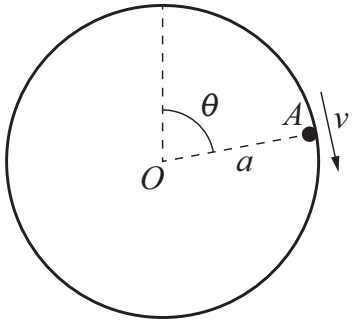

A fixed smooth spherical shell has centre O and radius a. A particle of mass m moves in complete vertical circles on the smooth inner surface of the shell, where the plane of the circular motion is vertical and passes through O. The particle has speed v when it is at point A, where OA makes an angle θ with the upward vertical through O, and $\cos\theta = \frac{1}{18}$ (see diagram).

(a) Show that $v \geqslant \frac{1}{3}\sqrt{26ag}$. [5]

It is given that $v = \frac{1}{3}\sqrt{26ag}$.

(b) Find, in terms of m and g, an expression for the greatest possible value of the normal reaction between the shell and the particle. [3]

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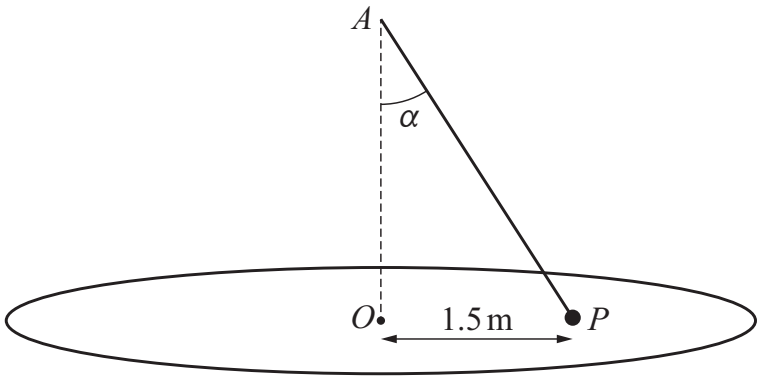

A rough horizontal disc, centre O, rotates with constant angular speed ω rad s^{-1}. A particle P of mass 1.6 kg lies on the disc at a distance 1.5 m from O, and is attached to a point A vertically above O by a light elastic string. The string has natural length 2 m, modulus of elasticity 32 N and makes an angle α with the vertical OA (see diagram). Particle P moves in a horizontal circle also at a constant angular speed ω rad s^{-1}. Particle P is on the point of slipping in the direction OP. The coefficient of friction between the particle and the disc is 0.5.

(a) Given that the tension in the string is 8 N, show that $\sin\alpha = 0.6$. [2]

(b) Find the number of revolutions per minute made by the disc and the particle P. [5]

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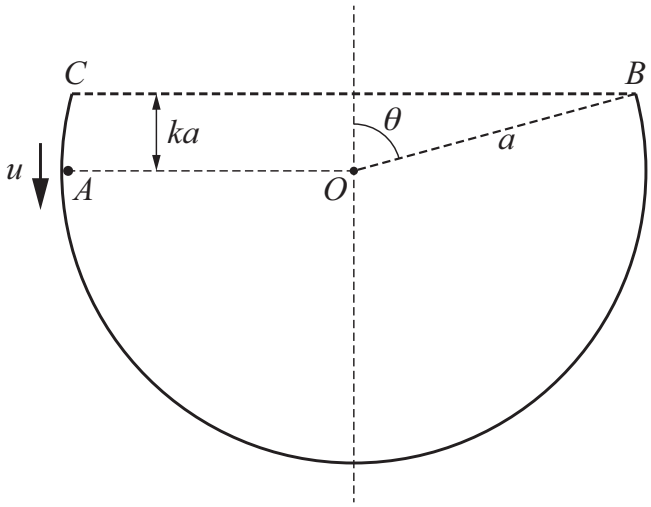

A fixed hollow sphere has radius a and centre O. The points A, B and C lie on the inner surface of the sphere with OA and BC horizontal. A portion of the sphere has been removed by a horizontal cut through points B and C at a vertical distance ka above the centre of the sphere, where k is a positive constant and $k < 1$. The points O, A, B and C all lie in the same vertical plane. OB makes an angle θ with the upward vertical through O (see diagram).

A particle P of mass m is free to move on the smooth inner surface of the sphere. The particle P is projected vertically downwards from A with speed u and begins to move in a vertical circle.

(a) In the case where $u = \sqrt{\frac{6}{5}ga}$, the reaction on P at B is half the reaction on P at A.

Find the value of k. [5]

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(b) Find an expression for u, in terms of a and g, in the case that the particle just reaches B. [1]

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(c) Find an expression for u, in terms of a and g, in the case that the particle passes through B and in its subsequent motion reaches C. [4]

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Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1 A particle P of mass m is attached to one end of a light inextensible string of length a. The other end of the string is attached to a fixed point O. The particle moves in a horizontal circle with constant angular speed ω and with the string inclined at an angle of θ to the downward vertical.

Given that $\tan\theta = \frac{4}{3}$, find ω in terms of a and g. [3]

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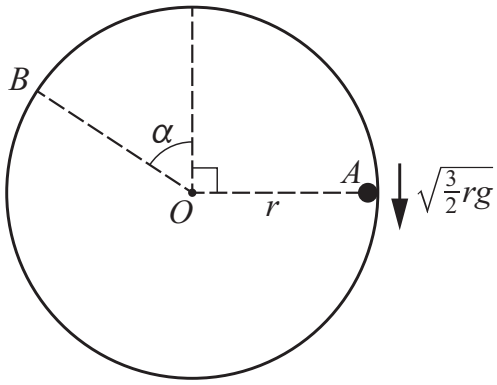


A hollow cylinder of radius r is fixed with its axis horizontal. Points A, B and O are in the same vertical plane perpendicular to the axis of the cylinder, with A and B on the smooth inner surface and O on the axis. OA and OB make angles 90° and α respectively with the upward vertical through O, with A and B on opposite sides of the vertical. A particle of mass m is projected vertically downwards from point A with speed $\sqrt{\frac{3}{2}rg}$ and moves in a vertical circle inside the cylinder (see diagram). The particle loses contact with the cylinder at point B.

(a) Find the value of α. [4]

(b) In the subsequent motion find, in terms of r, the greatest height above O reached by the particle. [4]

3.4 Hooke's law

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS elastic potential energy 弹性势能 • hooke's law 胡克定律 • equilibrium 平衡

Objective

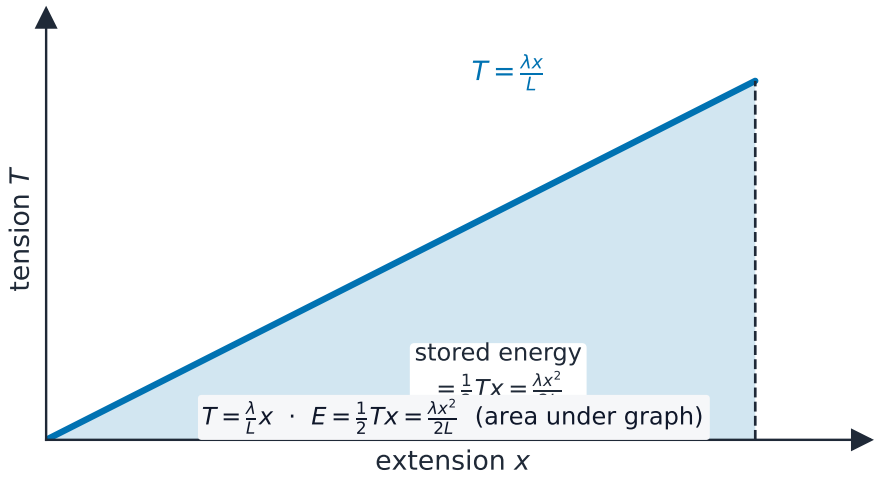
Build the skills to answer exam questions on **Hooke's law** 胡克定律 — the tension $T = \dfrac{\lambda x}{L}$ in an elastic string/spring, and the stored **elastic potential energy** 弹性势能 $E = \dfrac{\lambda x^2}{2L}$. Take $g = 10$ m s^{-2}.

You must be able to:

- use $T = \dfrac{\lambda x}{L}$ to find tension, extension, or λ
- find the stored energy $E = \dfrac{\lambda x^2}{2L}$
- combine Hooke's law with equilibrium or energy conservation

1 Worked examples

■ Tension and energy



Tension rises in proportion to extension; the shaded triangle is the stored elastic energy

Natural length 2 m, modulus 50 N, extension 0.5 m: $T = \dfrac{50 \times 0.5}{2} = 12.5$ N;

$$E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

2 Practice

2.1 A spring of natural length 1 m and modulus 30 N is stretched by 0.2 m. Find the tension. [2]

2.2 Find the energy stored in the spring in 2.1. [2]

3 Exam-style questions

3.1 Doubling the extension of an elastic string multiplies the stored energy by: [1]

- **A** 2
- **B** 4
- **C** $\frac{1}{2}$
- **D** 1

3.2 A particle of mass 2 kg hangs in equilibrium on an elastic string of natural length 1.5 m and modulus 60 N.

(a) Find the extension of the string. [3]

(b) Find the elastic energy stored. [2]

3.3 A particle of mass 0.5 kg is attached to one end of an elastic string of natural length 2 m and modulus 40 N, the other end fixed at O. The particle is released from rest at O

and falls vertically. Using energy, find its speed when the string has extension 0.5 m. [5]

Solutions

2.1 $T = \dfrac{30 \times 0.2}{1} = 6$ N.

2.2 $E = \dfrac{30 \times 0.2^2}{2 \times 1} = 0.6$ J.

3.1 B. $E \propto x^2$, so doubling x multiplies E by 4.

3.2 (a) equilibrium $T = mg = 20$ N; $\dfrac{60x}{1.5} = 20 \Rightarrow 40x = 20 \Rightarrow x = 0.5$ m.

(b) $E = \dfrac{60 \times 0.5^2}{2 \times 1.5} = \dfrac{15}{3} = 5$ J.

3.3 the particle falls $2 + 0.5 = 2.5$ m to reach extension 0.5 m; energy: $mg(2.5) = \frac{1}{2}mv^2 + \dfrac{\lambda x^2}{2L}$; $0.5(10)(2.5) = \frac{1}{2}(0.5)v^2 + \dfrac{40(0.25)}{2(2)}$; $12.5 = 0.25v^2 + 2.5 \Rightarrow v^2 = 40$; $v = 6.32$ m s^{-1}.

4 One end of a light elastic string of natural length a and modulus of elasticity $5mg$ is attached to a fixed point O. Two particles, P and Q, of masses m and $4m$ respectively are attached to the other end of the string and they hang vertically in equilibrium. Particle Q is then detached from the string, hence releasing particle P from rest.

Find, in terms of a, the length of the string when the speed of particle P is first equal to $\sqrt{\frac{7}{5}ag}$. [6]

2 One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point A on a rough horizontal surface. The other end of the string is attached to a particle P of mass m. The particle and string rest on the surface. The coefficient of friction between P and the surface is μ. The particle P is initially held in equilibrium at a distance $\frac{4}{3}a$ from A. The particle is then released from rest.

(a) Given that the string never becomes slack, find the minimum value of μ. [3]

It is now given that $\mu = \frac{1}{2}$.

(b) Find the extension of the string when the particle comes to rest. [3]

6 A and B are two fixed points at a distance $22a$ apart, with B vertically below A. A light elastic string of natural length $4a$ and modulus of elasticity $5mg$ has one end attached to A and the other end attached to a particle P of mass km. Another light elastic string of natural length $8a$ and modulus of elasticity $6mg$ has one end attached to B and the other end attached to P. Particle P is vertically above B.

(a) Show that, when the system is in equilibrium, $BP = \dfrac{57a - 2ak}{4}$. [5]

The particle P is pulled vertically upwards so that $BP = 18a$, and is then released from rest. In its subsequent motion, P first comes to instantaneous rest at the point where $BP = 8a$.

(b) Find the value of k. [4]

5 One end of a light elastic string of natural length 0.5 m and modulus of elasticity 14 N is attached to a fixed point A on a smooth plane. The plane makes an angle α to the horizontal, where $\tan\alpha = \frac{7}{24}$. A particle P of mass 2 kg is attached to the other end of the string. The string lies along a line of greatest slope of the plane. The particle P is initially held on the plane above the level of A, where $AP = 0.8$ m. The particle P is then released from rest.

Find the maximum velocity of P during the subsequent motion. [6]

2 A particle P of mass m is attached to one end of a light elastic string of natural length a and modulus of elasticity mg. The other end of the string is attached to a fixed point O on a rough plane inclined at an angle of 30° to the horizontal. The particle P is held at rest at point O before being released. The frictional force acting on P as it slides down the plane is $\frac{11}{30}mg$.

(a) Find, in terms of a, the distance that P moves down the plane before coming to rest. [5]

(b) It is given that P remains at rest in this new position.

Find, in terms of m and g, the magnitude of the frictional force in this position. [3]

3.5 Linear motion under a variable force

Name: _____ Class: _____ Date: _____ **Total: 14 marks**

KEY WORDS differential equation 微分方程 • variable force 变力

Objective

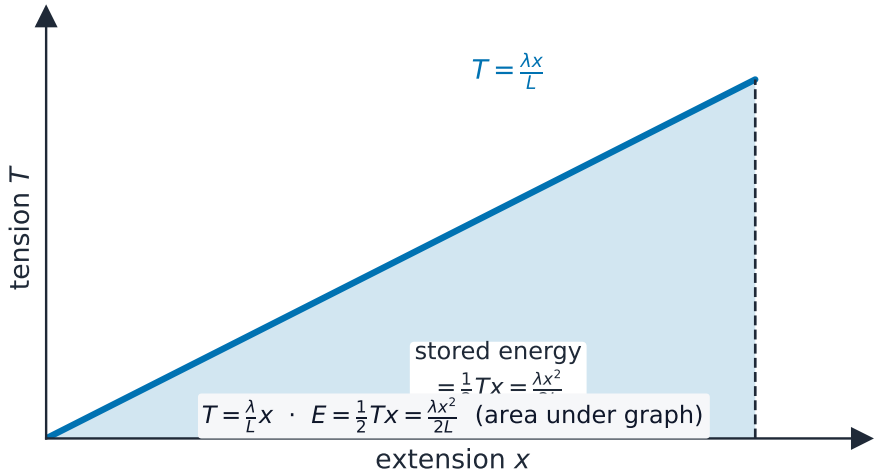
Build the skills to answer exam questions on **linear motion under a variable force** 变力下的直线运动 — using $a = v\dfrac{dv}{dx}$ (force depends on position) or $a = \dfrac{dv}{dt}$ (force depends on time). Take $g = 10$ m s^{-2}.

You must be able to:

- choose $a = v\dfrac{dv}{dx}$ for a position-dependent force
- set up and solve the resulting differential equation
- use $a = \dfrac{dv}{dt}$ for a time-dependent force

1 Worked examples

■ Force depending on position



A position-dependent force (e.g. a spring) makes Newton's law a differential equation in v and x

8 **kg under force** $(x^3 + 4x)$ **N,** $v = 1$ **at** $x = 0$**:** $8v\dfrac{dv}{dx} = x^3 + 4x$; integrate $\rightarrow$ $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4$, so $v = \frac{1}{4}x^2 + 1$.

2 Practice

2.1 State the form of acceleration to use when a force depends on **position** x. [1]

2.2 A particle has $v\dfrac{dv}{dx} = 2x$. Find v^2 in terms of x, given $v = 0$ at $x = 0$. [2]

3 Exam-style questions

3.1 For a force that depends on position, acceleration is best written as: [1]

- **A** $\dfrac{dv}{dt}$
- **B** $v\dfrac{dv}{dx}$
- **C** $\dfrac{dx}{dt}$
- **D** $\dfrac{d^2x}{dt^2}$ only

3.2 A particle of mass 2 kg moves along a straight line under a force $F = 6x$ N in the direction of motion. At $x = 0$ the speed is 2 m s^{-1}.

(a) Form a differential equation in v and x. [2]

(b) Find v when $x = 2$. [4]

3.3 A particle of mass 0.5 kg moves so that its acceleration is $\dfrac{dv}{dt} = 4 - 2t$. At $t = 0$ it is

at rest. Find its velocity when $t = 3$ and comment. [4]

Solutions

2.1 $a = v\dfrac{dv}{dx}$.

2.2 $\int v\,dv = \int 2x\,dx \Rightarrow \frac{1}{2}v^2 = x^2 + C$; $v = 0$ at $x = 0 \Rightarrow C = 0$, so $v^2 = 2x^2$.

3.1 B. Use $a = v\dfrac{dv}{dx}$ for a position-dependent force.

3.2 (a) $2v\dfrac{dv}{dx} = 6x$, i.e. $v\dfrac{dv}{dx} = 3x$.
(b) $\int v\,dv = \int 3x\,dx \Rightarrow \frac{1}{2}v^2 = \frac{3}{2}x^2 + C$; at $x = 0$, $v = 2 \Rightarrow C = 2$; at $x = 2$: $\frac{1}{2}v^2 = 6 + 2 = 8 \Rightarrow v^2 = 16$; $v = 4$ m s^{-1}.

3.3 $v = \int(4-2t)\,dt = 4t - t^2 + C$; $v = 0$ at $t = 0 \Rightarrow C = 0$; at $t = 3$: $v = 12 - 9 = 3$ m s^{-1}; the force has reversed (since $4 - 2t < 0$ for $t > 2$), so the particle is slowing after $t = 2$.

3 A particle P is moving in a straight horizontal line. At time t s, the displacement of P from a fixed point O on the line is x m and the velocity of P is $v\,\mathrm{m\,s^{-1}}$. The acceleration of P is $\frac{1}{2}(v^2+4)\,\mathrm{m\,s^{-2}}$ in the direction PO. Initially P is at O and is moving with velocity $2\,\mathrm{m\,s^{-1}}$.

(a) Find an expression for x in terms of t. [5]

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(b) Find the time when P next goes through O. [2]

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7 A particle P of mass m kg moving along a rough horizontal table has displacement x m from a fixed point O on the table and velocity $v\,\mathrm{m\,s^{-1}}$ at time t s. The particle P is subject to a resistive force of magnitude $mgkv$ N, where k is a positive constant, and a frictional force of magnitude μmg. The particle P is initially at O with speed $U\,\mathrm{m\,s^{-1}}$.

(a) Show that $t = \frac{1}{gk}\ln\left(\frac{kU+\mu}{kv+\mu}\right)$. [4]

It is given that $U = 10$, $k = 0.04$ and $\mu = 0.2$.

(b) Find the distance P moves before coming to rest. [4]

(c) Find the average speed of P over the period it is moving. [2]

Additional page

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5 A particle P of mass m kg is projected vertically upwards from a point O with an initial speed of $20\,\text{m s}^{-1}$ and moves under gravity. There is a resistive force of magnitude $0.025mv^2$ N, where $v\,\text{m s}^{-1}$ is the speed of P at time t s after projection. The displacement of P from O is x m at time t s after projection.

(a) Find an expression for v in terms of x, while P is moving upwards. [5]

(b) Find an expression for v in terms of t, while P is moving upwards. [4]

1 A particle P of mass 8 kg is moving in a straight horizontal line. At time t s, P has displacement x m from a fixed point O on the line and velocity v m s^{-1}. The only horizontal force acting on P has magnitude $(x^3 + 4x)$ N and acts in the direction OP. When $t = 0$, $x = 0$ and $v = 1$.

(a) Find an expression for v in terms of x, giving your answer in the form $v = ax^2 + b$, where a and b are constants to be determined. [3]

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(b) Find an expression for x in terms of t. [2]

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3 A ball of mass m kg is projected vertically upwards with initial speed U m s^{-1} and moves under gravity. At time t s after projection, the ball has travelled a distance x m and its speed is v m s^{-1}. There is a resistive force of magnitude mkv^2 N, where k is a positive constant.

(a) Show that the distance travelled by the ball when it is moving upwards is $x = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv^2}\right)$. [4]

It is given that $k = 0.025$ and that $U = 20$.

(b) Find the time taken for the ball to reach its maximum height. [4]

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

3.6 Momentum

Name: _____ Class: _____ Date: _____ **Total: 15 marks**

KEY WORDS coefficient of restitution 恢复系数 • conservation of momentum 动量守恒 • oblique impact 斜碰撞 • momentum and impacts 动量与碰撞 • moment 力矩

Objective

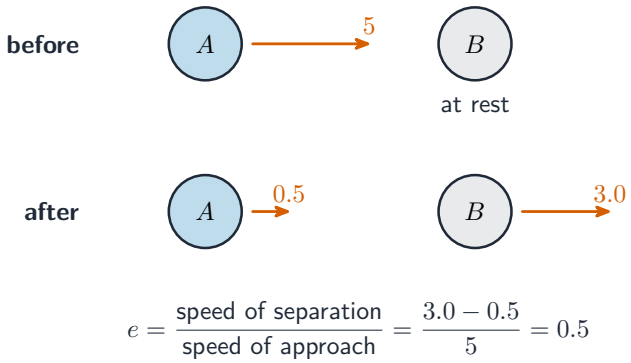
Build the skills to answer exam questions on **momentum and impacts** 动量与碰撞 — conservation of momentum, the **coefficient of restitution** 恢复系数 e, and **oblique impact** 斜碰撞. Take $g = 10$ m s^{-2}.

You must be able to:

- apply conservation of momentum to a collision
- use $e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$
- resolve velocities for an oblique impact

1 Worked examples

■ Restitution



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

e compares how fast the bodies separate with how fast they approached ($0 \leqslant e \leqslant 1$)

A (2 kg, 5 m s^{-1}) hits stationary B (3 kg), $e = 0.5$: momentum $2(5) = 2v_A + 3v_B$; restitution $v_B - v_A = 0.5(5) = 2.5$. Solve: $v_A = 0.5$, $v_B = 3.0$ m s^{-1}.

2 Practice

2.1 State the range of possible values of the coefficient of restitution e. [1]

2.2 Two balls approach at a combined closing speed of 8 m s^{-1} with $e = 0.25$. Find their separation speed. [2]

3 Exam-style questions

3.1 A perfectly elastic collision has a coefficient of restitution: [1]

- **A** 0
- **B** 0.5
- **C** 1
- **D** 2

3.2 A sphere P of mass 3 kg moving at 4 m s^{-1} collides directly with a sphere Q of mass 1 kg moving towards it at 2 m s^{-1}. The coefficient of restitution is 0.5.

(a) Write the momentum and restitution equations. [3]

(b) Find the velocities of P and Q after impact. [4]

3.3 A ball is dropped from a height of 2 m onto the floor, with coefficient of restitution 0.8 between ball and floor. Find the height of the first bounce. [4]

Solutions

2.1 $0 \leqslant e \leqslant 1$.

2.2 separation $= e\times$ approach $= 0.25 \times 8 = 2$ m s^{-1}.

3.1 C. A perfectly elastic collision has $e = 1$.

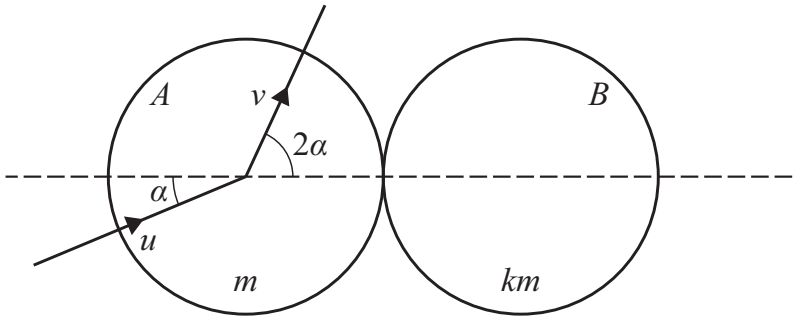
3.2 (a) take P's direction positive: momentum $3(4)+1(-2) = 3v_P+1v_Q$, i.e. $3v_P+v_Q = 10$; restitution $v_Q - v_P = 0.5(4-(-2)) = 3$.
(b) from the two equations: $v_Q = v_P+3$; $3v_P+v_P+3 = 10 \Rightarrow 4v_P = 7 \Rightarrow v_P = 1.75$ m s^{-1}; $v_Q = 4.75$ m s^{-1}.

3.3 speed hitting floor: $v = \sqrt{2g(2)} = \sqrt{40} = 6.32$ m s^{-1}; rebound speed $= ev = 0.8(6.32) = 5.06$ m s^{-1}; bounce height $= \dfrac{(ev)^2}{2g} = \dfrac{25.6}{20} = 1.28$ m.

1 Two uniform smooth spheres A and B of equal radii have masses $4m$ and m respectively. Sphere B is at rest on a smooth horizontal surface. Sphere A is moving on the surface with speed u and collides directly with sphere B. After the collision, the momentum of A is three times the momentum of B.

Find the value of the coefficient of restitution e. [4]

5 Two uniform smooth spheres, A and B, of equal radii are on a horizontal surface. They have masses of m and km respectively. Sphere A is moving with speed u at an angle α with the line of centres when it collides with sphere B which is stationary. Immediately after the collision, sphere A moves with speed v at an angle 2α with the line of centres (see diagram).



It is given that $\tan\alpha = \frac{3}{4}$.

(a) Find v in terms of u. [2]

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(b) Find the coefficient of restitution between the spheres in terms of k. [4]

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(c) Find the range of possible values of k. [3]

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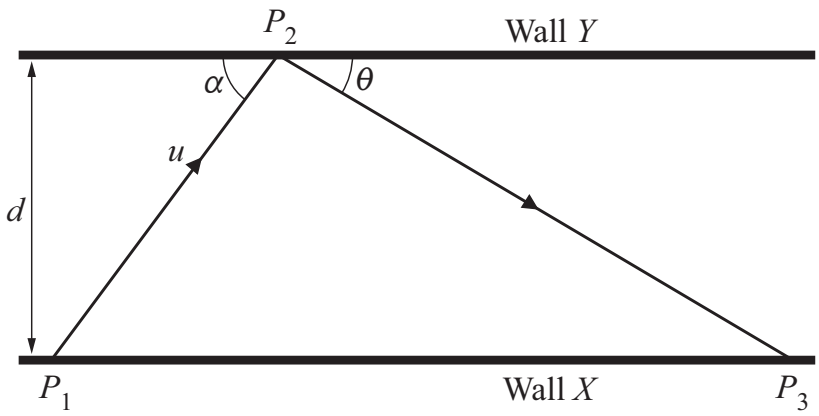

X and Y are two fixed smooth vertical walls on a smooth horizontal surface. The walls are parallel and at a distance d apart. The points P_1, P_2 and P_3 all lie on the surface.

A particle Q is projected horizontally from the point P_1 on Wall X with speed u, and moves along the surface. The particle Q strikes Wall Y at the point P_2. Immediately before the collision, the direction of motion of Q makes an angle α with Wall Y, where $\sin\alpha = \frac{4}{5}$. Immediately after the collision, the direction of motion of Q makes an angle θ with Wall Y. The particle Q then strikes Wall X at the point P_3 (see diagram).

The time that it takes Q to travel the distance P_1P_2 is T. The time that it takes Q to travel the distance P_2P_3 is kT.

Find, in terms of k, the coefficient of restitution between Q and Wall Y. [6]

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6 Two uniform smooth spheres A and B of equal radii have masses $2m$ and m respectively. Sphere A is moving in a straight horizontal line with speed u, and sphere B is stationary. Sphere A collides directly with B, and they both then move in the same direction with speeds v_A and v_B respectively. After the collision, the kinetic energy of B is $\frac{9}{2}$ times the kinetic energy of A.

(a) Show that $v_B = \frac{6}{5}u$. [3]

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Sphere B then collides with a fixed vertical barrier. Immediately before the collision, the direction of motion of B makes an angle α with the barrier. Immediately after the collision, the direction of motion of B makes an angle β with the barrier. The coefficient of restitution between B and the barrier is $\frac{4}{5}$. As a result of the collision, the velocity of B is reduced to $\frac{12}{25}\sqrt{5}u$.

(b) Find the value of $\sin(\alpha+\beta)$. [6]

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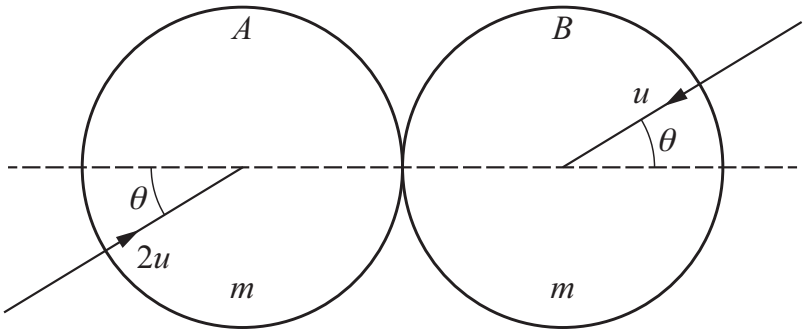

Two identical uniform smooth spheres A and B, each with mass m, are moving on a horizontal surface with speeds $2u$ and u respectively when they collide. Immediately before the collision, the spheres are moving parallel to each other in opposite directions such that their directions of motion each make an angle θ with the line of centres (see diagram). As a result of the collision, B moves in a direction which is perpendicular to its initial direction of motion. The coefficient of restitution between the spheres is e.

(a) Find an expression for $\tan\theta$ in terms of e. [6]

As a result of the collision, A moves in a direction which is perpendicular to the line of centres.

(b) Find the value of θ. [2]