Week 1

A-Level Further Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

A-Level Further Mathematics Name: ____________ Score: ______

1. For $x^2 - 5x + 6 = 0$, what is the sum of the roots $\alpha + \beta = -b/a$?

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2. A rational function has an oblique (slant) asymptote when the degree of the top is:
   - ☐ higher than the degree of the bottom
   - ☐ lower than the bottom
   - ☐ equal to the bottom
   - ☐ always zero

3. Using $\Sigma r = ½n(n+1)$, what is the sum of the first 5 integers?

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4. What is the determinant of the matrix [[3, 1], [2, 4]] (ad − bc)?

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5. In polar coordinates, the x-coordinate is given by:
   - ☐ $r \cos \theta$
   - ☐ $r \sin \theta$
   - ☐ $r \tan \theta$
   - ☐ r / θ

6. The vector product $a \times b$ gives a vector that is:
   - ☐ perpendicular to both a and b
   - ☐ parallel to a
   - ☐ equal to a + b
   - ☐ a scalar

7. The two steps of proof by induction are the base case and the:
   - ☐ inductive step (assume $n = k$, prove $n = k + 1$)
   - ☐ final check at $n = 100$
   - ☐ differentiation step
   - ☐ substitution

8. For $x^2 - 5x + 6 = 0$, what is the product $\alpha\beta = c/a$?

9. You find an oblique asymptote by:
   - ☐ dividing the top polynomial by the bottom
   - ☐ differentiating
   - ☐ setting $x = 0$
   - ☐ taking the modulus
10. Using $\Sigma r^2 = (1/6)n(n+1)(2n+1)$, what is the sum of the first 3 squares?

A-Level Further Mathematics

1. **5**

   $\alpha + \beta = -b/a = -(-5)/1 = 5$.

2. **higher than the degree of the bottom**

   When the numerator's degree exceeds the denominator's, dividing out leaves a slant line the curve approaches.

3. **15**

   ½ × 5 × 6 = 15 (= 1+2+3+4+5).

4. **10**

   det = 3×4 − 1×2 = 12 − 2 = 10.

5. **r cos θ**

   x = r cos θ and y = r sin θ.

6. **perpendicular to both a and b**

   a × b is perpendicular to both vectors; its length is |a||b|sin θ.

7. **inductive step (assume n = k, prove n = k + 1)**

   Induction needs a base case plus an inductive step from k to k+1.

8. **6**

   $\alpha\beta = c/a = 6/1 = 6$.

9. **dividing the top polynomial by the bottom**

   Polynomial division gives the straight-line part the curve approaches far from the origin.

10. **14**

    (1/6)(3)(4)(7) = 84/6 = 14 (= 1+4+9).

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

1.1 Roots of polynomial equations

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS roots 根 • coefficients 系数 • symmetric functions 对称函数 • roots of polynomial equations 多项式方
• substitution 代换

Objective

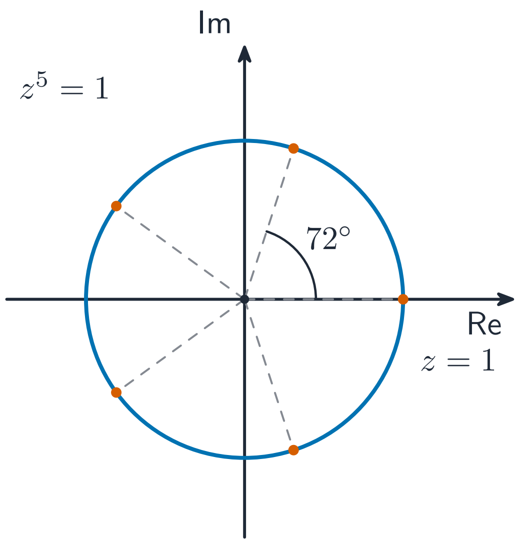
Build the skills to answer exam questions on **roots of polynomial equations 多项式方程的根** — the relations between roots and **coefficients 系数**, **symmetric functions 对称函数**, and forming a new equation by **substitution 代换**.

You must be able to:

- use $\sum\alpha$, $\sum\alpha\beta$, $\alpha\beta\gamma$ for quadratics and cubics
- evaluate symmetric functions such as $\sum\alpha^2$
- form an equation whose roots are a transformation of the originals

1 Worked examples

■ Roots and coefficients



The roots of a polynomial are tied to its coefficients by $\sum\alpha$, $\sum\alpha\beta$, …

$x^2 - 5x + 6 = 0$ **has roots** α, β. **Find the equation with roots** $\alpha + 1, \beta + 1$.
$\alpha + \beta = 5$, $\alpha\beta = 6$; new sum $= 7$, new product $= \alpha\beta + \alpha + \beta + 1 = 12$. So

> $x^2 - 7x + 12 = 0$.
> $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$.

2 Practice

2.1 For $x^2 - 7x + 10 = 0$ with roots α, β, write down $\alpha + \beta$ and $\alpha\beta$. [2]

2.2 Hence find $\alpha^2 + \beta^2$. [2]

3 Exam-style questions

3.1 For $2x^3 - 3x^2 + 4x - 5 = 0$ with roots α, β, γ, the value of $\alpha + \beta + \gamma$ is: [1]

- **A** $\frac{3}{2}$
- **B** $-\frac{3}{2}$
- **C** 2
- **D** $\frac{5}{2}$

3.2 The cubic $x^3 - 4x^2 + 5x - 2 = 0$ has roots α, β, γ.

(a) Write down $\sum \alpha$, $\sum \alpha\beta$ and $\alpha\beta\gamma$. [3]

(b) Find $\alpha^2 + \beta^2 + \gamma^2$. [3]

3.3 The equation $x^2 + 3x - 1 = 0$ has roots α, β. Find a quadratic equation with integer coefficients whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$. [4]

Solutions

2.1 $\alpha + \beta = 7$, $\alpha\beta = 10$.

2.2 $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 49 - 20 = 29$.

3.1 **A**. $\alpha + \beta + \gamma = -\dfrac{-3}{2} = \dfrac{3}{2}$.

3.2 (a) $\sum \alpha = 4$, $\sum \alpha\beta = 5$, $\alpha\beta\gamma = 2$.
(b) $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta = 16 - 10 = 6$.

3.3 $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{-3}{-1} = 3$; $\dfrac{1}{\alpha\beta} = \dfrac{1}{-1} = -1$; new equation $x^2 - 3x - 1 = 0$.

4 The quartic equation $x^4 + x^3 + x^2 + x + 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Show that a quartic equation with roots $2\alpha + 1, 2\beta + 1, 2\gamma + 1, 2\delta + 1$ is

$$y^4 - 2y^3 + 4y^2 + 2y + 11 = 0.$$ [4]

The sum $(2\alpha+1)^n+(2\beta+1)^n+(2\gamma+1)^n+(2\delta+1)^n$ is denoted by S_n.

(b) Find the value of S_2. [2]

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(c) Given that $S_3=-22$, find the value of S_4. [2]

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2 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β and γ. It is given that

$$\alpha + \beta + \gamma = 2,$$
$$\alpha^2 + \beta^2 + \gamma^2 = 3,$$
$$\alpha^4 + \beta^4 + \gamma^4 = 5.$$

(a) Find the values of b and c. [3]

(b) Find the value of d. [5]

2 The cubic equation $x^3 + bx^2 + cx + d = 0$, where $d \neq 0$, has roots α, β, γ such that $\gamma = \frac{1}{\beta}$.

(a) Show that $\beta + \dfrac{1}{\beta} = d - b$. [3]

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(b) Show also that $\beta + \dfrac{1}{\beta} = \dfrac{1-c}{d}$. [2]

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(c) It is given that $b = 3$, $c = -3$ and $d > 0$.

(i) Find the value of d. [2]

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

2 The cubic equation $x^3 + 2x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3 - 1$, $\beta^3 - 1$, $\gamma^3 - 1$. [3]

(b) Find the value of $(\alpha^3-1)^2+(\beta^3-1)^2+(\gamma^3-1)^2$. [2]

(c) Find the value of $(\alpha^3-1)^3+(\beta^3-1)^3+(\gamma^3-1)^3$. [2]

3 The quartic equation $x^4 + 7x^2 + 3x + 22 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. [2]

(b) Find the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$. [2]

(c) Use standard results from the list of formulae (MF19) to find the value of

$$\sum_{r=1}^{10}\left(\left(\alpha^2+r\right)^2+\left(\beta^2+r\right)^2+\left(\gamma^2+r\right)^2+\left(\delta^2+r\right)^2\right).$$ [5]

1.2 Rational functions and graphs

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS asymptotes 渐近线 • oblique 斜渐近线 • discriminant 判别式 • rational function 有理函数 • rational functions and graphs 有理函数与图像

Objective

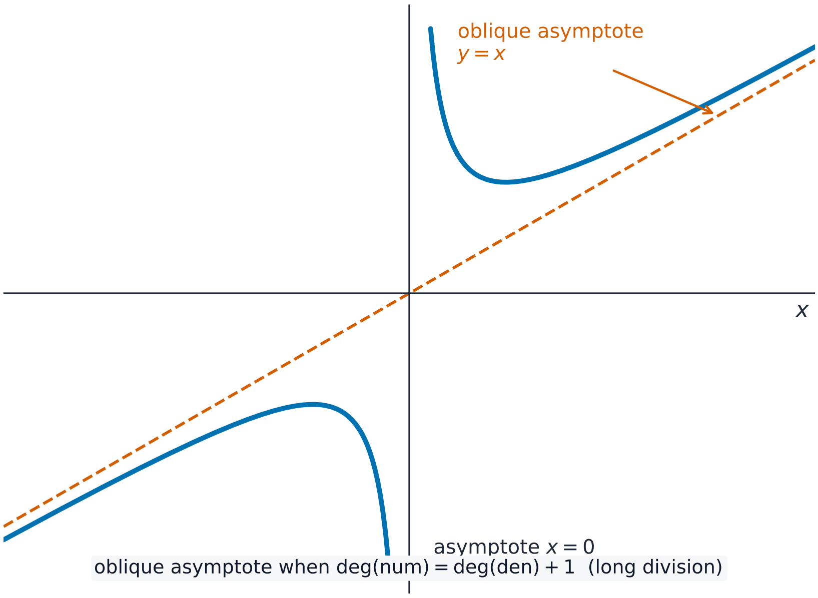
Build the skills to answer exam questions on **rational functions and graphs 有理函数与图像** — **asymptotes 渐近线** (including **oblique 斜渐近线**), the range via the **discriminant 判别式**, and related graphs.

You must be able to:

- find vertical and oblique asymptotes (divide out)
- find the set of values of a rational function using the discriminant
- relate $y = f(x)$ to $y = \frac{1}{f(x)}$, $y = |f(x)|$, $y^2 = f(x)$

1 Worked examples

Asymptotes & range



Far out the curve hugs the slant line; near a zero of the denominator it runs off along a vertical asymptote

$y = \dfrac{x^2+1}{x} = x + \dfrac{1}{x}$**:** vertical asymptote $x = 0$, oblique asymptote $y = x$.

Range of $y = \dfrac{x^2+1}{x}$**:** $yx = x^2 + 1 \Rightarrow x^2 - yx + 1 = 0$; real x needs $y^2 - 4 \geqslant 0$, so $y \leqslant -2$ or $y \geqslant 2$.

2 Practice

2.1 Write down the vertical asymptote of $y = \dfrac{2x+1}{x-3}$. [1]

2.2 Find the oblique asymptote of $y = \dfrac{x^2}{x-1}$. [2]

3 Exam-style questions

3.1 The curve $y = \dfrac{x^2-4}{x}$ has an oblique asymptote: [1]

- **A** $y = x$
- **B** $y = x - 4$

- **C** $y = 2x$
- **D** $y = -4$

3.2 A curve has equation $y = \dfrac{x^2 + 2}{x - 1}$.

(a) Find the equations of the two asymptotes. [3]

(b) Use the discriminant to find the set of values that y **cannot** take. [4]

3.3 The function is $y = \dfrac{x}{x^2 + 1}$. Find the set of values y can take. [4]

Solutions

2.1 $x = 3$.

2.2 $\dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$, so oblique asymptote $y = x + 1$.

3.1 **A**. $\dfrac{x^2-4}{x} = x - \dfrac{4}{x}$, so the oblique asymptote is $y = x$.

3.2 (a) vertical $x = 1$; divide: $\dfrac{x^2+2}{x-1} = x + 1 + \dfrac{3}{x-1}$, oblique $y = x + 1$.
(b) $y(x-1) = x^2+2 \Rightarrow x^2 - yx + (y+2) = 0$; real x needs $y^2 - 4(y+2) \geqslant 0 \Rightarrow y^2 - 4y - 8 \geqslant 0$; roots $y = 2 \pm 2\sqrt{3}$; so y cannot take values in $2 - 2\sqrt{3} < y < 2 + 2\sqrt{3}$.

3.3 $y(x^2+1) = x \Rightarrow yx^2 - x + y = 0$; for real x (with $y \neq 0$), $1 - 4y^2 \geqslant 0$; $-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}$.

7 The curve C has equation $y = \dfrac{x+2}{x^2+3x+1}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Show that C has no stationary points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x+2}{x^2+3x+1}\right|$. [2]

(e) Find in exact form the set of values of x for which $\left|\dfrac{x+2}{x^2+3x+1}\right| > 2$. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

7 The curve C has equation $y = \dfrac{x\quad + x + 1}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find the coordinates of any stationary points on C. [3]

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(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \dfrac{|x|^2 + |x| + 1}{|x| + 1}$. [2]

(e) Find, in exact form, the set of values of x for which $\dfrac{|x|^2 + |x| + 1}{|x| + 1} < 3$. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

7 The curve C has equation $y = \dfrac{10x^2 - 11x - 18}{10x - 18}$.

(a) Find the equations of the asymptotes of C. [3]

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(b) Show that C has no stationary points. [4]

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(c) Sketch C, stating the coordinates of the points of intersection with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{10x^2 - 11x - 18}{10x - 18}\right|$. [2]

(e) Given that $\left|\dfrac{10x^2-11x-18}{10x-18}\right| < 4$ for $\dfrac{-29-\sqrt{4441}}{20} < x < p$ and $\dfrac{-29+\sqrt{4441}}{20} < x < q$, find the values of p and q. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

7 The curve C has equation $y = \dfrac{2x^2 - 3x}{2x^2 - 7x - 4}$.

(a) Find the equations of the asymptotes of C. [2]

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(b) Find the coordinates of any stationary points on C. [4]

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(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - 5x}{2x^2 - 7x - 4}\right|$. [1]

(e) Find in exact form the set of values of x for which $\left|\dfrac{2x^2-5x}{2x^2-7x-4}\right| < \dfrac{1}{9}$. [5]

Additional page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

6 The curve C has equation $y = \dfrac{x^2 + a}{x+a}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

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(b) Find, in terms of a, the x-coordinates of the stationary points on C. [3]

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(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 + a}{x + a}\right|$. [1]

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(e) Find the set of values of a for which $\left|\dfrac{x^2+a}{x+a}\right| = a$ has two real solutions. [4]

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

1.3 Summation of series

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS method of differences 差分法 • sum to infinity 无穷和 • partial fractions 部分分式 • summation of se

Objective

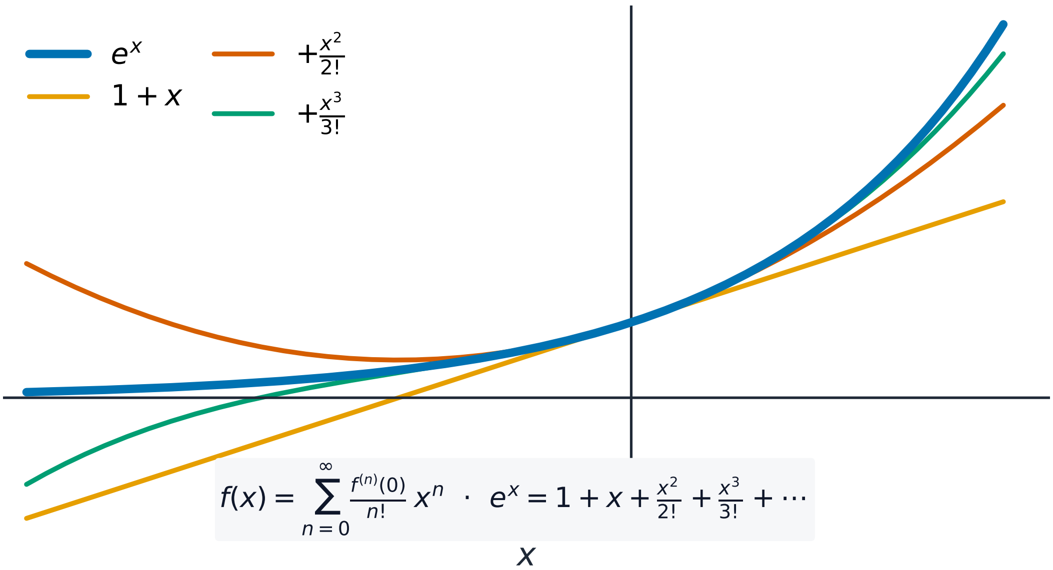
Build the skills to answer exam questions on the **summation of series** 级数求和 — the standard results for $\sum r$, $\sum r^2$, $\sum r^3$, and the **method of differences** 差分法.

You must be able to:

- use $\sum r = \frac{1}{2}n(n+1)$, $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum r^3 = \frac{1}{4}n^2(n+1)^2$
- sum a series by the method of differences (telescoping)
- decide convergence and find a sum to infinity

1 Worked examples

■ Standard results & differences



A finite sum builds from standard results; telescoping cancels all but the end terms

$$\sum_{r=1}^{n}(2r+1) = 2 \cdot \tfrac{1}{2}n(n+1) + n = n(n+2).$$

Differences: if $u_r = f(r) - f(r+1)$ then $\sum_{r=1}^{n} u_r = f(1) - f(n+1)$.

2 Practice

2.1 Evaluate $\displaystyle\sum_{r=1}^{n} r^2$ as a formula in n. [1]

2.2 Find $\displaystyle\sum_{r=1}^{10} (3r-1)$. [2]

3 Exam-style questions

3.1 $\displaystyle\sum_{r=1}^{n} r^3$ equals: [1]

- **A** $\frac{1}{4}n^2(n+1)^2$
- **B** $\frac{1}{6}n(n+1)(2n+1)$
- **C** $\frac{1}{2}n(n+1)$
- **D** n^3

3.2 (a) Express $\dfrac{1}{r(r+1)}$ in partial fractions. [2]

(b) Hence find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)}$ by the method of differences, and state its sum to infinity. [4]

3.3 Find $\sum_{r=1}^{n}(r^2 + r)$, giving your answer as a single factorised expression. [4]

Solutions

2.1 $\frac{1}{6}n(n+1)(2n+1)$.

2.2 $3 \cdot \frac{1}{2}(10)(11) - 10 = 165 - 10 = 155$.

3.1 **A**. $\sum r^3 = \frac{1}{4}n^2(n+1)^2$.

3.2 (a) $\dfrac{1}{r(r+1)} = \dfrac{1}{r} - \dfrac{1}{r+1}$.

(b) $\displaystyle\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$; as $n \to \infty$ the sum $\to 1$.

3.3 $\sum(r^2 + r) = \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$; $= \frac{1}{6}n(n+1)[(2n+1)+3]$; $= \frac{1}{6}n(n+1)(2n+4) = \frac{1}{3}n(n+1)(n+2)$.

1 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(8r^3+12r^2+4r+3)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1}{r^4} - \frac{1}{(r+1)^4} = \frac{4r^3 + 6r^2 + 4r + 1}{r^4(r+1)^4}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{4r^3 + 6r^2 + 4r + 1}{r^4(r+1)^4}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{4r^3 + 6r^2 + 4r + 1}{r^4(r+1)^4}$. [1]

1 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n}(r^3 - r)$ in terms of n, fully factorising your answer. [3]

(b) Express $\dfrac{r+3}{r^3-r}$ in the form $\dfrac{A}{r-1}+\dfrac{B}{r}+\dfrac{C}{r+1}$, where A, B and C are constants to be determined, and hence use the method of differences to find $\displaystyle\sum_{r=2}^{n}\frac{r+3}{r^3-r}$. [5]

(c) Deduce the value of $\displaystyle\sum_{r=2}^{\infty}\frac{r+3}{r^3-r}$. [1]

3 **(a)** Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(r+1)(r+2)(r+3)=\tfrac{1}{4}n\left(n^3+bn^2+cn+d\right),$$

where b, c and d are integers to be determined. [3]

(b) Express $\frac{2}{(r+1)(r+2)(r+3)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)(r+3)}.$$ [5]

(c) State the value of $\sum_{r=1}^{\infty} \frac{2}{(r+1)(r+2)(r+3)}$. [1]

1 **(a)** Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}(2-3r)(5-3r) = an^3 + bn^2 + cn,$$

where a, b and c are integers to be determined. [3]

(b) Use the method of differences to find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2-3r)(5-3r)}$ in terms of n. [4]

(c) Deduce the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2-3r)(5-3r)}$. [1]

4 Let $w_r = r(r+1)(r+2)\ldots(r+9)$.

(a) Show that

$$w_{r+1} - w_r = 10(r+1)(r+2)\ldots(r+9).$$ [2]

(b) Given that $u_r = (r+1)(r+2)\ldots(r+9)$, find $\sum_{r=1}^{n} u_r$ in terms of n. [3]

(c) Given that $v_r = x^{w_{r+1}} - x^{w_r}$, find the set of values of x for which the infinite series

$$v_1 + v_2 + v_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

1.4 Matrices

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS transformations 变换 • rotation 旋转 • inverse 逆矩阵 • enlargement 放大 • determinant 行列式

Objective

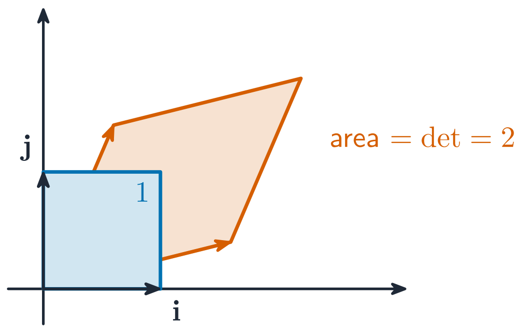
Build the skills to answer exam questions on **matrices** 矩阵 — multiplication, the **determinant** 行列式 and **inverse** 逆矩阵, and 2×2 matrices as **transformations** 变换.

You must be able to:

- multiply matrices and use I
- find $\det A$ and A^{-1} for a 2×2 matrix; use $(AB)^{-1} = B^{-1}A^{-1}$
- identify the transformation a matrix represents (and its area scale factor)

1 Worked examples

Inverse & transformation



A matrix maps the unit square to a parallelogram whose area equals $\det A$

$A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$: $\det A = 12 - 2 = 10$; $A^{-1} = \dfrac{1}{10}\begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}$.

Transformations: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a $90°$ rotation; $\det = 1$ (area preserved).

2 Practice

2.1 Find $\det \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$. [1]

2.2 Find the inverse of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. [2]

3 Exam-style questions

3.1 The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents: [1]

- **A** a rotation
- **B** an enlargement, scale factor 2
- **C** a reflection
- **D** a shear

3.2 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

(a) Find AB. [2]

(b) Find $\det(AB)$ and verify it equals $\det A \times \det B$. [3]

3.3 The transformation T has matrix $M = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the area scale factor of T. [1]

(b) Find M^{-1} and hence the image-to-object matrix that undoes T. [3]

Solutions

2.1 det $= 5(4) - 2(3) = 14$.

2.2 det $= 2 - 1 = 1$; inverse $= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

3.1 B. A scalar multiple of I is an enlargement; here scale factor 2.

3.2 (a) $AB = \begin{pmatrix} 1\cdot2+2\cdot1 & 1\cdot0+2\cdot3 \\ 3\cdot2+4\cdot1 & 3\cdot0+4\cdot3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 10 & 12 \end{pmatrix}$.
(b) $\det(AB) = 48 - 60 = -12$; $\det A = -2$, $\det B = 6$, product $= -12$ ✓.

3.3 (a) area scale factor $= \det M = 6$.
(b) $M^{-1} = \dfrac{1}{6}\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$; this matrix undoes T.

2 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [2]

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(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

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The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [2]

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(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

4 Let k and m be non-zero constants. The matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} k & 0 \\ 0 & m \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by the matrix $\mathbf{B}$ in each of the following cases.

(i) $m = 1$ [2]

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(ii) $m = k$ [2]

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(b) Show that the matrix $\mathbf{ABC}$ is singular. [6]

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4 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, where $0 < \theta < 2\pi$.

(a) The matrix **M** represents a sequence of two geometrical transformations in the x–y plane.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Find the value of θ for which the transformation represented by **M** has a line of invariant points. [7]

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1 The matrix $\mathbf{M}$ represents the sequence of two transformations in the x-y plane given by a stretch parallel to the x-axis, scale factor 14, followed by a rotation anticlockwise about the origin through angle $\frac{1}{3}\pi$.

(a) Show that $2\mathbf{M} = \begin{pmatrix} 14 & -\sqrt{3} \\ 14\sqrt{3} & 1 \end{pmatrix}$. [4]

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(b) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by $\mathbf{M}$. [5]

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The unit square S in the x-y plane is transformed by $\mathbf{M}$ onto the rectangle P.

(c) Find the matrix which transforms P onto S. [2]

1.5 Polar coordinates

Name: ______________ Class: __________ Date: ____________ **Total: 15 marks**

KEY WORDS polar coordinates 极坐标 • polar curves 极坐标曲线 • cartesian coordinates 直角坐标

Objective

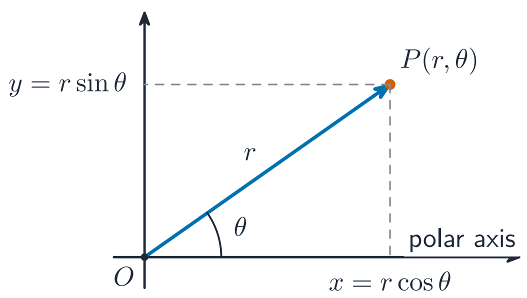
Build the skills to answer exam questions on **polar coordinates 极坐标** — converting to/from Cartesian, sketching **polar curves 极坐标曲线**, and the **sector area** $\frac{1}{2}\int r^2\,d\theta$.

You must be able to:

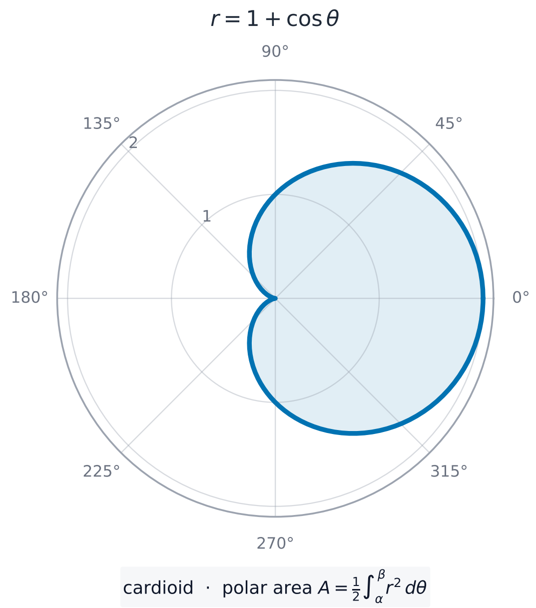
- convert between (r, θ) and (x, y)
- sketch a simple polar curve (e.g. a cardioid)
- find an area with $\frac{1}{2}\int r^2\,d\theta$

1 Worked examples

Conversion & area



$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$$



Polar curves are traced as the angle varies

Convert $r = 4\cos\theta$**:** $\times r \to r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, i.e. $(x-2)^2 + y^2 = 4$ (circle).

Area: $A = \frac{1}{2}\displaystyle\int_\alpha^\beta r^2\,d\theta$.

2 Practice

2.1 Convert the point $(r,\theta) = \left(2, \frac{\pi}{3}\right)$ to Cartesian coordinates. [2]

2.2 Write down the formula for the area of a polar sector. [1]

3 Exam-style questions

3.1 The polar equation $r = 2\sin\theta$ represents: [1]

- **A** a line
- **B** a circle
- **C** a parabola
- **D** a spiral

3.2 A curve has polar equation $r = 1 + \cos\theta$ (a cardioid).

(a) Find r when $\theta = 0$ and $\theta = \frac{\pi}{2}$. [2]

(b) Find the area enclosed by the curve, using $A = \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2 \, d\theta$. [5]

3.3 Convert $r = \dfrac{2}{1+\cos\theta}$ to Cartesian form. [4]

Solutions

2.1 $x = 2\cos\frac{\pi}{3} = 1$, $y = 2\sin\frac{\pi}{3} = \sqrt{3}$.

2.2 $A = \frac{1}{2}\displaystyle\int r^2\,d\theta$.

3.1 **B**. $r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y$, a circle.

3.2 (a) $\theta = 0$: $r = 2$; $\theta = \frac{\pi}{2}$: $r = 1$.
(b) $(1+\cos\theta)^2 = 1+2\cos\theta+\cos^2\theta = \frac{3}{2}+2\cos\theta+\frac{1}{2}\cos 2\theta$; $A = \frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$;
$= \frac{1}{2}\left(\frac{3}{2}\cdot 2\pi\right) = \frac{3\pi}{2}$.

3.3 $r + r\cos\theta = 2 \Rightarrow r = 2 - x$ (since $r\cos\theta = x$); $r^2 = (2-x)^2 \Rightarrow x^2 + y^2 = 4 - 4x + x^2$;
$y^2 = 4 - 4x$, i.e. $y^2 = -4(x-1)$.

6 The curve C has polar equation $r = \sin 3\theta$, for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$.

(a) Sketch C and state the equation of the line of symmetry. [3]

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(b) Find the exact value of the area of the region enclosed by C. [4]

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In parts **(c)** and **(d)** you may use the identity $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$.

(c) Find the maximum distance of a point on C from the initial line. [5]

(d) Find a Cartesian equation for C. [3]

5 The curve C has polar equation $r^2 = \tan 2\theta$, where $0 \leqslant \theta \leqslant \frac{1}{8}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

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(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{8}\pi$. [4]

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(c) Show that C has Cartesian equation $x^4 - 2xy - y^4 = 0$ given that $0 \leqslant x \leqslant \cos\left(\frac{1}{8}\pi\right)$ and $0 \leqslant y \leqslant \sin\left(\frac{1}{8}\pi\right)$. [4]

(d) Using your answer to **(b)**, deduce the exact value of the area bounded by C, the x-axis and the line $x = \cos\left(\frac{1}{8}\pi\right)$. [2]

6 The curve C has polar equation $r = \cos\frac{1}{2}\theta$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C. [2]

In parts **(b)** and **(c)** you may use the identities $\sin\theta \equiv 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta$ and $\cos\theta \equiv 2\cos^2\frac{1}{2}\theta - 1$.

(b) Find the exact value of the area of the region enclosed by C and the initial line. [4]

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(c) Find the maximum distance of a point on C from the initial line. Give your answer in the form $p\sqrt{q}$, where p and q are rational numbers to be determined. [6]

5 The curve C has polar equation $r = \theta e^{\frac{1}{8}\theta}$, for $0 \leqslant \theta \leqslant 2\pi$.

(a) Sketch C. [2]

(b) Find the area of the region bounded by C and the initial line, giving your answer in the form $\left(p\pi^2 + q\pi + r\right)e^{\frac{1}{2}\pi} + s$, where p, q, r and s are integers to be determined. [6]

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(c) Show that, at the point of C furthest from the initial line,

$$\theta\cos\theta + \left(\tfrac{1}{8}\theta + 1\right)\sin\theta = 0$$

and verify that this equation has a root between 5 and 5.05. [5]

7 The curve C has polar equation $r^2 = \mathrm{e}^{\sin\theta}\cos\theta$, for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole, giving your answers correct to 3 decimal places. [5]

(b) Find the polar coordinates of the point on C that is furthest from the half-line $\theta = \frac{1}{2}\pi$, giving your answers correct to 3 decimal places. [5]

(c) Sketch C. [3]

(d) Find the area of the region bounded by C, giving your answer in exact form. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

1.6 Vectors

Name: ____________ Class: ________ Date: __________ **Total: 17 marks**

KEY WORDS plane 平面 • vectors 向量 • vector product 向量积

Objective

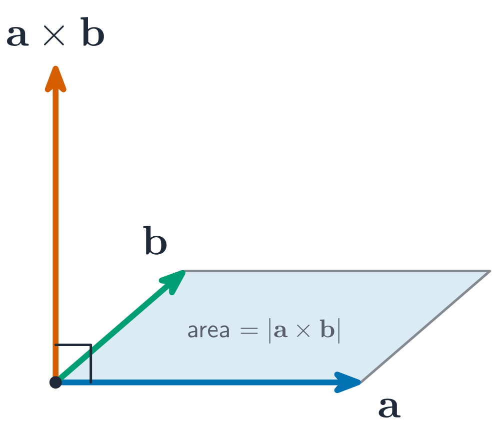
Build the skills to answer exam questions on **vectors** 向量 in 3-D — the **vector product** 向量积, equations of a **plane** 平面, and distances/angles between lines and planes.

You must be able to:

- compute $\mathbf{a} \times \mathbf{b}$ and use it for a normal/area
- write a plane as $\mathbf{r} \cdot \mathbf{n} = p$ or $ax + by + cz = d$
- find the angle between, and distance from, lines and planes

1 Worked examples

■ Vector product & plane



$\mathbf{a} \times \mathbf{b}$ *is perpendicular to both; its length is the area of the parallelogram*

$\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$: $\mathbf{a} \times \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

Plane through A **with normal n:** $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$.

2 Practice

2.1 Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$. [3]

2.2 Write the plane through $(1, 2, 3)$ with normal $\mathbf{n} = (1, 1, 1)$ in the form $ax + by + cz = d$. [2]

3 Exam-style questions

3.1 $\mathbf{a} \times \mathbf{b}$ is a vector that is: [1]

- **A** parallel to $\mathbf{a}$
- **B** perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- **C** equal to $\mathbf{a} \cdot \mathbf{b}$
- **D** of length $|\mathbf{a}||\mathbf{b}| \cos\theta$

3.2 Points $A(1, 0, 2)$, $B(2, 1, 0)$ and $C(0, 2, 1)$ are given.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [4]

(b) Hence find the equation of the plane through A, B, C. [3]

3.3 Find the acute angle between the planes $x + 2y + 2z = 5$ and $2x - y + 2z = 1$. [4]

Solutions

2.1 $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(2) + \mathbf{k}(-1) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

2.2 $x + y + z = 1 + 2 + 3 = 6$.

3.1 **B**. The vector product is perpendicular to both vectors.

3.2 (a) $\overrightarrow{AB} = (1, 1, -2)$, $\overrightarrow{AC} = (-1, 2, -1)$; cross $= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = \mathbf{i}(-1 + 4) - \mathbf{j}(-1 - 2) + \mathbf{k}(2 + 1) = (3, 3, 3)$.
(b) normal $(1, 1, 1)$ through $A(1, 0, 2)$: $x + y + z = 3$.

3.3 normals $(1, 2, 2)$ and $(2, -1, 2)$; dot $= 2 - 2 + 4 = 4$; $|\mathbf{n}_1| = 3$, $|\mathbf{n}_2| = 3$; $\cos\theta = \dfrac{4}{9} \Rightarrow \theta = 63.6°$.

5 The plane Π_1 has equation $\mathbf{r} = -3\mathbf{i} - \mathbf{j} - \mathbf{k} + \lambda(\mathbf{j} + 2\mathbf{k}) + \mu(\mathbf{i} + 3\mathbf{j} + \mathbf{k})$.

(a) Find an equation for Π_1 in the form $ax + by + cz = d$. [4]

(b) Find the perpendicular distance from the point with position vector $-\mathbf{i} - 2\mathbf{k}$ to Π_1. [3]

(c) The plane Π_2 has equation $3x+2y-z=14$.

Find a vector equation of the line of intersection of Π_1 and Π_2. [4]

6 The plane Π has equation $x + 3y + 2z = 1$.

(a) Find the perpendicular distance from the origin O to the plane Π. [2]

Relative to O, the points A, B, C have position vectors

$$-\mathbf{j} + 2\mathbf{k}, \quad 2\mathbf{i} - \mathbf{k}, \quad 2\mathbf{i} - \mathbf{j} - \mathbf{k},$$

respectively.

(b) Find the acute angle between the planes OAB and Π. [4]

(c) Find an equation for the common perpendicular to the lines OC and AB. [8]

4 The points A, B, C have position vectors

$$3\mathbf{i}+5\mathbf{j}+5\mathbf{k}, \qquad 2\mathbf{i}+2\mathbf{j}+2\mathbf{k}, \qquad 2\mathbf{i}+\mathbf{j}-\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) The point D has position vector $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$.

Find the shortest distance between the lines AB and CD. [5]

6 The points A, B, C have position vectors

$$\mathbf{i} - 2\mathbf{k}, \quad \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}, \quad 2\mathbf{i} - \mathbf{j} - \mathbf{k},$$

respectively.

(a) Find the equation of the plane ABC, giving your answer in the form $ax + by + cz = d$. [5]

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A point D has position vector $\mathbf{i} + t\mathbf{k}$, where $t \neq -2$.

(b) Find the acute angle between the planes ABC and ABD. [4]

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(c) Find the values of t such that the shortest distance between the lines AB and CD is $\sqrt{2}$. [7]

5 The plane Π has equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + \mu(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The point P has position vector $4\mathbf{i} + 2\mathbf{j} + 9\mathbf{k}$.

(b) Find the position vector of the foot of the perpendicular from P to Π. [4]

(c) The line l is parallel to the vector $3\mathbf{i} + 5\mathbf{j} - \mathbf{k}$.

Find the acute angle between l and Π. [3]

1.7 Proof by induction

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS proof by induction 数学归纳法 • inductive proof 归纳证明

Objective

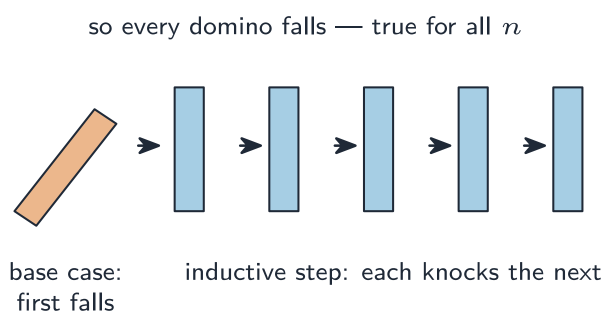
Build the skills to answer exam questions on **proof by induction 数学归纳法** — the base case, the inductive step, and a clear concluding statement, for series, divisibility and matrices.

You must be able to:

- prove a summation formula by induction
- prove a divisibility result by induction
- write the base case, inductive step and conclusion clearly

1 Worked examples

The two steps



Base case topples the first; the inductive step makes each one knock over the next

Prove $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. Base $n = 1$: $1 = \frac{1}{2}(1)(2)$ ✓. Assume true for k; then $\sum_{1}^{k+1} = \frac{1}{2}k(k+1) + (k+1) = \frac{1}{2}(k+1)(k+2)$ — the formula at $n = k+1$. So true for all n.

2 Practice

2.1 State the two steps of a proof by induction. [2]

2.2 Verify the base case $n = 1$ for $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$. [1]

3 Exam-style questions

3.1 In an inductive proof, the inductive step assumes the result for $n = k$ and proves it for: [1]

- **A** $n = 1$
- **B** $n = k - 1$
- **C** $n = k + 1$
- **D** all n at once

3.2 Prove by induction that $\displaystyle\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$. [5]

3.3 Prove by induction that $7^n - 1$ is divisible by 6 for all positive integers n. [5]

Solutions

2.1 base case (true for $n = 1$); inductive step (assume $n = k$, prove $n = k + 1$).

2.2 LHS $= 1$, RHS $= \frac{1}{6}(1)(2)(3) = 1$ ✓.

3.1 **C**. The inductive step proves the result for $n = k + 1$.

3.2 base $n = 1$: LHS $= 2$, RHS $= \frac{1}{3}(1)(2)(3) = 2$ ✓; assume $\sum_1^k r(r+1) = \frac{1}{3}k(k+1)(k+2)$; $\sum_1^{k+1} = \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$; $= (k+1)(k+2)\left(\frac{k}{3} + 1\right) = \frac{1}{3}(k+1)(k+2)(k+3)$; this is the formula at $n = k + 1$, so true for all n.

3.3 base $n = 1$: $7^1 - 1 = 6$, divisible by 6 ✓; assume $7^k - 1 = 6m$ for integer m; $7^{k+1} - 1 = 7 \cdot 7^k - 1 = 7(6m + 1) - 1 = 42m + 6 = 6(7m + 1)$; divisible by 6, so by induction true for all n.

3 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\cos x) = (-1)^n(x\sin x - (2n-1)\cos x).$$ [7]

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{6u_n + 5}{u_n + 2}.$$

(a) By considering $5 - u_{n+1}$, prove by mathematical induction that $u_n < 5$ for all positive integers n. [5]

(b) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. [3]

1 Prove by mathematical induction that $\left(\frac{6}{5}\right) \geqslant 1 + \frac{1}{5}n$ for all positive integers n. [5]

3 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = 6u_n + 5$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 6^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

2 Prove by mathematical induction that $2025^n + 47^n - 2$ is divisible by 46 for all positive integers n. [6]