

4.5 Probability generating functions

Name: _____ Class: _____ Date: _____ Total: 14 marks

KEY WORDS probability generating functions 概率母函数

Objective

Build the skills to answer exam questions on **probability generating functions** 概率母函数 — building $G(t)$, getting the mean and variance from its derivatives, and combining independent variables.

You must be able to:

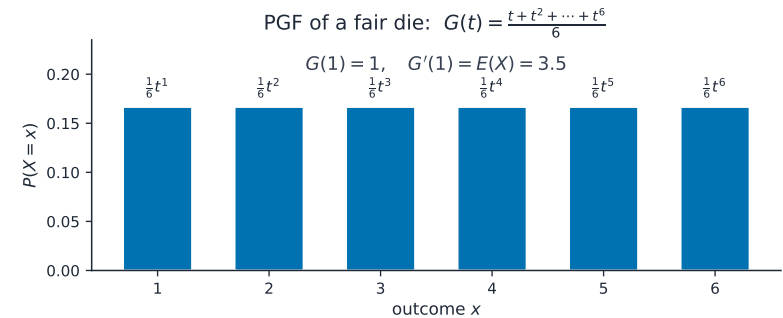
- write $G(t) = E(t^X) = \sum P(X = x)t^x$
- use $E(X) = G'(1)$ and $\text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$
- multiply PGFs for a sum of independent variables

1 Worked examples

■ Mean from the PGF



A PGF packs a whole discrete distribution into one function $G(t)$



$$G(t) = E(t^X) \cdot E(X) = G'(1) \cdot \text{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$$

The PGF of a fair die: $P(X = x)$ is the coefficient of t^x , and $G'(1) = E(X) = 3.5$

$P(X = 0) = 0.5, P(X = 1) = 0.3, P(X = 2) = 0.2: G(t) = 0.5 + 0.3t + 0.2t^2;$
 $G'(t) = 0.3 + 0.4t$, so $E(X) = G'(1) = 0.7$.

2 Practice

2.1 Write down $G(t)$ for a variable with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$. [2]

2.2 State the formula for $E(X)$ in terms of G . [1]

3 Exam-style questions

3.1 For any PGF, $G(1)$ equals: [1]

- | | |
|---|-----------------|
| A | 0 |
| B | $E(X)$ |
| C | 1 |
| D | $\text{Var}(X)$ |

3.2 A variable has $G(t) = \frac{1}{4}(1+t)^2$.

(a) Find the probability distribution of X .

[3]

(b) Find $E(X)$ and $\text{Var}(X)$.

[4]

3.3 X and Y are independent with PGFs $G_X(t) = \frac{1}{2}(1+t)$ and $G_Y(t) = \frac{1}{3}(1+t+t^2)$. Find the PGF of $X+Y$ and hence $P(X+Y=0)$.

[3]
