A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.3 Differentiation (Maclaurin, hyperbolic, parametric)

Name: ____________ Class: ________ Date: __________ **Total: 37 marks**

KEY WORDS maclaurin's series 麦克劳林级数 • hyperbolic functions 双曲函数 • further differentiation 微分

Objective

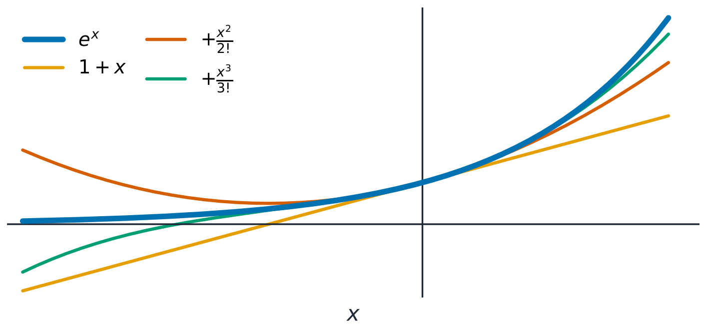
Build the skills to answer exam questions on **further differentiation 微分** — derivatives of inverse and **hyperbolic functions 双曲函数**, second derivatives of implicit/parametric curves, and **Maclaurin's series 麦克劳林级数**.

You must be able to:

- differentiate $\sin^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$
- find $\dfrac{d^2y}{dx^2}$ for implicit/parametric curves
- find a Maclaurin series for a function

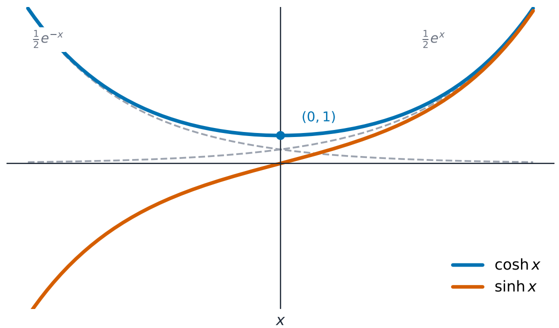
1 Worked examples

■ Maclaurin series



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Adding more terms makes the polynomial match the function over a wider stretch around $x = 0$



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \cdot \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cdot \quad \cosh^2 x - \sinh^2 x = 1$$

The hyperbolic functions whose derivatives appear here

Maclaurin: $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$; e.g. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots$.

Derivatives: $\dfrac{d}{dx}\sinh x = \cosh x$; $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1+x^2}$.

■ **The derivatives to know, and the substitutions they unlock**

Hyperbolic functions, defined from the exponential:

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \qquad \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

The identity is $\cosh^2 x - \sinh^2 x = 1$ — note the **minus**, which is what makes the substitutions differ from the trigonometric ones.

y	$\frac{dy}{dx}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (no minus — unlike cos)
$\tanh x$	$\operatorname{sech}^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$

Reading the integral to choose the substitution — the sign under the root decides:

Integrand contains	**Substitute**	**Because**
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2 = \cos^2$
a^2+x^2	$x = a\tan\theta$	$1+\tan^2 = \sec^2$
$\sqrt{a^2+x^2}$	$x = a\sinh u$	$1+\sinh^2 = \cosh^2$
$\sqrt{x^2-a^2}$	$x = a\cosh u$	$\cosh^2 -1 = \sinh^2$

Worked example. Differentiate $y = \operatorname{arsinh}(3x)$ and hence write down $\int \frac{1}{\sqrt{1+9x^2}}\,dx$.

1. By the chain rule, $\frac{dy}{dx} = \frac{3}{\sqrt{1+(3x)^2}} = \frac{3}{\sqrt{1+9x^2}}$.
2. So $\int \frac{3}{\sqrt{1+9x^2}}\,dx = \operatorname{arsinh}(3x) + C$.
3. Dividing by 3: $\int \frac{1}{\sqrt{1+9x^2}}\,dx = \frac{1}{3}\operatorname{arsinh}(3x) + C$.

That is the pattern behind almost every "differentiate, hence integrate" question: the integral you are asked for is the derivative you just found, scaled.

2 Practice

2.1 Differentiate $y = \cosh(3x)$. [2]

2.2 Differentiate $y = \sin^{-1}(2x)$. [2]

2.3 Differentiate $y = \cosh(4x)$ and $y = \arctan(2x)$. [3]

2.4 State the substitution you would use for $\int \frac{1}{\sqrt{x^2-9}}\,dx$, and why. [2]

2.5 Show that $\cosh^2 x - \sinh^2 x = 1$ using the exponential definitions. [3]

3 Exam-style questions

3.1 $\frac{d}{dx}\tanh x$ equals: [1]

A B
C D

A $\operatorname{sech}^2 x$
B $\operatorname{cosech}^2 x$
C $\cosh^2 x$
D $\sinh^2 x$

3.2 Find the Maclaurin series for $\ln(1+x)$ up to and including the term in x^3. [4]

3.3 A curve is given by $x = t^2$, $y = t^3$. Find $\dfrac{d^2y}{dx^2}$ in terms of t. [5]

3.4 (a) Differentiate $y = \operatorname{arcosh}(2x)$ with respect to x. [3]

(b) Hence find $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx$. [2]

(c) Use the substitution $x = 3\sinh u$ to show that $\displaystyle\int \frac{1}{\sqrt{9+x^2}}\,dx = \operatorname{arsinh}\left(\frac{x}{3}\right) + C$. [5]

(d) Differentiate $y = x\arctan x$ and hence evaluate $\displaystyle\int_0^1 \arctan x\,dx$, giving your answer in exact form. [5]