

2.3 Differentiation (Maclaurin, hyperbolic, parametric)

Name: _____ Class: _____ Date: _____

Total: 37 marks

KEY WORDS maclaurin's series 麦克劳林级数 • hyperbolic functions 双曲函数
• further differentiation 微分

Objective

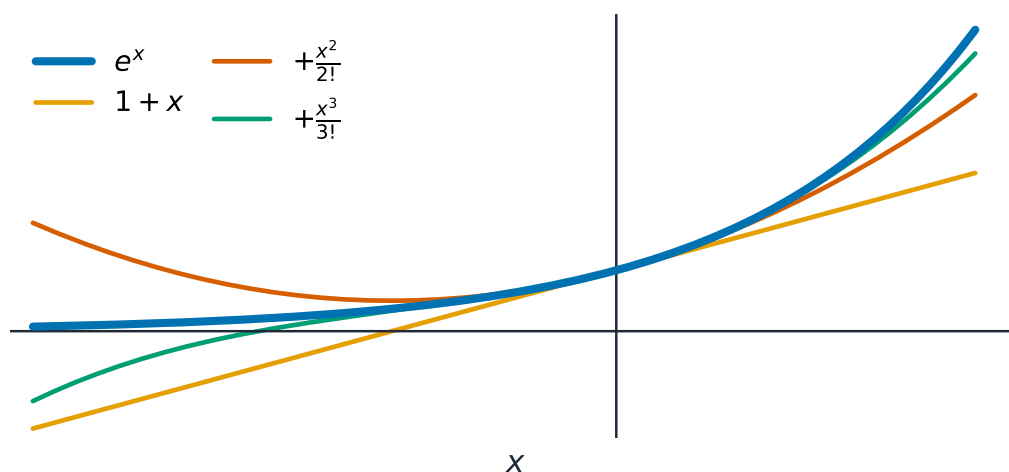
Build the skills to answer exam questions on **further differentiation** 微分 — derivatives of inverse and **hyperbolic functions** 双曲函数, second derivatives of implicit/parametric curves, and **Maclaurin's series** 麦克劳林级数.

You must be able to:

- differentiate $\sin^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$
- find $\frac{d^2y}{dx^2}$ for implicit/parametric curves
- find a Maclaurin series for a function

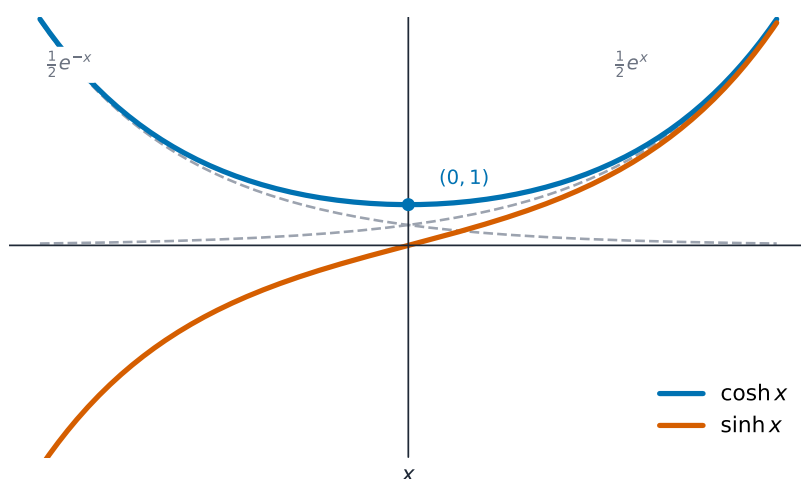
1 Worked examples

■ Maclaurin series



$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \cdot \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Adding more terms makes the polynomial match the function over a wider stretch around $x = 0$



$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \cdot \quad \sinh x = \frac{e^x - e^{-x}}{2} \quad \cdot \quad \cosh^2 x - \sinh^2 x = 1$$

The hyperbolic functions whose derivatives appear here

Maclaurin: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots$; e.g. $e^x = 1 + x + \frac{x^2}{2!} + \dots$.

Derivatives: $\frac{d}{dx} \sinh x = \cosh x$; $\frac{d}{dx} \tanh^{-1} x = \frac{1}{1+x^2}$.

■ The derivatives to know, and the substitutions they unlock

Hyperbolic functions, defined from the exponential:

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \tanh x = \frac{\sinh x}{\cosh x}$$

The identity is $\cosh^2 x - \sinh^2 x = 1$ — note the **minus**, which is what makes the substitutions differ from the trigonometric ones.

y	$\frac{dy}{dx}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$ (no minus — unlike $\cos$)
$\tanh x$	$\operatorname{sech}^2 x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$\frac{1}{1+x^2}$
$\operatorname{arsinh} x$	$\frac{1}{\sqrt{1+x^2}}$
$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$

Reading the integral to choose the substitution — the sign under the root decides:

Integrand contains	Substitute	Because
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 = \cos^2$
$a^2 + x^2$	$x = a \tan \theta$	$1 + \tan^2 = \sec^2$
$\sqrt{a^2 + x^2}$	$x = a \sinh u$	$1 + \sinh^2 = \cosh^2$
$\sqrt{x^2 - a^2}$	$x = a \cosh u$	$\cosh^2 - 1 = \sinh^2$

Worked example. Differentiate $y = \operatorname{arsinh}(3x)$ and hence write down $\int \frac{1}{\sqrt{1+9x^2}} dx$.

1. By the chain rule, $\frac{dy}{dx} = \frac{3}{\sqrt{1+(3x)^2}} = \frac{3}{\sqrt{1+9x^2}}$.
2. So $\int \frac{3}{\sqrt{1+9x^2}} dx = \operatorname{arsinh}(3x) + C$.
3. Dividing by 3: $\int \frac{1}{\sqrt{1+9x^2}} dx = \frac{1}{3} \operatorname{arsinh}(3x) + C$.

That is the pattern behind almost every “differentiate, hence integrate” question: the integral you are asked for is the derivative you just found, scaled.

2 Practice

2.1 Differentiate $y = \cosh(3x)$. [2]

2.2 Differentiate $y = \sin^{-1}(2x)$. [2]

2.3 Differentiate $y = \cosh(4x)$ and $y = \arctan(2x)$. [3]

2.4 State the substitution you would use for $\int \frac{1}{\sqrt{x^2 - 9}} dx$, and why. [2]

2.5 Show that $\cosh^2 x - \sinh^2 x = 1$ using the exponential definitions. [3]

3 Exam-style questions

3.1 $\frac{d}{dx} \tanh x$ equals: [1]

A	B
C	D

- A** $\operatorname{sech}^2 x$
B $\operatorname{cosech}^2 x$
C $\cosh^2 x$
D $\sinh^2 x$

3.2 Find the Maclaurin series for $\ln(1+x)$ up to and including the term in x^3 . [4]

3.3 A curve is given by $x = t^2$, $y = t^3$. Find $\frac{d^2y}{dx^2}$ in terms of t . [5]

3.4 (a) Differentiate $y = \operatorname{arcosh}(2x)$ with respect to x . [3]

(b) Hence find $\int \frac{1}{\sqrt{4x^2 - 1}} dx$. [2]

(c) Use the substitution $x = 3 \sinh u$ to show that $\int \frac{1}{\sqrt{9 + x^2}} dx = \operatorname{arsinh}\left(\frac{x}{3}\right) + C$. [5]

(d) Differentiate $y = x \arctan x$ and hence evaluate $\int_0^1 \arctan x \, dx$, giving your answer in exact form. [5]