Solutions

Each answer shows the steps, then the **result**.

2.1

- $\det(A - \lambda I) = 0$

2.2

- $\lambda = 4$ and $\lambda = -2$ (the diagonal entries)

2.3

- $\operatorname{tr} A = 7$ and $\det A = 10$, so $\lambda^2 - 7\lambda + 10 = 0$

2.4

- P has the eigenvectors as its columns; D is diagonal with the matching eigenvalues, in the same order

2.5

- $A^2 - 6A + 8I = \mathbf{0}$, so $A^2 = 6A - 8I$

3.1 B

- $A\mathbf{v} = \lambda\mathbf{v}$ defines an eigenvector

3.2

(a)

- $\det\begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$

(b)

- $\lambda = 3$: $(A - 3I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3.3

- $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, so $M^2 = QD^2Q^{-1}$ with $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$; this gives $M^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$; direct: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ ✓

3.4

(a)

- $\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 6$, and setting it to zero gives the result

(b)

- $(\lambda - 6)(\lambda - 1) = 0$, so $\lambda = 6$ or $\lambda = 1$

(c)

- For $\lambda = 6$: $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \mathbf{v} = \mathbf{0}$ gives $v_1 = 2v_2$, so $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For $\lambda = 1$: $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0}$ gives $2v_1 = -v_2$, so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(d)

- $P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

(e)

- $A^n = PD^nP^{-1}$; D^n is found by raising each diagonal entry to the power n, so $A^n = P \begin{pmatrix} 6^n & 0 \\ 0 & 1 \end{pmatrix} P^{-1}$, one multiplication instead of n