A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

4.5 Probability generating functions

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS probability generating functions 概率母函数

Objective

Build the skills to answer exam questions on **probability generating functions 概率母函数** — building $G(t)$, getting the mean and variance from its derivatives, and combining independent variables.

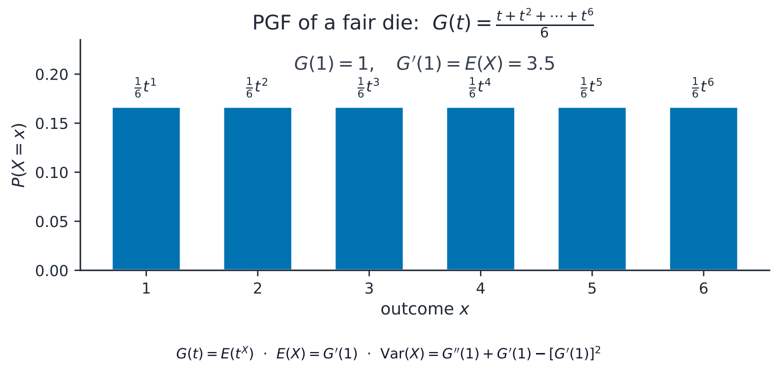
You must be able to:

- write $G(t) = E(t^X) = \sum P(X = x)t^x$
- use $E(X) = G'(1)$ and $\mathrm{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$
- multiply PGFs for a sum of independent variables

1 Worked examples

■ Mean from the PGF

A PGF packs a whole discrete distribution into one function $G(t)$



The PGF of a fair die: $P(X = x)$ is the coefficient of t^x, and $G'(1) = E(X) = 3.5$

$P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$: $G(t) = 0.5 + 0.3t + 0.2t^2$; $G'(t) = 0.3 + 0.4t$, so $E(X) = G'(1) = 0.7$.

2 Practice

2.1 Write down $G(t)$ for a variable with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$. [2]

2.2 State the formula for $E(X)$ in terms of G. [1]

3 Exam-style questions

3.1 For any PGF, $G(1)$ equals: [1]

- **A** 0
- **B** $E(X)$
- **C** 1
- **D** $\mathrm{Var}(X)$

3.2 A variable has $G(t) = \frac{1}{4}(1+t)^2$.

(a) Find the probability distribution of X. [3]

(b) Find $E(X)$ and $\mathrm{Var}(X)$. [4]

3.3 X and Y are independent with PGFs $G_X(t) = \frac{1}{2}(1+t)$ and $G_Y(t) = \frac{1}{3}(1+t+t^2)$. Find the PGF of $X+Y$ and hence $P(X+Y=0)$. [3]

Solutions

2.1 $G(t) = \frac{1}{3}(t+t^2+t^3)$.

2.2 $E(X) = G'(1)$.

3.1 C. $G(1) = \sum P(X=x) = 1$.

3.2 (a) $G(t) = \frac{1}{4}(1+2t+t^2) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$, so $P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{1}{2}$, $P(X=2) = \frac{1}{4}$.
(b) $G'(t) = \frac{1}{2} + \frac{1}{2}t \Rightarrow E(X) = G'(1) = 1$; $G''(t) = \frac{1}{2} \Rightarrow \mathrm{Var}(X) = \frac{1}{2} + 1 - 1^2 = \frac{1}{2}$.

3.3 $G_{X+Y}(t) = G_X(t)G_Y(t) = \frac{1}{2}(1+t)\cdot\frac{1}{3}(1+t+t^2) = \frac{1}{6}(1+t)(1+t+t^2)$; $P(X+Y=0)$ is the constant term $= \frac{1}{6}$.